An optimal control problem of FBSDE with noisy observation

Guangchen WANG

School of Control Science and Engineering, Shandong University
Jinan 250061, China
E-mail: wguangchen@sdu.edu.cn

第十二届控制科学与工程前沿论坛

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Outline

What is optimal filtering?

- The filtering problem is to obtain the best estimate $\hat{x}_t$ of an unobservable state x_t based on the observation data $Y_0^t = \{Y_s; 0 \leq s \leq t\}$.
- If $\mathbb{E}x_t^2 < \infty$, then $\hat{x}_t = \mathbb{E}[x_t|\mathscr{F}_t^Y]$ with $\mathscr{F}_t^Y = \sigma\{Y_s; 0 \leq s \leq t\}$.
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Linear filtering

- Kalman-Bucy filtering is the most successful result in linear filtering theory, which was obtained by Kalman and Bucy [TASME Ser. D, 1961].
- The most famous application of Kalman-Bucy filtering is the Apollo Project, which was used to estimate the trajectories of manned spaceship going to Moon and back.

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Nonlinear filtering

There have been two essentially different approaches so far.

- The innovation process approach $\Longrightarrow$ FKK equation.
  The theory achieved its culmination with the celebrated paper of Fujisaki, Kallianpur and Kunita [Osaka J. Math., 1972].
- The Kallianpur-Striebel formula approach $\Longrightarrow$ Zakai's equation.
  The approach was introduced by Duncan [1967], Mortensen [1966], and Zakai [Z. Wahrsch. Geb., 1969], independently.

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What is optimal control with noisy observation?

- This kind of problem is always hard to study. An intrinsic difficulty is the circular dependence between control and observation, which results in the unavailability of classical variation.
- The difficulty is often omitted in the early literature.
- There have been two techniques to overcome the difficulty.
  - Decomposition technique.
  - Probability measure transformation technique.
  - See, e.g., Bensoussan [CUP, 1992], Wang, Wu and Xiong [Springer, 2018] for a systematic account.

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Certainty equivalence principle

Kalman and Koepcke [TASME, 1958] raised a question of “whether the separate optimization of statistical prediction and control system performance yields a system which is optimal in the over-all sense.”

- Joseph and Tou [TAIEEAI, 1961] answered the question by a discrete time combined problem of estimate and LQG control.
- The optimal control is designed as the linear feedback of the filtering of optimal state. One feature is that the coefficients of the feedback are the same as the full information case.
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(Traditional) separation principle

- Wonham [SIAM J. Control, 1968] proved a more general separation result within the framework of a continuous time Gaussian system observed via a noisy linear system with more general cost functional.
- The result is improved by many scholars under various setups and is referred to as the (traditional) separation principle.
- In general, the traditional separation principle is not applicable for non-Gaussian control system with noisy observation.
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Backward separation principle

Wang and Wu [JMAA, 2008] initiated a new backward separation method, which decouples control and estimate by first (formally) deducing optimal control and then computing optimal filtering.

- The method is different from the traditional separation principle, and avoids lots of stochastic calculus in infinite-dimensional spaces.
- The method is applicable for a broad class of non-Gaussian control systems. See, e.g., Wu [SCF, 2010], Wang and Wu [IEEE TAC, 2009], Wang, Zhang and Zhang [IEEE TAC, 2014], Hu, Nualart and Zhou [arXiv, 2014], Wu and Liu [EJC, 2017], Ma and Liu [AJC, 2017].

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What is FBSDE?

Forward-backward stochastic differential equation (FBSDE):

$$\begin{cases} dx_t = b(t,\omega,x_t)dt + \sigma(t,\omega,x_t)dw_t, \\ -dy_t = f(t,\omega,x_t,y_t,z_t)dt - z_t dw_t, \\ x_0 = a, \quad y_T = g(x_T), \end{cases}$$

where

- $b : [0,T] \times \Omega \times \mathbb{R}^n \to \mathbb{R}^n$, $\sigma : [0,T] \times \Omega \times \mathbb{R}^n \to \mathbb{R}^{n\times r}$,
- $f : [0,T] \times \Omega \times \mathbb{R}^{n+m+m\times r} \to \mathbb{R}^m$, $g : \mathbb{R}^n \to \mathbb{R}^m$.

The Eq. has a unique solution (x,y,z) under usual assumptions.

