An optimal control problem of backward stochastic differential equation with incomplete information

Guangchen Wang

School of Control Science and Engineering, Shandong University
Jinan 250061, China
Email: wguangchen@sdu.edu.cn

第十三届控制科学与工程前沿论坛 · 临沂

2021.05.07-09

Content

Probability space

$\left(\Omega, \mathscr{F}^W, \left(\mathscr{F}_t^W\right)_{0\le t\le T}, \mathbb{P}\right)$: complete filtered propability space

W_t : n-D standard Brownian motion

$\mathscr{F}_t^W$: $\sigma\{W_s; 0 \le s \le t\}$

$[0, T]$: fixed time horizon

SDE, BSDE and FBSDE

(Forward) stochastic differential equation (SDE, in short)

$$\begin{cases} dx_t = b(t, \omega, x_t)dt + \sigma(t, x_t)dW_t, & t \in [0, T], \\ x_0 = a. \end{cases}$$

Backward stochastic differential equation (BSDE, in short)

$$\begin{cases} -dy_t = f(t, \omega, y_t, z_t)dt - z_t dW_t, & t \in [0, T], \\ y_T = \xi. \end{cases}$$

Bond and stock price

$$\begin{cases} dS_{0,t} = r_t S_{0,t} dt, \\ dS_{i,t} = S_{i,t}\left[\mu_{i,t} dt + \sum_{j=1}^{m} \sigma_{ij,t} dW_{j,t}\right], \quad (i = 1, 2, \cdots m). \end{cases}$$

Market assumption

The interest rate r, the vector of the stocks appreciation rates μ and the volatility matrix σ are deterministic bounded processes. Moreover, σ has full rank for all $t \in [0, T]$, and the inverse matrix σ^{-1} is also bounded.

Decision set

Recall that $\mathscr{F}_t^W$ denotes the full information in the market at time t. Let $\mathscr{G}_t \subseteq \mathscr{F}_t^W$ be the information available to an agent. A typical form of $\mathscr{G}_t$ is

$$\mathscr{G}_t = \mathscr{F}^W_{(t-\delta)^+}$$

with $\delta > 0$. Set $C = L^2_{\mathscr{G}}(0, T; \mathbb{R}^+ \cup \{0\})$. Each element of C is interpreted as an admissible consumption rate.

Triple goals

- Minimize the initial endowment y_0.
- Reach the wealth goal ξ at time T.
- Maximize the expected utility from consumption $U(c)$.

Triple goals

- Minimize the initial endowment y_0.
- Reach the wealth goal ξ at time T.
- Maximize the expected utility from consumption $U(c)$.

Triple goals

- Minimize the initial endowment y_0.
- Reach the wealth goal ξ at time T.
- Maximize the expected utility from consumption $U(c)$.

Wealth process

According to the BSDE theory, the agent's wealth is described by

$$\begin{cases} dy_t^c = \left[r_t y_t^c + b_t^\top (\sigma_t^{-1})^\top (z_t^c)^\top - c_t\right] dt + z_t^c dW_t, \\ y_T^c = \xi, \end{cases} \tag{1}$$

where $b = \mu - r\mathbb{1}$, and c is the consumption rate.

Portfolio and consumption selection problem

Problem (PC). Ensuring to achieve the goal ξ, the agent wants to maximize her utility

$$J[c] = \mathbb{E} \int_0^T L e^{-\beta t} \frac{c_t^{1-R}}{1-R} dt - K y_0^c, \tag{2}$$

where $R \in (0, 1)$ is a constant, β, L and K are positive constants.

This is an optimal control problem of BSDE with incomplete information.

Problem formulation: Control system

The control system is described by the BSDE

$$\begin{cases} -dy_t^v = f(t, y_t^v, z_t^v, v_t)dt - z_t^v dW_t, \\ \quad y_T^v = \xi, \end{cases} \tag{3}$$

where v is the control process.

