On Networked Dynamical Systems with Heterogeneous Constraints: Equilibrium Points and Stability

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Abstract: In this paper, we investigate multi-agent systems with heterogeneous constraints, which can represent the heuristic beliefs of the agents towards an issue or the physical constraints of the agents. Heterogeneous constraints are frequently observed however have rarely been characterized. As a result of the heterogeneity of the constraints, an equilibrium point may not exist and is presumably to be a dissensus point when it exists. We investigate the existence of equilibrium points from the perspective of Kakutani's fixed point theory for a set-valued map. We also prove the local/global stability of certain equilibrium points. Then, a special case that the constraints are homogeneous is taken into account. The constrains are assumed to take interval forms with nonempty intersection. It is proved that consensus can be achieved globally and asymptotically for this case. Numerical examples are designed to illustrate our theoretical findings.

Key Words: Multi-agent systems, heterogeneous constraints, equilibrium point and its stability, constrained consensus

1 Introduction

Multi-agent systems (MASs), which can be seen to underlie various fields including distributed coordination and social network modeling and analysis, have been extensively studied over the past decades [1–5]. To some extent, *constraints* of the agents are indispensable for MASs. They are intended to characterize specific properties of the agents. In general, the constraint can be any form that regulates or guides the collective behavior of the agents.

One popular kind of the constraints frequently observed is that imposed on agents' states, which arises probably due to, e.g., the physical requirement (e.g., storage capacity of an agent) or heuristic belief of an agent towards an issue [6–10, 20]. To achieve constrained consensus, each agent simply discards the states of its neighbors that are not in its own constraint interval [7]. Ref. [8] designs a potential function incorporating an explicit constraint term and uses its gradient to seek consensus while guarantee state constraint. Time-varying state constraints for a group of full state coupled nonlinear MASs are considered in [9]. It is shown via auxiliary variables that constrained consensus is equivalent to bounded consensus [9]. Projection-based method commonly used in optimization is also applied to the state constrained consensus problem, see for instance [10]. The above works [7–10] impose *strict* constraint on states. In contrast, ref. [6] proposes an *elastic* interval consensus model, which is later revisited in [20]. By being elastic means that the agents are guaranteed to satisfy their constraints *in the limit*.

Beyond existing constraints that have been widely investigated, there exist many others that are held by the agents on states but have not been fully characterized, one of which can be observed from the following facts. (i) Before a decision making or issue discussing process, an agent may have a heuristic preference or belief [11–14] towards the final decision or opinion. Such a heuristic belief/preference has multiple origins, one of which is probably the stereotype of an individual, which represents each individual's stored category-level information that greatly influences the judgment. (ii) During a discussion of an issue, there might exist very convinced individuals with extreme opinions, called extremists, and also moderate individuals being less certain on the issue [18]. (iii) In addition, in the seeking for a common point of a group of sets, it is usually the case that the sets consist of convex and non-convex ones. This motivate the agents to impose heterogeneous constraints on their states.

With the above observation as a motivation, we develop a multi-agent system having heterogeneous constraints. Specifically, each agent maps the combination of its neighbors' states into a set, which can be either an interval or a set with finite number of elements. Different kinds of the sets reflect the heterogeneity of the constraints for the agents. Moreover, the states of agents do not need to satisfy their constraints all the time. We only require that the converged values of the states satisfy their constraints. Thus, the proposed model can be seen as imposing *elastic* constraints on states.

The following contributions are made in this paper. (i) First, since the mapped sets can be either an interval or a set of discrete points, the agent dynamics involves non-continuous and non-contractive terms. In view of this, the Kakutani's fixed point theory is applied to describe the equilibrium points. An interesting observation is that heterogeneity of the constraints makes it more likely that dissensus of the agents appears. (ii) Second, local stability for a class of continuous equilibrium point is proved. We also discuss a di-

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rected ring communication graph and show that an equilibrium point exists uniquely and is globally stable under certain conditions. (iii) Finally, we deal with the case that the constraints are homogeneous and take interval forms. It is shown that if the intersection of the intervals is non-empty and the agents communicate over a strongly connected graph, then they reach an agreement asymptotically. This indicates that consensus is possible provided that the constraints have certain similarities. The constrained MAS proposed in this paper can be seen as an alternative for interval consensus of multi-agent systems, which has been investigated in [6].

The remainder of the paper consists of five sections. Some notations and definitions are provided in Section 2. The problem is formulated and the model of the MAS with heterogeneous constraints is proposed in Section 3. Section 4 presents the main results, followed by numerical examples in Section 5. The paper is finally concluded in Section 6.

2 Preliminary

2.1 Notation

Let $\|\cdot\|$ be the Euclidean norm of a finite dimensional vector. $\mathbb{R}$ is the set of real numbers. Denote by $\mathbf{I}_n$ the identity matrix in $\mathbb{R}^{n\times n}$ (if the subscript is dropped, then $\mathbf{I}$ denotes the identity matrix of a compatible dimension) and by $\mathbf{0}_{m\times n}$ the zero matrix in $\mathbb{R}^{m\times n}$. For two scalar-valued functions g_1 and g_2, $g_2 \circ g_1 = g_2(g_1)$. $[a,b]^N$ denotes the Cartesian product of N intervals $[a,b]$'s. $\rho(\mathbf{M})$ represents the spectral radius of $\mathbf{M} \in \mathbb{R}^{N\times N}$.

2.2 Algebraic Graph Theory

Let $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathbf{A})$ be a weighted directed graph with a set of nodes $\mathcal{V} = \{1, 2, \cdots, N\}$, a set of edges $\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$. If $(i,j) \in \mathcal{E}$, then j can receive the information from i. A weighted adjacency matrix $\mathbf{A} = [a_{ij}] \in \mathbb{R}^{N\times N}$ is defined such that a_{ij} is the weight of the directed edge (j,i) and $a_{ij} \neq 0$ if $(j,i) \in \mathcal{E}$; $a_{ij} = 0$ otherwise. Assume that $a_{ii} = 0$, $\forall i \in \mathcal{V}$. If $(j,i) \in \mathcal{E}$, j is said to be a neighbor of i. Denote by $\mathcal{N}_i = \{k | (k,i) \in \mathcal{E}\}$ the set of node i's in-neighbors in $\mathcal{G}$.

A directed path from node i to node j in the graph $\mathcal{G}$ is a sequence of distinct edges of the following form: $(i, s_1), (s_1, s_2), \ldots, (s_n, j) \in \mathcal{E}$. If between any distinct nodes i and j of $\mathcal{G}$, there is a directed path from i to j, then we call $\mathcal{G}$ *strongly connected*.

2.3 Kakutani's Fixed Point

Consider a map $\mathbf{X} : \mathbb{R}^d \to \mathbb{R}^d$, which can be discontinuous. The Filippov set-valued map $\mathbf{F}[\mathbf{X}]$ with respect to the map $\mathbf{X}$ is defined by [16]

$$\mathbf{F}[\mathbf{X}](\mathbf{x}) \triangleq \cap_{\delta>0} \cap_{\mu(S)=0} \bar{co}\{\mathbf{X}(B(\mathbf{x},\delta) \setminus S)\}, \quad \mathbf{x} \in \mathbb{R}^d,$$

where $\bar{co}$ denotes the convex closure, μ is the Lebesgue measure, and $B(\mathbf{x},\delta) = \{\mathbf{y} \in \mathbb{R}^d | \|\mathbf{y} - \mathbf{x}\| \leq \delta\}$.