Probability space

$$\left(\Omega, \mathscr{F}^{w,\bar{w}}, \left(\mathscr{F}_t^{w,\bar{w}}\right)_{0\leq t\leq T}, \mathbb{P}\right): \quad \text{complete filtered propability space}$$

$$(w_t, \bar{w}_t): \quad \text{2-dimensional standard BM}$$

$$\mathscr{F}_t^{w,\bar{w}}: \quad \sigma\{(w_s, \bar{w}_s); 0 \leq s \leq t\}$$

$$[0, T]: \quad \text{fixed time horizon}$$

Liability process

Consider a firm whose liability process $\bar{l}_t^v$ is governed by

$$-d\bar{l}_t^v = \left(b_t v_t - \bar{b}_t\right) dt + c_t dw_t + \bar{c}_t d\bar{w}_t,$$

where

$$\begin{aligned} v_t &- \text{ premium rate, a control process,} \\ \bar{b}_t > 0 &- \text{ expected liability rate,} \\ c_t > 0, \bar{c}_t > 0 &- \text{ liability risk.} \end{aligned}$$

Cash flow

Then the cash balance process x_t^v of the firm is

$$x_t^v = e^{\int_0^t a_s ds}\left(e_0 - \int_0^t e^{-\int_0^s a_r dr} d\bar{l}_s^v\right),$$

whose differential form reads

$$\begin{cases} dx_t^v = (a_t x_t^v + b_t v_t - \bar{b}_t)dt + c_t dw_t + \bar{c}_t d\bar{w}_t, \\ x_0^v = e_0, \end{cases}$$

where

$$a_t > 0 - \text{ interest rate,}$$

$$e_0 - \text{ initial endowment.}$$

Observation equation

Assume that x^v can only be partially observed via a factor process

$$\begin{cases} dY_t^v = (f_t x_t^v + g_t)dt + h_t d w_t, \\ \;\; Y_0^v = 0, \end{cases}$$

where

f_t, g_t, h_t – uniformly bounded and deterministic,

h_t – larger than zero.

Stock price equation

A typical interpretation of Y_t^v in reality is the logarithm of the stock as follows. To be specific,

$$\begin{cases} dS_t^v = S_t^v \left[\left(f_t x_t^v + g_t + \frac{1}{2} h_t^2 \right) dt + h_t dw_t \right], \\ S_0^v = 1. \end{cases}$$

where

h_t – the volatility coefficient of the stock, a positive constant,

$f_t x_t^v + g_t + \frac{h_t^2}{2}$ – the appreciation rate of return of the stock.

Observable filtration

Then $\sigma\{S_s^v; 0 \le s \le t\}$, rather than $\mathscr{F}_t^{w,\bar{w}}$, is the information available to the firm at time t. Moreover, it follows from

$$Y_t^v = \log S_t^v$$

that

$$\begin{aligned}\mathscr{F}_t^{Y^v} &= \sigma\{Y_s^v; 0 \le s \le t\} \\ &= \sigma\{S_s^v; 0 \le s \le t\}.\end{aligned}$$

Cost functional

$$J[v] = \frac{1}{2}\mathbb{E}\left[\int_0^T R_t(v_t - r_t)^2 dt + M(x_T^v - m)^2 - 2y_0^v\right],$$

where

$$r_t, m - \text{ benchmark, pre-set target,}$$

$$y_0^v - \text{ recursive utility from } v:$$

$$\begin{cases} -dy_t^v = (\tilde{B}_t y_t^v + \tilde{D}_t v_t)dt - z_t^v dw_t - \bar{z}_t^v d\bar{w}_t, \\ \quad y_T^v = x_T^v. \end{cases}$$

Recursive utility problem

Our problem is to find an $\mathscr{F}_t^{Y^v}$-adapted v to minimize

$$J[v] = \frac{1}{2}\mathbb{E}\left[\int_0^T R_t(v_t - r_t)^2 dt + M(x_T^v - m)^2 - 2y_0^v\right],$$

subject to x^v and y^v.

Since the firm can only get information from the stock, we are facing a special LQ problem driven by FBSDE with noisy observation.

Economic implication

The model applies to the case the firm has three objectives:

- to minimize the difference between v and r;
- to minimize the risk of the terminal cash balance;
- to maximize the recursive utility from v.