Assumption (H1)

f is continuously differentiable in (y, z, v). Moreover, f_y, f_z and f_v are uniformly bounded.

Decision set

Let U be a nonempty convex subset of $\mathbb{R}^r$, and let $\mathscr{G}_t \subseteq \mathscr{F}_t^W$ be a given sub-filtration which represents the information available to a controller at time $t \in [0, T]$. We introduce the decision set

$$\mathcal{U} = L^2_{\mathscr{G}}(0, T; U).$$

Each element of $\mathcal{U}$ is called an admissible control.

Utility functional

Besides ensuring to achieve the goal ξ, the controller also has her benefit described by utility functional

$$J[v] = \mathbb{E} \int_0^T l(t, y_t^v, z_t^v, v_t)dt + \phi(y_0^v), \tag{4}$$

where l and ϕ are given functions satisfying the condition $l(\cdot, y^v, z^v, v) \in L^1_{\mathscr{F}^W}(0, T; \mathbb{R})$ for all $v \in \mathcal{U}$.

Optimal control problem

The controller hopes to maximize her utility functional by selecting an appropriate admissible control.

Problem (OC) To look for a $u \in \mathcal{U}$ such that

$$J[u] = \max_{v \in \mathcal{U}} J[v]. \tag{5}$$

Each u satisfying (5) is called an optimal control of Problem (OC).

Dual method with incomplete information

Since $\mathcal{U}$ is convex, we can adopt an idea developed by Peng [AMO, 1993] to analyze the control problem. Nevertheless, it should be mentioned that the problem is different from Peng [AMO, 1993] mainly due to the incomplete information as well as the non-markov framework.

Assumption (H2)

l and ϕ are continuously differentiable with respect to (y, z, v). Moreover, the derivatives of l are bounded by $C(1 + |y| + |z| + |v|)$.

Maximum principle

Theorem 2.1.

Let (H1)-(H2) hold. Suppose that u is an optimal control of Problem (OC) and (y, z) is the corresponding trajectory. Then we have

$$\mathbb{E}\left[\langle H_v(t, y_t, z_t, u_t; p_t), v - u_t\rangle \middle| \mathcal{G}_t\right] \le 0$$

for all $v \in U$, where p is the solution of adjoint equation

$$\begin{cases} dp_t = -H_y(t, y_t, z_t, u_t; p_t)dt - H_z(t, y_t, z_t, u_t; p_t)dW_t, \\ p_0 = -\phi_y(y_0) \end{cases} \tag{6}$$

with the Hamiltonian function $H(t, y, z, v; p) = l(t, y, z, v) - \langle f(t, y, z, v),\ p\rangle$.

Some remarks

- If $\mathscr{G}_t = \mathscr{F}_t$, the maximum condition is reduced to the standard case $\langle H_v(t, y_t, z_t, u_t; p_t), v - u_t \rangle \leq 0$ for all $v \in U$.
- Theorem 2.1 is unavailable to the case that l is not quadratic growth with (y, z, v), say, $l(t, y, z, v) = v^r$ with $0 < r < 1$ being a constant.
- The result covers Lim and Zhou [SICON, 2001] as a special case.

Some remarks

- If $\mathscr{G}_t = \mathscr{F}_t$, the maximum condition is reduced to the standard case $\langle H_v(t, y_t, z_t, u_t; p_t), v - u_t \rangle \leq 0$ for all $v \in U$.
- Theorem 2.1 is unavailable to the case that l is not quadratic growth with (y, z, v), say, $l(t, y, z, v) = v^r$ with $0 < r < 1$ being a constant.
- The result covers Lim and Zhou [SICON, 2001] as a special case.

Some remarks

- If $\mathscr{G}_t = \mathscr{F}_t$, the maximum condition is reduced to the standard case $\langle H_v(t, y_t, z_t, u_t; p_t), v - u_t \rangle \leq 0$ for all $v \in U$.
- Theorem 2.1 is unavailable to the case that l is not quadratic growth with (y, z, v), say, $l(t, y, z, v) = v^r$ with $0 < r < 1$ being a constant.
- The result covers Lim and Zhou [SICON, 2001] as a special case.