Example 1 (Filippov set-valued map of the sign function [16])**.** *Consider the sign function* $\mathbf{X} : \mathbb{R} \to \mathbb{R}$

$$\mathbf{X}(x) = sign(x) = \begin{cases} 1, & x > 0 \\ 0, & x = 0 \\ -1, & x < 0. \end{cases}$$

The Filippov set-valued map of $\mathbf{X}$ *is as follows*

$$\mathbf{F}[\mathbf{X}](x) = \begin{cases} 1, & x > 0 \\ [-1,1], & x = 0 \\ -1, & x < 0. \end{cases}$$

Lemma 1 (Kakutani's Fixed Point [15])**.** *For any given positive integer n, let Ω be a nonempty, closed, bounded and convex subset of $\mathbb{R}^n$. If F is a convex-valued self-correspondence on Ω that has a closed graph, then F has a fixed point, that is, there exists an $x \in \Omega$ with $x \in F(x)$.*

3 Problem Formulation

3.1 System Model

Consider a group of N agents evolving according to the following rule:

$$\mathbf{x}_i(k+1) = \alpha \mathbf{x}_i(k) + \beta \mathbf{f}_i\left(\sum_{j\in\mathcal{N}_i} a_{ij}\mathbf{x}_j(k)\right), \; i = 1, \ldots, N. \tag{1}$$

The variable $\mathbf{x}_i(k) \in \mathbb{R}$ denotes the state of the i-th agent. $\mathbf{f}_i : \mathbb{R} \to \mathbb{R}$ is a nonlinear function, which represents the constraint of agent i. $\alpha \geq 0$ and $\beta > 0$ are update parameters satisfying $\alpha + \beta = 1$. The edge weights are normalized such that $\sum_{j=1}^N a_{ij} = 1$ for all i.

Two types of constraints, i.e., two forms of $\mathbf{f}_i$, are considered. One maps its argument into an interval and the other maps its argument into a set with finite number of elements. They are specified as follows (refer to Figure 1 for a simple illustration).

1) $\mathbf{f}_m = \chi_m$ is continuous and piecewise linear:

$$\chi_m(z) = \begin{cases} q_m, & \text{if } z > q_m \\ z, & \text{if } p_m \leq z \leq q_m \\ p_m, & \text{if } z < p_m. \end{cases}$$

2) $\mathbf{f}_m = \phi_m$ is discontinuous:

$$\phi_m(z) = \begin{cases} 1, & \text{if } z > \kappa_{m1} \\ \epsilon_{m1} \in [0,1], & \text{if } z = \kappa_{m1} \\ 0, & \text{if } z \in (-\kappa_{m2}, \kappa_{m1}) \\ \epsilon_{m2} \in [-1,0], & \text{if } z = -\kappa_{m2} \\ -1, & \text{if } z \leq -\kappa_{m2}, \end{cases}$$

where $\kappa_{m1}, \kappa_{m2} > 0$.

χ_m maps the argument into an interval $\Psi_m = [p_m, q_m]$. In contrast, ϕ_n maps its argument to a set with finite number of elements, which is $\Psi_n = \{1, \epsilon_{n1}, 0, \epsilon_{n2}, -1\}$. That Ψ_n can be a set of finite points has the following interpretation. If $\mathbf{x}_i$ represents the opinion of an individual and lies in $[-1,1]$, then $-1, 0, 1$ can be seen to denote the negative, neutral, and positive attitudes, respectively, of individual i towards an issue. The final decision or judgment of agent i (a convinced individual with extreme opinions [18]) towards an issue is driven to one of $\{-1, 0, 1\}$ based on the observation $\sum_j a_{ij}\mathbf{x}_j(k)$ at

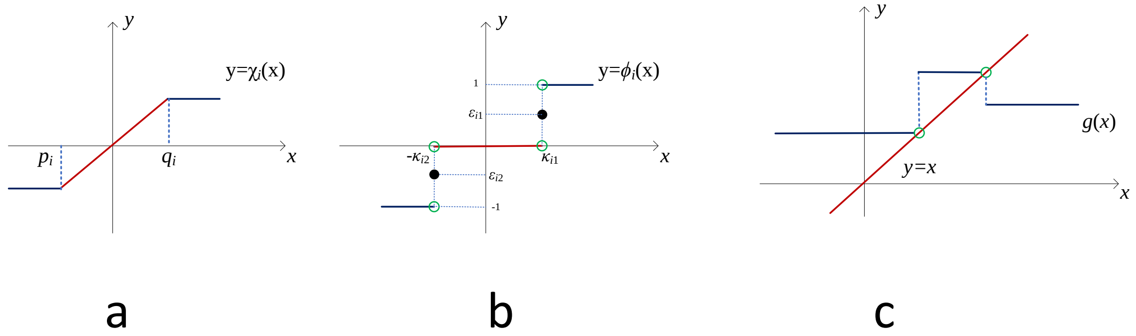

Fig. 1: Subfigure (a): the graph of χ_i; subfigure (b): the graph of ϕ_i; subfigure (c): an example where a fixed point does not exist for a discontinuous function $g(x)$.

time k and its preference represented by function $\mathbf{f}_i$. There also exist many other choices of $\Psi_i, i = 1, \dots, N$, according to the context. $\epsilon_{m1}, \epsilon_{m2}, \kappa_{m1}$, and κ_{m2} are parameters making MAS (1) more flexible.

For simplicity, we call the agents with interval constraints *moderate agents* while the rest *extreme agents*. Without loss of generality, let $-1 \le p_m \le q_m \le 1$ for all m.

Problem of Interest. Characterize the equilibrium points of multi-agent system (1) and prove their global/local stability.

3.2 Concepts of Stability and Consensus

We present the definitions of consensus and stability for subsequent analysis in what follows. Let $\mathbf{x}(k) = [\mathbf{x}_1(k), \dots, \mathbf{x}_N(k)]^\top$.

Definition 1 (Stability of an Equilibrium Point). *An equilibrium point $\mathbf{x}^*$ of (1) is globally asymptotically stable if for any initial value $\mathbf{x}_0 \in \mathbb{R}^N$, there holds that $\lim_{k\to\infty} \mathbf{x}(k) - \mathbf{x}^* = \mathbf{0}$; locally asymptotically stable if there exists a δ such that whenever $\|\mathbf{x}_0 - \mathbf{x}^*\| \le \delta$, there holds that $\lim_{k\to\infty} \mathbf{x}(k) - \mathbf{x}^* = \mathbf{0}$.*

Definition 2 (Global Consensus). *Consensus is said to be achieved for system (1) globally and asymptotically if for any initial value $\mathbf{x}_0 \in \mathbb{R}^N$, there holds that $\lim_{k\to\infty}(\mathbf{x}_i(k) - \mathbf{x}_j(k)) = 0$ for all $i, j \in \mathcal{V}$.*

4 Equilibrium Points and Stability

In this section, $\mathbf{f}_i$ can be either ϕ_i or χ_i for agent i and both extreme and moderate agents exist. That is, the constraints of the agents are heterogeneous. We will characterize the equilibrium points of MAS (1) from the viewpoint of Kakutani's fixed point theory since $\mathbf{f}_i$ can be discontinuous. Then, we investigate the stability of certain equilibrium points.