State equation

$$
\begin{cases}
dx_t^v = \left(a_t x_t^v + b_t v_t + \bar{b}_t\right) dt + c_t dw_t + \bar{c}_t d\bar{w}_t, \\
-dy_t^v = \left(A_t x_t^v + B_t y_t^v + C_t z_t^v + \bar{C}_t \bar{z}_t^v + D_t v_t + \bar{D}_t\right) dt \\
\qquad - z_t^v dw_t - \bar{z}_t^v d\bar{w}_t, \\
x_0^v = e_0, \quad y_T^v = F x_T^v + G,
\end{cases}
\tag{1}
$$

where

$$v_t - \text{ control process.}$$

Observation equation

$$
\begin{cases} dY_t^v = (f_t x_t^v + g_t)dt + h_t dw_t, \\ Y_0^v = 0, \end{cases} \tag{2}
$$

where

$$
\begin{aligned} w_t &- \text{ correlated noise,} \\ f_t x_t^v + g_t &- \text{ observation coefficient, unbounded.} \end{aligned}
$$

Problem (LQC)

Find an $\mathscr{F}_t^{Y^v}$-adapted control u such that

$$J[u] = \inf_v J[v]$$

subject to (1), (2), and

$$J[v] = \frac{1}{2}\mathbb{E}\left\{\int_0^T \left[L_t(x_t^v)^2 + O_t(y_t^v)^2 + R_t v_t^2 + 2l_t x_t^v + 2o_t y_t^v + 2r_t v_t\right] dt \right. \\ \left. + M(x_T^v)^2 + 2m x_T^v + N(y_0^v)^2 + 2n y_0^v \right\}. \tag{3}$$

- Cover the above motivating example as a special case.

Assumption

Assumption 1

The coefficients a_t, b_t, $\bar{b}_t$, c_t, $\bar{c}_t$, f_t, g_t, h_t, $1/h_t$, A_t, B_t, C_t, $\bar{C}_t$, D_t and $\bar{D}_t$ are uniformly bounded, deterministic functions. e_0 and F are constants, and $G \in \mathscr{L}^2_{\mathscr{F}_T^{w,\bar{w}}}(\mathbb{R})$.

Assumption 2

$L_t \geq 0$, $O_t \geq 0$, $R_t \geq 0$, l_t, o_t and r_t are uniformly bounded, deterministic functions. $M \geq 0$, $N \geq 0$, m and n are constants.

What are the new features?

- The observation noise is correlated with the state noise.
- The observation coefficient is linear with respect to the state.
- The diffusion coefficients are any large constants.

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What are the difficulties?

Partially observed stochastic control problems are always hard to study:

- Circular dependence between control and observation results in an intrinsic difficulty to study Problem (LQC).
- Classical separation principle is usually invalid to deal with Problem (LQC)/ results in stochastic control with infinite-dimensional space.
- How to solve some practical problems?

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Decomposition of the state and observation

To solve Problem (LQC), we separate $(x^v, y^v, z^v, \bar{z}^v)$ and Y^v into

$$
\begin{aligned}
(x^v, y^v, z^v, \bar{z}^v) &= \left(x^0, y^0, z^0, \bar{z}^0\right) + \left(x^1, y^1, z^1, \bar{z}^1\right), \\
Y^v &= Y^0 + Y^1,
\end{aligned}
$$

where $\left(x^0, y^0, z^0, \bar{z}^0\right)$ and Y^0 are independent of v. Set

$$
\mathscr{F}_t^{Y^v} = \sigma\{Y_s^v; 0 \le s \le t\}, \quad \mathscr{F}_t^{Y^0} = \sigma\left\{Y_s^0; 0 \le s \le t\right\}.
$$

Two decision sets

Definition 4.1

$$\mathscr{U}_{ad}^{0} = \left\{ v | v_t \text{ is } \mathscr{F}_t^{Y^0}\text{-adapted with values in } \mathbb{R} \text{ such that } \mathbb{E} \sup_{0 \le t \le T} v_t^2 < \infty \right\}.$$

Definition 4.2

$$\mathscr{U}_{ad} = \left\{ v | v_t \text{ is } \mathscr{F}_t^{Y^v}\text{-adapted and is an element of } \mathscr{U}_{ad}^{0} \right\}.$$

Two LQ problems

Problem (LQC). Find a $u \in \mathscr{U}_{ad}$ such that

$$J[u] = \inf_{v \in \mathscr{U}_{ad}} J[v].$$

Preliminary problem. Find a $u \in \mathscr{U}_{ad}^0$ such that

$$J[u] = \inf_{v \in \mathscr{U}_{ad}^0} J[v].$$

Equivalence

Proposition 4.1

Under Assumptions 1 and 2,

$$\inf_{v\in\mathscr{U}_{ad}} J[v] = \inf_{v\in\mathscr{U}_{ad}^{0}} J[v].$$

- One key point of its proof is that $\mathscr{U}_{ad}$ is dense in $\mathscr{U}_{ad}^{0}$ under the metric of $\mathscr{L}^2_{\mathscr{F}^{Y^0}}(0,T;\mathbb{R})$.