Verification theorem

Assumption (H2) is weaken to

Assumption (H3)

For each $v \in \mathcal{U}$, $l(\cdot, y^v, z^v, v) \in L^1_{\mathscr{F}^W}(0,T;\mathbb{R})$, moreover, l and ϕ are differentiable with respect to (y,z).

Theorem 2.2.

Let (H1), (H3) hold. Let $u \in \mathcal{U}$ be given such that $l_y(\cdot, y, z, u)$, $l_z(\cdot, y, z, u)$, $l_v(\cdot, y, z, u) \in L^2_{\mathscr{F}^W}(0,T)$. $\forall\ (t,v) \in [0,T] \times U$, $l_v(t, y_t, z_t, v) \in L^1(\Omega, \mathscr{F}^W, \mathbb{P})$. Suppose (6) admits a solution $p \in L^2_{\mathscr{F}^W}(0,T;\mathbb{R}^n)$. Suppose

$$\mathbb{E}\left[H(t, y_t, z_t, u_t; p_t)\middle|\mathscr{G}_t\right] = \max_{v \in U} \mathbb{E}\left[H(t, y_t, z_t, v; p_t)\middle|\mathscr{G}_t\right]$$

holds for all $t \in [0,T]$.

Verification theorem (continued)

Moreover, suppose $\mathbb{E}\Big[H_v(t, y_t, z_t, v; p_t)\Big|\mathscr{G}_t\Big]$ is continuous in $v = u_t$ for any $t \in [0, T]$. Suppose for all $(t, y, z) \in [0, T] \times \mathbb{R}^n \times \mathbb{R}^{n\times m}$,

$$(y, z, v) \to H(t, y, z, v; p_t)$$

is concave, and $y \to \phi(y)$ is concave. Then u is an optimal control of Problem (OC).

Step 1: Candidate optimal consumption rate

Although Assumption (H2) does not hold in this case, we formally use the maximum principle to guess a candidate optimal control

$$\hat{c}_t = \left[L e^{-\beta t} \right]^{\frac{1}{R}} \mathbb{E}\left[p_t^{-\frac{1}{R}} \middle| \mathscr{G}_t \right],$$

where p is the unique solution of adjoint equation

$$\begin{cases} dp_t = -r_t p_t dt - b_t^\top (\sigma_t^{-1})^\top p_t dW_t, \\ p_0 = K. \end{cases}$$

It is easy to know that

$$p_t = K\Gamma_0(t)$$

with

$$\Gamma_t(s) = \exp\left\{-\int_t^s \left[r_u + \frac{1}{2}\left|\sigma_u^{-1}b_u\right|^2\right]du - \int_t^s (\sigma_u^{-1}b_u)^\top dW_u\right\},$$

then

$$\hat{c}_t = \left[\frac{L}{K}\right]^{\frac{1}{R}} e^{-\frac{\beta}{R}t}\mathbb{E}\left[\Gamma_0(t)^{-\frac{1}{R}}\middle|\mathcal{G}_t\right]. \tag{7}$$

Step 2: Optimal solution

Since (H1) and (H3) hold, we use the verification theorem to check $\hat{c}$ defined by (7) is optimal. Moreover, we get the initial wealth and the utility

$$y_0 = \mathbb{E}\left[\xi\Gamma_0(T) + \int_0^T \left[\frac{L}{K}\right]^{\frac{1}{R}} e^{-\frac{\beta}{R}t}\Gamma_0(t)\mathbb{E}\left[\Gamma_0(t)^{-\frac{1}{R}}\middle|\mathscr{G}_t\right]\right]dt \tag{8}$$

and

$$\begin{aligned} J[\hat{c}] = & -K\mathbb{E}[\xi\Gamma_0(T)] \\ & + K\mathbb{E}\int_0^T \left[\frac{1}{1-R}\left(\mathbb{E}\left[\Gamma_0(t)^{-\frac{1}{R}}\middle|\mathscr{G}_t\right]\right)^{-R} - \Gamma_0(t)\right]\left[\frac{L}{K}\right]^{\frac{1}{R}} e^{-\frac{\beta}{R}t}\mathbb{E}\left[\Gamma_0(t)^{-\frac{1}{R}}\middle|\mathscr{G}_t\right]dt. \end{aligned} \tag{9}$$

We conclude the discussion of this section with the following

Proposition 3.1.