By $\alpha + \beta = 1$ and $\beta > 0$, the equilibrium point $\mathbf{x}^*$ of the multi-agent system (1) satisfies that

$$\mathbf{x}_i^* = \alpha \mathbf{x}_i^* + \beta \mathbf{f}_i\left(\sum_{j\in\mathcal{N}_i} a_{ij}\mathbf{x}_j^*\right), \forall i$$

$$\iff \quad \mathbf{x}_i^* = \mathbf{f}_i\left(\sum_{j\in\mathcal{N}_i} a_{ij}\mathbf{x}_j^*\right), \forall i.$$

Consequently, $\mathbf{x}^*$ is a fixed point of

$$\mathbf{X}(\mathbf{x}) = \mathbf{f}(\mathbf{A}\mathbf{x}) = \left[\mathbf{f}_1([\mathbf{A}\mathbf{x}]_1), \dots, \mathbf{f}_N([\mathbf{A}\mathbf{x}]_N)\right]^\top,$$

where $[\mathbf{A}\mathbf{x}]_i$ denotes the i-element of $\mathbf{A}\mathbf{x}$. However, $\mathbf{f}_i$ is discontinuous for some i. This suggests that an equilibrium point may not exist (refer to an illustration in Fig. 1). In view of this fact, we introduce a concept of generalized equilibrium point, with $\mathbf{f}_i$ replaced by a set-valued map at the discontinuous points.

Definition 3 (Generalized Equilibrium Point). *A point $\mathbf{x}^*$ is termed a generalized equilibrium point (GEP) of system (1) if $\mathbf{x}^* \in \mathbf{F}[\mathbf{X}](\mathbf{x}^*)$, i.e., $\mathbf{x}^*$ is a fixed point of the Filippov set-valued map $\mathbf{F}[\mathbf{X}]$ of $\mathbf{X}$.*

With $\mathbf{F}[\mathbf{X}]$, a generalized equilibrium point is guaranteed to exist.

Theorem 1 (Generalized Equilibrium Points). *Consider the multi-agent system (1) with $\mathbf{f}_i$ being either χ_i or ϕ_i for all i. Then, $\mathbf{x}_i(k)$ are bounded for all i and for any initial state, and there exists at least one GEP.*

Proof. (i) We first show that $\mathbf{x}_i(k)$ are bounded for $i = 1, \dots, N$. It follows from (1) that

$$\mathbf{x}(k) = \alpha^k \mathbf{x}(0) + \sum_{j=0}^{k-1} \alpha^j \beta \mathbf{f}(*\,(k-j-1)), \tag{2}$$

where $*\,(k-j-1)$ are unspecified and bounded terms. The above equality gives $\|\mathbf{x}(k)\| \le \alpha^k\|\mathbf{x}(0)\| + b\sum_{j=0}^{k-1}\alpha^j$, where b is a positive constant satisfying that $b \ge \beta\|\mathbf{f}(\cdot)\|$. Therefore, $\|\mathbf{x}(k)\|$ is bounded as a result of the fact that $0 \le \alpha < 1$.

(ii) Now, we show the existence of at least one GEP using Kakutani's fixed point theory. We need to show that $\mathbf{F}[\mathbf{X}](\mathbf{x})$ has a closed graph and is a convex-valued self-correspondence on $[-1,1]^N$, where $\mathbf{X}(\mathbf{x}) = [\mathbf{X}_1(\mathbf{x}), \dots \mathbf{X}_N(\mathbf{x})]^\top \in \mathbb{R}^N$ and

$$\mathbf{X}_i(\mathbf{x}) = \mathbf{f}_i([\mathbf{A}\mathbf{x}]_i) = \mathbf{f}_i\left(\sum_{j\in\mathcal{N}_i} a_{ij}\mathbf{x}_j\right).$$

Suppose that $\mathbf{x}$ is not a continuous point of $\mathbf{X}$ and further the i-th component $\mathbf{X}_i$ of $\mathbf{X}$ is not continuous at $\mathbf{x}$. Then,

$\mathbf{X}_i(\mathbf{x}) = \phi_i([\mathbf{Ax}]_i)$, where $[\mathbf{Ax}]_i$ denotes the i-component of $\mathbf{Ax}$.

According to definitions of ϕ_i and the Filippov set-valued map, given a sufficiently small $\delta > 0$, by calculation, one has $\mathbf{X}_i(\mathbf{y}) \in [a_i, b_i]$ for $\mathbf{y} \in B(\mathbf{x}, \delta)$. For instance, $a_i = 1$ and $b_i = 0$ when $[\mathbf{Ax}]_i = \kappa_{i1}$. Moreover, a_i and b_i can be attained for some $\mathbf{y} \in B(\mathbf{x}, \delta)$. Hence, one can write $\mathbf{F}[\mathbf{X}_i](\mathbf{x}) = [a_i, b_i]$, where $a_i = b_i$ if $\mathbf{x}$ is a continuous point of $\mathbf{X}_i$ or $[a_i, b_i]$ is a nontrivial closed interval as specified above if $\mathbf{x}$ is a discontinuous point of $\mathbf{X}_i$. Consequently, one concludes that $\mathbf{F}[\mathbf{X}]$ is convex-valued.

By the definition, for a discontinuous point $\mathbf{x}$ of $\mathbf{X}_i$, the Filippov set-valued map $\mathbf{F}[\mathbf{X}_i](\mathbf{x})$ is the intersection of a collective of closed set and is closed as well. By the two possible forms of $\mathbf{f}_i$, i.e., χ_i and ϕ_i, it is easily verified that for $(\mathbf{v}_n, \mathbf{w}_n) \in ([-1,1]^N, \mathbf{F}[\mathbf{X}]([-1,1]^N))$ and $\mathbf{v}_n \to \mathbf{v}, \mathbf{w}_n \to \mathbf{w}$ as $n \to \infty$, one has

$$(\mathbf{v}, \mathbf{w}) \in \left([-1,1]^N, \mathbf{F}[\mathbf{X}]([-1,1]^N)\right).$$

Therefore, $\mathbf{F}[\mathbf{X}]$ has a closed graph on $[-1,1]^N$.

We proceed to show that $\mathbf{F}[\mathbf{X}]$ is a self-correspondence on $[-1,1]^N$. Let $\mathbf{x}$ be any point in $[-1,1]^N$. Then, $-1 \le \mathbf{x}_i \le 1$. This implies that

$$-1 = -\sum_{j\in\mathcal{N}_i} a_{ij} \le \sum_{j\in\mathcal{N}_i} a_{ij}\mathbf{x}_j \le \sum_{j\in\mathcal{N}_i} a_{ij} = 1.$$

Hence, $-1 \le \mathbf{f}_i([\mathbf{Ax}]_i) \le 1$. This yields that $\mathbf{F}[\mathbf{f}_i]([\mathbf{Ax}]_i) \subset [-1,1]$. We prove that $\mathbf{F}[\mathbf{X}]$ is a self-correspondence on $[-1,1]^N$.

According to Lemma 1, there exists a point $\mathbf{z} \in [-1,1]^N$ such that $\mathbf{z} \in \mathbf{F}[\mathbf{X}](\mathbf{z})$. That is to say, there exists at least one GEP. □

Let $\mathbf{x}^*$ be a GEP. Due to heterogeneity of the constraints, an equilibrium point probably does not exist. Note that if agent i is moderate, $\mathbf{x}_i^* = \mathbf{f}_i([\mathbf{Ax}^*]_i)$; while $\mathbf{x}_i^* \in \mathbf{F}[\mathbf{f}_i]([\mathbf{Ax}^*]_i)$ for an extreme agent i. Hence, the existence of extreme agents generally causes that an equilibrium point does not exist. An intuitive explanation of Theorem 1 is as follows. If some agents have the preference to be extreme towards an issue, then it is likely that the discussion of the agents on that issue may never end with a meaningful result, unless the extreme agents change their preference. In addition, even if an equilibrium point exists, it is more likely to be a dissensus point, as a result of heterogeneity of Ψ_m or the emptiness of the intersection of Ψ_m. Finally, note that we do not impose any connectivity condition on $\mathcal{G}$ in Theorem 1.

The function ϕ_i is neither contractive or non-expansive. This makes the analysis of global behavior of the agents challenging. In what follows, we analyze the local behavior of the agents around the continuous equilibrium points. A *continuous equilibrium point* is defined as an equilibrium point where $\mathbf{X}(\mathbf{x}) = \mathbf{f}(\mathbf{Ax})$ is continuous.

Theorem 2 (Local Stability of Continuous EPs)**.** *Consider the multi-agent system* (1) *with* $\mathbf{f}_i$ *being either* χ_i *or* ϕ_i *for all* i *and there exists at least one extreme agent. Suppose that* $\mathcal{G}$ *is strongly connected. Then, any continuous equilibrium point is locally stable.*

We need some intermediate results for the proof of Theorem 2.

Definition 4. *[19] Given* $\mathbf{M} \in \mathbb{R}^{n\times n}$, $\Gamma(\mathbf{M})$ *is defined to be the directed graph on* n *nodes such that there is a directed edge in* $\Gamma(\mathbf{M})$ *from node* i *to node* j *if and only if* $\mathbf{M}_{ij} \neq 0$.

Lemma 2. *[19, Theore 8.3.1] If* $\mathbf{M} \in \mathbb{R}^{n\times n}$ *is nonnegative, then* $\rho(\mathbf{M})$*, the spectral radius of* $\mathbf{M}$*, is an eigenvalue of* $\mathbf{M}$ *and there is a nonnegative nonzero vector* $\mathbf{v}$ *such that* $\mathbf{Mv} = \rho(\mathbf{M})\mathbf{v}$.

Lemma 3. *[19, Corollary 6.2.9, Theorem 6.2.14] Given* $\mathbf{M} \in \mathbb{R}^{n\times n}$*, if the directed graph* $\Gamma(\mathbf{M})$ *underlying* $\mathbf{M}$ *is strongly connected,* $\mathbf{M}$ *is diagonally dominant, and there is a* k *such that*

$$\mathbf{M}_{kk} > \sum_{j=1, j\neq k}^{N} |\mathbf{M}_{kj}|,$$

then $\mathbf{M}$ *is nonsingular.*

Proof of Theorem 2. Let $\mathbf{x}^*$ be a continuous equilibrium point. Then, there exists a $\delta > 0$ such that $\mathbf{X}$ is smooth on $B(\mathbf{x}^*, \delta) = \{\mathbf{y} \in \mathbb{R}^N \mid \|\mathbf{y} - \mathbf{x}^*\| \le \delta\}$. Linearizing $\mathbf{X}$ around $\mathbf{x}^*$ gives a Jacobian matrix

$$DX_{\mathbf{x}^*} = \begin{bmatrix} \alpha & \beta\delta_{12} & \cdots & \beta\delta_{1N} \\ \beta\delta_{21} & \alpha & \cdots & \beta\delta_{2N} \\ \vdots & \cdots & \ddots & \vdots \\ \beta\delta_{N1} & \cdots & \beta\delta_{N,N-1} & \alpha \end{bmatrix},$$

where $\delta_{ij} = \frac{\mathrm{d}\mathbf{f}_i([\mathbf{Ax}^*]_i)}{\mathrm{d}\mathbf{x}_j}$ and $\frac{\mathrm{d}\mathbf{f}_i([\mathbf{Ax}^*]_i)}{\mathrm{d}\mathbf{x}_j} \in \{0, a_{ij}\}$. All we need is that $DX_{\mathbf{x}^*}$ is Schur stable [19].

Note that δ_{ij} are nonnegative. The i-th row sum of $DX_{\mathbf{x}^*}$ is

$$\sum_{j=1}^{N} [DX_{\mathbf{x}^*}]_{ij} = \alpha + \beta \sum_{j\neq i, j=1}^{N} \delta_{ij} \le \alpha + \beta \sum_{j\neq i, j=1}^{N} a_{ij} \le 1.$$

Hence, $DX_{\mathbf{x}^*}$ is a nonnegative matrix with each row sum being not greater than 1. As a result, the norm of any eigenvalue of $DX_{\mathbf{x}^*}$ is not greater than 1 [19]. We require that at least one agent, say k, exists such that $\mathbf{f}_k = \phi_k$. By definition, ϕ_k must be constant around $\mathbf{x}^*$. Then, $\frac{\mathrm{d}\mathbf{f}_k([\mathbf{Ax}^*]_k)}{\mathrm{d}\mathbf{x}_j} = 0, j \neq k, j = 1, \dots, N$. Consequently,

$$\sum_{j=1}^{N} [DX_{\mathbf{x}^*}]_{kj} = \alpha + \beta \sum_{j\neq k, j=1}^{N} \delta_{kj} = \alpha < 1.$$

Next, we show that any eigenvalue λ of $DX_{\mathbf{x}^*}$ has a norm strictly less than 1, i.e., $|\lambda| < 1$. By Lemma 2, suppose that there is an eigenvalue of $DX_{\mathbf{x}^*}$ having norm 1, then $\rho(DX_{\mathbf{x}^*}) = 1$ and there exists a nonnegative nonzero vector $\mathbf{v}$ such that $DX_{\mathbf{x}^*}\mathbf{v} = \mathbf{v}$. Let $\mathbf{B} = \mathbf{I} - DX_{\mathbf{x}^*}$ whose spectrum consists of $1 - \lambda(DX_{\mathbf{x}^*})$. Then, $\mathbf{B}$ has an eigenvalue 0 and is a singular matrix.