Problem (PC) admits an optimal consumption rate $\hat{c}$ which is defined by (7). Moreover, the initial wealth and the utility are given by (8) and (9).

Remark 3.1.

The case with multi agents can be studied in a stochastic differential game framework. See, e.g., Wang and Yu [Automatica, 2012], Wang et al. [Automatica, 2018] for more details.

An LQC problem

In what follows, we will illustrate the theoretical results obtained before.

Problem (LQC). To minimize

$$J[v] = \frac{1}{2}\mathbb{E}\left\{\int_0^T \left[Q_t(y_t^v)^2 + S_t(z_t^v)^2 + \bar{S}_t(\bar{z}_t^v)^2 + R_t v_t^2\right] dt + \phi(y_0^v)^2\right\},$$

subject to $\mathcal{U} = L^2_{\mathscr{F}^{W_1}}(0, T; \mathbb{R})$ and

$$\begin{cases} dy_t^v = (A_t y_t^v + B_t v_t + C_t z_t^v + \bar{C}_t \bar{z}_t^v)dt + z_t^v dW_{1,t} + \bar{z}_t^v dW_{2,t}, \\ y_T^v = \xi. \end{cases}$$

Here $A, B, C, \bar{C}, Q, S, \bar{S}, R$ are uniformly bounded, deterministic functions; $Q, S, \bar{S}, \phi \geq 0$; $R > 0$; $\xi \in L^2_{\mathscr{F}}(\Omega, \mathscr{F}_T; \mathbb{R})$.

What are the difficulties?

- Control and estimate can not be separated in the sense of traditional separation principle.

- The existence and uniqueness of optimal filtering for FBSDE is not an immediate result.

What are the difficulties?

- Control and estimate can not be separated in the sense of traditional separation principle.
- The existence and uniqueness of optimal filtering for FBSDE is not an immediate result.

What is the solution method?

- Wang and Wu [JMAA, 2008] initiated a new backward separation method, which decouples control and estimate by first deducing optimal control and then computing optimal filtering.
  - The method is different from the traditional separation principle, and avoids lots of stochastic calculus in infinite-dimensional spaces.
  - The method is applicable for a broad class of non-Gaussian control systems, especially for the underlying system.

    See, e.g., Wang et al. [Springer, 2018], Hu et al. [arXiv, 2014], Agram and Øksendal [Stochastics, 2019], Wu and Liu [IJC, 2020].
- An equivalent linear transformation technique is introduced to overcome the second difficulty.

What is the solution method?

- Wang and Wu [JMAA, 2008] initiated a new backward separation method, which decouples control and estimate by first deducing optimal control and then computing optimal filtering.
  - The method is different from the traditional separation principle, and avoids lots of stochastic calculus in infinite-dimensional spaces.
  - The method is applicable for a broad class of non-Gaussian control systems, especially for the underlying system.
    See, e.g., Wang et al. [Springer, 2018], Hu et al. [arXiv, 2014], Agram and Øksendal [Stochastics, 2019], Wu and Liu [IJC, 2020].
- An equivalent linear transformation technique is introduced to overcome the second difficulty.

What is the solution method?