Recall that $DX_{\mathbf{x}^*}$ is nonnegative with each row sum being not greater than 1. As a result, $\mathbf{B}$ has nonnegative diagonal entries and nonpositive off-diagonal entries. Moreover, $\mathbf{B}$ is diagonally dominant, i.e.,

$$1 - \alpha \ge \beta \sum_{j\neq i, j=1}^{N} \delta_{ij},\ \forall i \neq k,$$

and

$$1-\alpha > \beta \sum_{j\neq k, j=1}^{N} \delta_{kj}.$$

We have that $\mathcal{G}$ is strongly connected by the conditions in the theorem. Assume that $\Gamma(\mathbf{B})$ is also strongly connected. By Lemma 3, $\mathbf{B}$ is nonsingular – a contradiction. If $\Gamma(\mathbf{B})$ is not strongly connected, then $\Gamma(\mathbf{B})$ consists of multiple strongly connected components. Similar analysis can be applied to each strongly connected component by writing $\mathbf{B}$ in the Frobenius normal form [17] as follows:

$$\mathbf{B} = \begin{bmatrix} \mathbf{B}_1 & & & \\ \mathbf{B}_{21} & \mathbf{B}_2 & & \\ \vdots & \ddots & \ddots & \\ \mathbf{B}_{k1} & \cdots & \cdots & \mathbf{B}_k \end{bmatrix}.$$

Note that $\mathbf{B}_i, i = 1, \dots, k$, are all diagonally dominant and for each i, there exists ℓ such that $[\mathbf{B}_i]_{\ell\ell} > \sum_j [\mathbf{B}_i]_{\ell j}$, where $[\cdot]_{ij}$ denotes the (i,j)-th entry of a matrix. Thus we prove that the norms of all the eigenvalues of $DX_{\mathbf{x}^*}$ are strictly less than 1. The proof is complete. □

4.1 Directed Ring Communication Graph

In this subsection, $\mathcal{G}$ is assumed to be a directed ring graph as shown in Fig. 3, i.e., the neighbor of the $(i+1)$-th agent is i for $i = 1, \dots, N-1$ and the neighbor of agent 1 is N. In this case, we characterize the equilibrium point and its *global stability*.

Assume without loss of generality that $\mathbf{f}_1 = \chi_1$. In a directed ring graph, if an equilibrium point exists, then

$$\mathbf{f}_1 \circ \mathbf{f}_N \circ \cdots \circ \mathbf{f}_2(x) = x, \; x \in [p_1, q_1].$$

In what follows, let $\mathbf{f}^\circ = \mathbf{f}_1 \circ \mathbf{f}_N \circ \cdots \circ \mathbf{f}_2$ be defined on $[p_1, q_1]$. According to the definition of an equilibrium point and the network structure, one can further conclude that $\mathbf{f}^\circ$ has a fixed point belonging to the interval $[p_1, q_1]$ if and only if an equilibrium point exists. Following this idea, we have

Theorem 3. *Consider the multi-agent system* (1) *communicating over graph $\mathcal{G}$ and both moderate and extreme agents exist.*[1] *Let $\kappa_{mi} = 0.5$ and $\epsilon_{mi} = 0$ for $i = 1,2$ and any extreme agent m. Assume that $\mathcal{G}$ is a directed ring graph. If $[p_i, q_i] \cap \{\pm 0.5\} = \emptyset$ for any moderate agent i, then there exists a unique equilibrium point and it is globally asymptotically stable.*

Before presenting the proof of Theorem 3, we first show that $\mathbf{x}_i(k)$ approaches a certain interval as k tends to infinity.

Lemma 4. *Consider the multi-agent system* (1) *with $\mathbf{f}_i$ being either χ_i or ϕ_i for all i. Then, for any i, $\mathbf{x}_i(k) \to [p_i, q_i]$ if $\mathbf{f}_i = \chi_i$; while $\mathbf{x}_i(k) \to [-1, 1]$ if $\mathbf{f}_i = \phi_i$.*

Proof. Let agent i be endowed with χ_i. One obtains from (2) that

$$\mathbf{x}_i(k+1) = \alpha^{k+1}\mathbf{x}_i(0) + \sum_{j=0}^{k} \alpha^j \beta \chi_i(*(k-j)).$$

[1] If all the agents are extreme, then more than one equilibrium point exists, e.g., $\pm\mathbf{1}_N$ and $\mathbf{0}_N$, under the conditions of this theorem.

Recall that $p_i \leq \chi_i(*(k)) \leq q_i$. By letting $\chi_i(*(k-j))$ be any fixed value in $[p_i, q_i]$, one easily has $\mathbf{x}_i(k) \to [p_i, q_i]$ as $k \to \infty$.

The case that agent i is endowed with ϕ_i can be analyzed similarly by noting that $\phi_i(\cdot) \in [-1, 1]$. We omit the details for brevity. □

Proof of Theorem 3. (*Existence & Uniqueness*) We start by proving that $\mathbf{f}^\circ$ has a unique fixed point.

Without loss of generality, assume that $\mathbf{f}_1 = \chi_1$ is a continuous function and let $\mathbf{x}_1 \in [p_1, q_1]$. If $\mathbf{f}_2$ is a continuous function, then $\mathbf{f}_2(\mathbf{x}_1) \in [p_2, q_2]$. Otherwise, $\mathbf{x}_1$ is mapped to $\{-1, 0, 1\}$, e.g., $\mathbf{f}_2(\mathbf{x}_1) = -1$ if $[p_1, q_1] \subset [-1, -0.5)$. In the latter case, $\mathbf{f}_2(\mathbf{x}_1)$ has the same value for any $\mathbf{x}_1 \in [p_1, q_1]$ according to $[p_m, q_m] \cap \{\pm 0.5\} = \emptyset$ for any moderate agent m. Continuing this analysis and by the condition in the theorem statement, $\mathbf{x}_i$ is a continuous point of $\mathbf{f}_{i+1}$ for $i = 1, \dots, N-1$ with $\mathbf{x}_1 \in [p_1, q_1]$. That is to say, $\mathbf{f}^\circ$ is continuous on $[p_1, q_1]$. Moreover,

$$\mathbf{f}^\circ = \mathbf{f}_1 \circ \mathbf{f}_N \circ \cdots \circ \mathbf{f}_2([p_1, q_1]) \subset [p_1, q_1].$$

As a result, $\mathbf{f}^\circ$ is a self-correspondence on $[p_1, q_1]$. By Kakutani's fixed point theorem in Lemma 1, there exists at least one *equilibrium point*.