- Wang and Wu [JMAA, 2008] initiated a new backward separation method, which decouples control and estimate by first deducing optimal control and then computing optimal filtering.
  - The method is different from the traditional separation principle, and avoids lots of stochastic calculus in infinite-dimensional spaces.
  - The method is applicable for a broad class of non-Gaussian control systems, especially for the underlying system.
    See, e.g., Wang et al. [Springer, 2018], Hu et al. [arXiv, 2014], Agram and Øksendal [Stochastics, 2019], Wu and Liu [IJC, 2020].
- An equivalent linear transformation technique is introduced to overcome the second difficulty.

What is the solution method?

- Wang and Wu [JMAA, 2008] initiated a new backward separation method, which decouples control and estimate by first deducing optimal control and then computing optimal filtering.
  - The method is different from the traditional separation principle, and avoids lots of stochastic calculus in infinite-dimensional spaces.
  - The method is applicable for a broad class of non-Gaussian control systems, especially for the underlying system.
    See, e.g., Wang et al. [Springer, 2018], Hu et al. [arXiv, 2014], Agram and Øksendal [Stochastics, 2019], Wu and Liu [IJC, 2020].
- An equivalent linear transformation technique is introduced to overcome the second difficulty.

Step 1: Candidate optimal control

If u is an optimal control, then it follows from the maximum principle and the backward separation method that

$$u_t = -R_t^{-1} B_t \hat{p}_t,$$

where $\hat{p}_t = \mathbb{E}\left[p_t \middle| \mathscr{F}_t^{W_1}\right]$ satisfies the filtering equation

$$\begin{cases} d\hat{y}_t = (A_t \hat{y}_t + B_t u_t + C_t \hat{z}_t + \bar{C}_t \hat{\hat{z}}_t)dt + \hat{z}_t dW_{1,t}, \\ d\hat{p}_t = -(Q_t \hat{y}_t + A_t \hat{p}_t)dt - (S_t \hat{z}_t + C_t \hat{p}_t)dW_{1,t}, \\ \hat{y}_T = \hat{\xi}, \quad \hat{p}_0 = -\phi y_0. \end{cases} \tag{10}$$

The existence and uniqueness of (10) is not an immediate result because an additional $\hat{\hat{z}}$ appears in the BSDE of (10).

Step 2: Existence and uniqueness of (11)

Introduce an additional FBSDE

$$\begin{cases} d\hat{y}_t = \left\{A_t\hat{y}_t - \left[R_t^{-1}B_t^2 + (1+\Sigma_t\bar{S}_t)^{-1}\Sigma_t\bar{C}_t^2\right]\hat{p}_t + C_t\hat{z}_t\right\}dt + \hat{z}_t dW_{1,t}, \\ d\hat{p}_t = -(Q_t\hat{y}_t + A_t\hat{p}_t)dt - (S_t\hat{z}_t + C_t\hat{p}_t)dW_{1,t}, \\ \hat{y}_T = \hat{\xi}, \quad \hat{p}_0 = -\phi y_0 \end{cases} \tag{11}$$

with

$$\begin{cases} \dot{\Sigma}_t - 2A_t\Sigma_t - Q_t\Sigma_t^2 + \tilde{C}_t(I+\Sigma_t\tilde{S}_t)^{-1}\tilde{C}_t^\tau\Sigma_t + R_t^{-1}B_t^2 = 0, \\ \Sigma_T = 0. \end{cases}$$

It is easy to see that FBSDE (11) admits a unique solution $(\hat{p}, \hat{y}, \hat{z})$.

Step 3: Equivalence between (10) and (11)

- Firstly, we prove that the solution $(\hat{p}, \hat{y}, \hat{z})$ to (11) is a solution to (10).
- Secondly, we prove that for fixed u, the solution $(\hat{p}, \hat{y}, \hat{z}, \hat{\hat{z}})$ to (10) is a solution to (11).

Step 3: Equivalence between (10) and (11)

- Firstly, we prove that the solution $(\hat{p}, \hat{y}, \hat{z})$ to (11) is a solution to (10).

- Secondly, we prove that for fixed u, the solution $(\hat{p}, \hat{y}, \hat{z}, \hat{\bar{z}})$ to (10) is a solution to (11).