We now further prove that the equilibrium point is unique. Let $\mathbf{x}^*$ and $\mathbf{y}^*$ be two equilibrium points. Let the k-th agent be an extreme one with $\mathbf{f}_k = \phi_k$ and the j-th agent be endowed with χ_j for $1 \leq j < k$. Since $[p_m, q_m] \cap \{\pm 0.5\} = \emptyset$ for any moderate agent m, one has

$$\mathbf{f}_k \circ \mathbf{f}_{k-1} \circ \cdots \circ \mathbf{f}_2(\mathbf{x}_1^*) = \mathbf{f}_k \circ \mathbf{f}_{k-1} \circ \cdots \circ \mathbf{f}_2(\mathbf{y}_1^*).$$

This is because $\mathbf{f}_k = \phi_k$ maps any $\mathbf{x}_{k-1} \in [p_{k-1}, q_{k-1}]$ to the same value belonging to $\{-1, 0, 1\}$. As a result, $\mathbf{x}_i^* = \mathbf{y}_i^*$ for $N \geq i \geq k$. Since $\mathcal{G}$ is a directed ring graph, $\mathbf{f}_1(\mathbf{x}_N^*) = \mathbf{f}_1(\mathbf{y}_N^*)$. Repeating this procedure to $k-1$, one concludes that $\mathbf{x}_i^* = \mathbf{y}_i^*$ for $i = 1, \dots, k-1$. To summarize, $\mathbf{x}^* = \mathbf{y}^*$. There is a unique equilibrium point.

(*Stability*) Next, we prove that for system (1), $\mathbf{x}(k) \to \mathbf{x}^*$ as $k \to \infty$ given any initial point $\mathbf{x}(0)$. By Lemma 4, $\mathbf{x}_i(k) \to [p_i, q_i]$ as $k \to \infty$ for moderate agent i. Then, for some sufficiently small $\delta > 0$, there exists $K > 0$ such that for $k > K$, one has $\mathbf{x}_i(k) \in [p_i - \delta, q_i + \delta] \cap \{-0.5, 0.5\} = \emptyset$ for any moderate agent i. Hence, given an extreme agent, say agent s, with a moderate neighbor, there holds that $\phi_s(\mathbf{x}_{s-1}(k)) = \hat{x}$ for all $k > K$, where $\hat{x} \in \{-1, 0, 1\}$. By the idea in the proof of Lemma 4, this implies that $\mathbf{x}_s(k) \to \hat{x}$ as $k \to \infty$.

Further, if agent $s+1$ is a moderate agent, then

$$\begin{aligned} \mathbf{x}_{s+1}(k+1) &= \alpha \mathbf{x}_{s+1}(k) + \beta \chi_{s+1}(\hat{x} + \delta(k)) \\ &= \alpha^{k-K}\mathbf{x}_{s+1}(K+1) + \sum_{j=K+1}^{k} \alpha^{k-j} \left[\beta \chi_{s+1}(\hat{x}) + \beta \epsilon(j)\right] \end{aligned}$$

with $k > K$ and $\delta(k) \to 0$ as $k \to \infty$. Write

$$\chi_{s+1}(\hat{x} + \delta(k)) = \chi_{s+1}(\hat{x}) + \epsilon(k).$$

Thus, $\epsilon(k) \to 0$ as $k \to \infty$ due to the continuity of χ_{s+1}. Note that $\inf_{j \leq k} \epsilon(j) \leq \epsilon(k) \leq \sup_{j \geq k} \epsilon(j)$. By letting $k \to \infty$, one has

$$\limsup_{k\to\infty} \mathbf{x}_{s+1}(k+1) \le \chi_{s+1}(\hat{x}) + \lim_{K\to\infty} \sup_{j\ge K+1} \epsilon(j).$$

Since $\epsilon(k) \to 0$ as $k \to \infty$, $\lim_{k\to\infty} \sup_{j\ge k} \epsilon(j) = 0$. Hence,

$$\limsup_{k\to\infty} \mathbf{x}_{s+1}(k+1) \le \chi_{s+1}(\hat{x}).$$

Similarly,

$$\liminf_{k\to\infty} \mathbf{x}_{s+1}(k+1) \ge \chi_{s+1}(\hat{x}).$$

As a result,

$$\mathbf{x}_{s+1}(k) \to \chi_{s+1}(\hat{x}) \quad \text{as} \quad k \to \infty.$$

The proof for agent $s+1$ being extreme and $\mathbf{x}_{s+1}(k) \to \phi_{s+1}(\hat{x})$ as $k \to \infty$ can be completed in a similar way.

Following the above idea, it can be easily obtained that $\lim_{k\to\infty} \mathbf{x}(k) = \mathbf{x}^*$ with $\mathbf{x}^*$ being the unique equilibrium point and taking the following form:

$$\mathbf{x}_i^* = \mathbf{f}_i \circ \cdots \circ \mathbf{f}_1 \circ \mathbf{f}_N \circ \mathbf{f}_{N-1} \circ \cdots \circ \mathbf{f}_{s+1}(\hat{x})$$

if $i < s$, and

$$\mathbf{x}_i^* = \mathbf{f}_i \circ \mathbf{f}_{i-1} \circ \cdots \circ \mathbf{f}_{s+1}(\hat{x})$$

if $i > s$, and $\mathbf{x}_s^* = \hat{x}$. The proof is complete. □

4.2 Homogeneous Interval Constraints

In this subsection, we take into account the case that all the agents are moderate. We will show that in this case, if the intersection of Ψ_m is non-empty and the agents communicate over a strongly connected graph, then they reach consensus asymptotically and globally. As a result, the multi-agent system (1) can be seen as an alternative for achieving interval consensus, which has been investigated in [6].

Theorem 4 (Constrained Consensus)**.** *Consider the multi-agent system* (1) *communicating over graph $\mathcal{G}$. Suppose that all the agents are moderate. If $\mathcal{G}$ is strongly connected and $\cap_{j=1}^N \Psi_j \neq \emptyset$, then consensus can be achieved globally asymptotically.*

Proof. Let $p^* = \max_j p_j$ and $q^* = \min_j q_j$. One has $p^* \le q^*$ because $\cap_{j=1}^N \Psi_j \neq \emptyset$. We first show that

$$\lim_{k\to\infty} dist\left(\mathbf{x}_i(k), \left[p^*, q^*\right]\right) = 0$$

for all i, where

$$dist\left(\mathbf{x}_i(k), [p^*, q^*]\right) = \inf_{y\in[p^*,q^*]} |\mathbf{x}_i(k) - y|.$$

Define $\Gamma_{\min}(k) = \min_i \mathbf{x}_i(k)$ and $\Gamma_{\max}(k) = \max_i \mathbf{x}_i(k)$. Note that if $\Gamma_{\min}(0), \Gamma_{\max}(0) \in [p^*, q^*]$, then it follows directly from (1) that $\Gamma_{\min}(k), \Gamma_{\max}(k) \in [p^*, q^*]$ for all $k \ge 0$.

Let $\Gamma_{\max}(k) = \mathbf{x}_m(k) > q^*$. [If this is not the case, we can consider the case that $\Gamma_{\min}(k) < p^*$.] Then, for any i,

$$\mathbf{x}_i(k+1) = \alpha \mathbf{x}_i(k) + \beta \chi_i\left(\sum_{j\in\mathcal{N}_i} a_{ij}\mathbf{x}_j(k)\right) \le \mathbf{x}_m(k),$$

where the inequality follows from that χ_i is non-decreasing and $\sum_j a_{ij}\mathbf{x}_j(k) \le \mathbf{x}_m(k)$. Consequently, $\Gamma_{\max}(k)$ is non-increasing. In what follows, we prove that

$$\lim_{k\to\infty} dist(\Gamma_{\max}(k), (-\infty, q^*]) = 0.$$

Suppose that $\lim_{k\to\infty} \Gamma_{\max}(k) = \Gamma_{\max}^* > q^*$. Note that $\mathbf{x}_i(k)$ are bounded for all i. Hence,

$$\lim_{k\to\infty} dist(\mathbf{x}(k), \Omega) = 0,$$

where Ω is the positive limit set of $\mathbf{x}(k)$ and is bounded [21]. Moreover, Ω is invariant with respect to system (1). Since $\lim_{k\to\infty} \Gamma_{\max}(k) = \Gamma_{\max}^* > q^*$, there exists $\mathbf{x}^0 \in \Omega$ and $w \in \mathcal{V}$ such that $\mathbf{x}_w^0 = \Gamma_{\max}^*$.

It follows from system (1) that the evolution of $\mathbf{x}_i(k)$ in reverse time can be implicitly written as follows for $i = 1, \ldots, N$,

$$\alpha \mathbf{x}_i(k-1) + \beta \mathbf{f}_i\left(\sum_{j\in\mathcal{N}_i} a_{ij}\mathbf{x}_j(k-1)\right) = \mathbf{x}_i(k). \tag{3}$$

Let $\mathbf{x}(0) = \mathbf{x}^0$. As a result of the invariance of Ω, $\mathbf{x}_i(-1) \le \mathbf{x}_w(0) = \mathbf{x}_w^0$ for all i. Recall that χ_w is non-decreasing. To guarantee $\mathbf{x}_w(0) = \mathbf{x}_w^0 = \Gamma_{\max}^*$, it follows from (3) that $\mathbf{x}_j(-1) = \mathbf{x}_w(-1) = \mathbf{x}_w(0)$ for all $j \in \mathcal{N}_w$ and

$$\sum_{j\in\mathcal{N}_w} a_{wj}\mathbf{x}_j(-1) \in [p_w, q_w].$$

Since $\mathcal{G}$ is strongly connected, there exists $s \in \mathcal{N}_w$ such that $\mathcal{N}_s \setminus \mathcal{N}_w \neq \emptyset$. Applying the same arguments to $\mathbf{x}_s(-1)$ and $\mathbf{x}_w(-1)$, it is reached that $\mathbf{x}_j(-2) = \mathbf{x}_w(0)$ for $j \in \mathcal{N}_w \cup \mathcal{N}_s$ and $\sum_i a_{ji}\mathbf{x}_i(-3) \in [p_j, q_j]$. Continuing the above analysis, one finally concludes that for some $0 < h \le N$, $\mathbf{x}_i(-h) = \mathbf{x}_j(-h) = \mathbf{x}_w(0)$ for all i, j and $\mathbf{x}_i(-h) \in [p_i, q_i]$ for any i. This gives that $\mathbf{x}_i(-h) \in [p^*, q^*]$, which in turn yields that $\mathbf{x}_i^0 \in [p^*, q^*]$ for all i. This is against the fact that $\mathbf{x}_w^0 > q^*$. Therefore, $\lim_{k\to\infty} dist\left(\mathbf{x}_i(k), (-\infty, q^*]\right) = 0$. Similarly, one has $\lim_{k\to\infty} dist\left(\mathbf{x}_i(k), [p^*, +\infty]\right) = 0$. As a result,

$$\lim_{k\to\infty} dist\left(\mathbf{x}_i(k), \left[p^*, q^*\right]\right) = 0, \quad i = 1, \ldots, N.$$

The above analysis suggests that for $i = 1, \ldots, N$,

$$\chi_i\left(\sum_{j\in\mathcal{N}_i} a_{ij}\mathbf{x}_j(k)\right) = \sum_{j\in\mathcal{N}_i} a_{ij}\mathbf{x}_j(k) + \epsilon_i(k)$$

where $\lim_{k\to\infty} \epsilon_i(k) = 0$. Then, system (1) can be written in a compact form as follows:

$$\mathbf{x}(k+1) = \mathbf{W}\mathbf{x}(k) + \epsilon(k), \tag{4}$$

where $\mathbf{W}_{ij} = \beta a_{ij}, i \neq j$, $\mathbf{W}_{ii} = \alpha$ for all i, and $\epsilon(k) = [\epsilon_1(k), \ldots, \epsilon_N(k)]^\top$.

Let

$$\Delta\mathbf{x}(k) = \left[\mathbf{x}_1(k), \mathbf{x}_1(k) - \mathbf{x}_2(k), \cdots, \mathbf{x}_1(k) - \mathbf{x}_N(k)\right]^\top$$

with

$$\Delta = \begin{bmatrix} 1 & \mathbf{0}_{1\times(N-1)} \\ \mathbf{1}_{N-1} & -\mathbf{I}_{N-1} \end{bmatrix}.$$

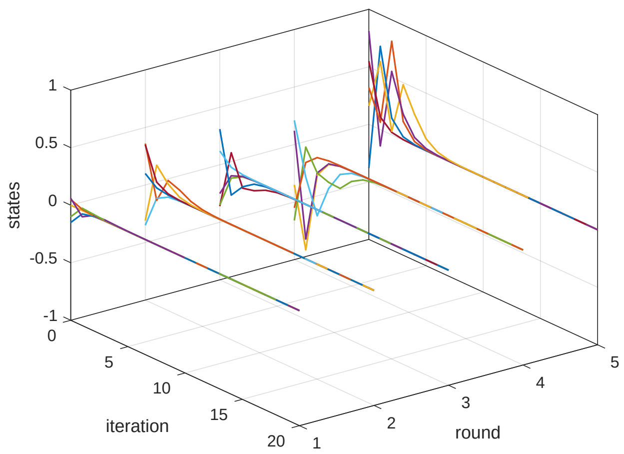

Fig. 2: Local stability of a continuous equilibrium point $\mathbf{0}_5$. In each round, an initial value near $\mathbf{0}_5$ is chosen.

Moreover,

$$\Delta \mathbf{W} \Delta = \begin{bmatrix} 1 & * \\ \mathbf{0} & \tilde{\mathbf{W}} \end{bmatrix}.$$

Note that $\mathbf{W}$ is a stochastic matrix and $\Gamma(\mathbf{W})$ (refer to Definition 4) is strongly connected. Then, $\tilde{\mathbf{W}}$ is Schur stable [17]. Let $\delta(k) = [\mathbf{0}_{N-1}, \mathbf{I}_{N-1}]\Delta\mathbf{x}(k)$ be the agreement error and note that $\Delta^2 = \mathbf{I}$. One has

$$\delta(k+1) = \tilde{\mathbf{W}}\delta(k) + [\mathbf{0}_{N-1}, \mathbf{I}_{N-1}]\Delta\epsilon(k),$$

which gives

$$\delta(k) = \tilde{\mathbf{W}}^k\delta(0) + \sum_{j=1}^{k} \tilde{\mathbf{W}}^{k-j}[\mathbf{0}_{N-1}, \mathbf{I}_{N-1}]\Delta\epsilon(j) \to \mathbf{0},$$

as $k \to \infty$. This finishes the proof. □

5 Numerical Examples

In this section, we present three examples to illustrate the findings of Theorems 2–4.