Step 4: Feedback representation of optimal control

The optimal control is denoted by

$$u_t = -R_t^{-1} B_t(\Pi_t \hat{y}_t + \hat{q}_t), \tag{12}$$

where Π and $\hat{q}$ satisfy

$$\begin{cases} \dot{\Pi} + 2A\Pi + [R^{-1}B^2 + \tilde{C}(I + \Sigma\tilde{S})^{-1}\tilde{C}^\tau\Sigma]\Pi^2 - Q = 0, \\ \Pi_0 = \phi, \end{cases}$$

$$\begin{cases} d\hat{q} = \Big\{C(1 + \Sigma S)^{-1}\Pi\hat{\eta} - [A + (R^{-1}B^2 + \tilde{C}(I + \Sigma\tilde{S})^{-1}\tilde{C}^\tau\Sigma)\Pi]\hat{q}\Big\}dt \\ \qquad + [(\Pi - S)(1 + \Sigma S)^{-1}\hat{\eta} + C(1 + \Sigma S)^{-1}(\Pi\hat{h} - \hat{q})]dW_1, \\ \hat{q}_0 = 0 \end{cases}$$

with

$$\begin{cases} d\hat{h} = [(A + \Sigma Q)\hat{h} + (1 + \Sigma S)^{-1}C\hat{\eta}]dt + \hat{\eta}dW_1, \\ \hat{h}_T = -\hat{\xi}. \end{cases}$$

We conclude the discussion of this section with the following

Proposition 4.1.

Problem (LQC) admits a unique optimal control u, which is defined by (12).

Remark 4.1.

- The above example is taken from Huang et al. [SCIS, 2019]. If the coefficients satisfy $C = \bar{C} = 0$, Proposition 4.1 is reduced to the case of Wang et al. [IEEE TAC, 2015].
- A research motivation for the observable filtration can be found in Wang et al. [arXiv, 2017].

We conclude the discussion of this section with the following

Proposition 4.1.

Problem (LQC) admits a unique optimal control u, which is defined by (12).

Remark 4.1.

- The above example is taken from Huang et al. [SCIS, 2019]. If the coefficients satisfy $C = \bar{C} = 0$, Proposition 4.1 is reduced to the case of Wang et al. [IEEE TAC, 2015].
- A research motivation for the observable filtration can be found in Wang et al. [arXiv, 2017].

Numerical simulation of optimal control u

Let $T = 1, A = 2, B = 3t + 2, C = t - 2, Q = e^{-0.05t}, S = t(T - t), \bar{S} = 2, R = 2t + 1, \phi = 2$ and $\xi = T + \sin(W_{1,T}) + \cos(2W_{2,T})$.

Using a numerical method of BSDE, we get the dynamic simulation of u. The result is taken from Wang et al. [AMC, 2021].

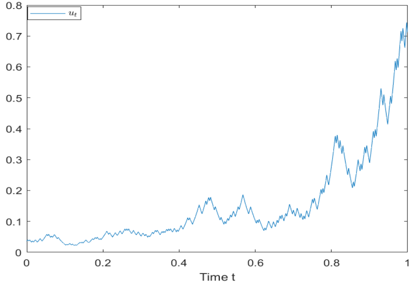

Figure: Numerical simulation of u.

Concluding remarks

- An optimal control problem of BSDE is reformulated from a new viewpoint of portfolio and consumption choice.

- A backward separation method is used to overcome the difficulties caused here. The method is also applicable for dynamic optimization of stochastic large population system with partial information. See, e.g., Huang et al. [submitted to IEEE TCYB, 2021].

Concluding remarks

- An optimal control problem of BSDE is reformulated from a new viewpoint of portfolio and consumption choice.
- A backward separation method is used to overcome the difficulties caused here. The method is also applicable for dynamic optimization of stochastic large population system with partial information. See, e.g., Huang et al. [submitted to IEEE TCYB, 2021].

Thank you for your attention!