Example 2 (Local stability)**.** *This example is designed to illustrate Theorem 2. Consider a network of six agents with constraints being defined as follows. Let the first three agents be extreme with the same set of discrete points. Specifically,* $\epsilon_{i1} = -\epsilon_{i2} = 0.5$ *and* $\kappa_{i1} = \kappa_{i2} = 0.5$ *for* $i = 1, 2, 3$. *In addition, let the last three agents be moderate, and* $\Psi_4 = [-0.2, 0.7]$, $\Psi_5 = [-0.6, 1.6]$, $\Psi_6 = [-2, 8]$. *The adjacency matrix of the graph* $\mathcal{G}$ *that is strongly connected is given follows*

$$\mathbf{A} = \begin{bmatrix} 0 & 0 & 0 & 2/3 & 1/3 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1/4 & 0 & 3/4 & 0 \\ 0 & 0 & 1/6 & 0 & 5/6 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

It is easy to verify that $\mathbf{x}^* = \mathbf{0}_5$ *is a continuous equilibrium point. It can be observed from Figure 2 that if the initial state is chosen to be near* $\mathbf{0}$, *then* $\mathbf{x}(k) \to \mathbf{0}$ *as* $k \to \infty$.

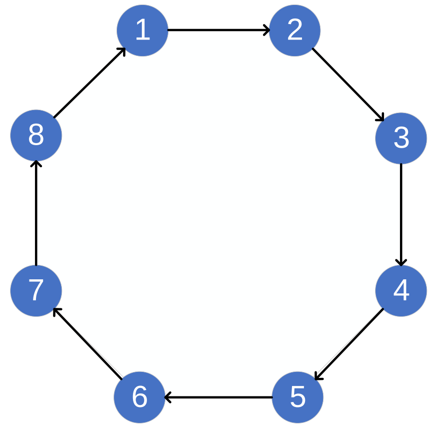

Fig. 3: A directed ring graph with eight nodes.

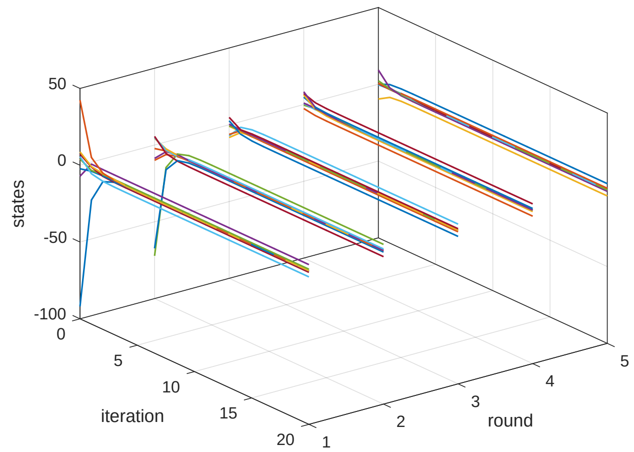

Fig. 4: Global stability of the equilibrium point for a directed ring graph. In each round, an arbitrary initial value is chosen.

Example 3 (Directed ring graph)**.** *In this example, we consider a directed ring graph to illustrate Theorem 3 and show global stability of the equilibrium point. The graph is given in Fig. 3 with eight nodes and the weight of each edge is 1. Let* $\epsilon_{i1} = -\epsilon_{i2} = 0.5$ *and* $\kappa_{i1} = \kappa_{i2} = 0.5$ *for* $i = 1, 3, 5, 7$ *and* $\Psi_2 = [0, 3]$, $\Psi_4 = [4, 7]$, *and* $\Psi_6 = [-7, -4]$. *It can be observed from Figure 4 that* $\mathbf{x}(k)$ *converges to the same equilibrium point with the initial state chosen arbitrarily.*

Example 4 (Constrained consensus)**.** *Consider the a random undirected graph with 200 nodes, which is strongly connected as seen in Fig. 5.*

Set $\Psi_i = [-0.5 - r_1, 1 + r_2]$ *where* r_1, r_2 *are positive random numbers sampled from the interval* $[0, 1]$. *Note that* $\cap_{j=1}^{N}\Psi_j \subset [-0.5, 1]$, *i.e., the intersection is nonempty. It can be easily verified that* $c\mathbf{1}_N$ *for* $c \subset [-0.5, 1]$ *are equilibrium points. As* $k \to \infty$, *it is seen from Figure 6 that consensus is achieved asymptotically from different initial values.*

6 Conclusion

In this paper, we have developed a multi-agent system with heterogeneous constraints. They (i.e., the constraints) are adopted to describe the heterogeneous heuristic beliefs of the

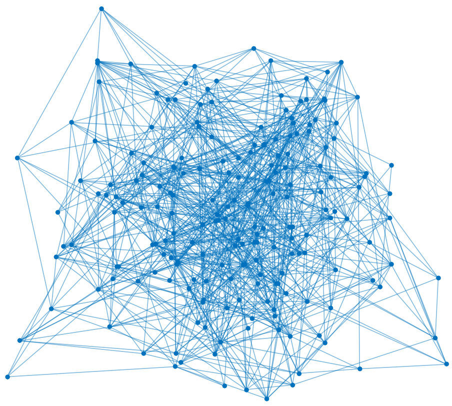

Fig. 5: A random graph with 200 nodes that is strongly connected.

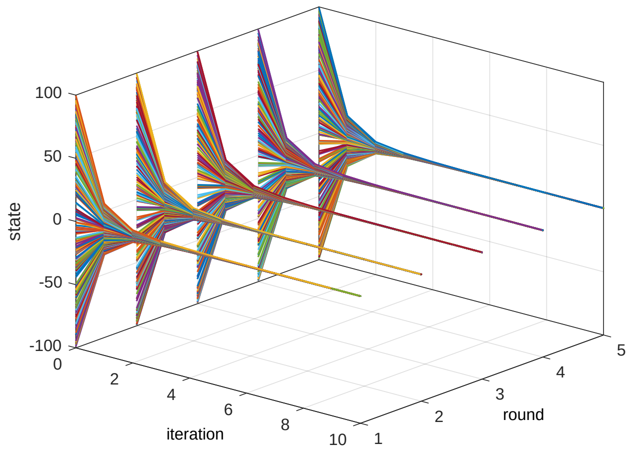

Fig. 6: Consensus is achieved for a group of agents having nonempty intersection of $\Psi_m, m = 1, \ldots, 200$. From arbitrary initial values in different rounds, consensus is achieved asymptotically.

agents or heterogeneity of the sets for which the intersection is being sought. We characterize the generalized equilibrium point from the perspective of Kakutani's fixed point for a set-valued map. We also prove the local/global stability of certain equilibrium points. Two special cases are further discussed with respect to the structure of the communication graph and the constraint forms, respectively. It is observed that whether dissensus or consensus appears is closely related to the constraints themselves. Future works may consider a time-varying deterministic or stochastic heterogeneous constraints.

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