Maximum principle

Theorem 4.1

Let Assumptions 1 and 2 hold. Suppose that $(u, x, y, z, \bar{z})$ is the optimal solution. Then the FBSDE

$$\begin{cases} dp_t = (B_t p_t - O_t y_t - o_t)dt + C_t p_t dw_t + \bar{C}_t p_t d\bar{w}_t, \\ -dq_t = (a_t q_t - A_t p_t + L_t x_t + l_t)dt - k_t dw_t - \bar{k}_t d\bar{w}_t, \\ p_0 = -Ny_0 - n, \quad q_T = -Fp_T + Mx_T + m \end{cases}$$

admits a unique solution $\left(p, q, k, \bar{k}\right) \in \mathscr{L}^2_{\mathscr{F}^{w,\bar{w}}}\left(0, T; \mathbb{R}^4\right)$ such that

$$R_t u_t - D_t \mathbb{E}\left[p_t \middle| \mathscr{F}_t^Y\right] + b_t \mathbb{E}\left[q_t \middle| \mathscr{F}_t^Y\right] + r_t = 0$$

with $\mathscr{F}_t^Y = \sigma\{Y_s^u; 0 \leq s \leq t\}$.

Verification theorem

Theorem 4.2

Assume Assumptions 1 and 2 hold. Let $u \in \mathscr{U}_{ad}$ satisfy

$$R_t u_t - D_t \mathbb{E}\left[p_t \middle| \mathscr{F}_t^Y\right] + b_t \mathbb{E}\left[q_t \middle| \mathscr{F}_t^Y\right] + r_t = 0,$$

where $(x, y, z, \bar{z}, p, q, k, \bar{k})$ is a solution to the Hamiltonian system

$$\begin{cases} dx_t = (a_t x_t + b_t u_t + \bar{b}_t)dt + c_t dw_t + \bar{c}_t d\bar{w}_t, \quad x_0 = e_0, \\ -dy_t = (A_t x_t + B_t y_t + C_t z_t + \bar{C}_t \bar{z}_t + D_t u_t + \bar{D}_t)dt - z_t dw_t - \bar{z}_t d\bar{w}_t, \\ dp_t = (B_t p_t - O_t y_t - o_t)dt + C_t p_t dw_t + \bar{C}_t p_t d\bar{w}_t, \quad p_0 = -N y_0 - n, \\ -dq_t = (a_t q_t - A_t p_t + L_t x_t + l_t)dt - k_t dw_t - \bar{k}_t d\bar{w}_t, \\ y_T = F x_T + G, \quad q_T = -F p_T + M x_T + m. \end{cases}$$

Then u is an optimal control of Problem (LQC).

Uniqueness

Assumption 3

$R_t > 0$ and $1/R_t$ are uniformly bounded and deterministic functions.

Proposition 4.1

Let Assumptions 1, 2 and 3 hold. If u is an optimal control of Problem (LQC), then u is unique.

Optimal filtering of state equation

Proposition 4.2

Let Assumption 1 hold. For any $v \in \mathscr{U}_{ad}$, the optimal filtering of $(x_t^v, y_t^v, z_t^v, \bar{z}_t^v)$ with respect to $\mathscr{F}_t^{Y^v}$ satisfies an FBSDE

$$\begin{cases} d\hat{x}_t^v = \left(a_t\hat{x}_t^v + b_t v_t + \bar{b}_t\right) dt + \left(c_t + \dfrac{P_t f_t}{h_t}\right) d\hat{w}_t, \\ -d\hat{y}_t^v = \left(A_t\hat{x}_t^v + B_t\hat{y}_t^v + C_t\hat{z}_t^v + \bar{C}_t\hat{\bar{z}}_t^v + D_t v_t + \bar{D}_t\right) dt - \hat{Z}_t^v d\hat{w}_t, \\ \hat{x}_0^v = e_0, \quad \hat{y}_T^v = F\hat{x}_T^v + \hat{G}, \end{cases} \tag{4}$$

where the mean square error P_t of the estimate $\hat{x}_t^v$ is the unique solution of

- A special case of (4) is derived originally in Huang, Wang and Xiong [SICON, 2009].

Optimal filtering of state equation

Continuation of Proposition 4.2

$$\begin{cases} \dot{P}_t - 2a_t P_t + \left(c_t + \dfrac{P_t f_t}{h_t}\right)^2 - (c_t + \bar{c}_t)^2 = 0, \\ P_0 = 0, \end{cases}$$

$$\hat{w}_t = \int_0^t \frac{f_s}{h_s}(x_s^v - \hat{x}_s^v)ds + w_t \tag{5}$$

is a standard BM with values in $\mathbb{R}$, and

$$\hat{Z}_t^v = \hat{z}_t^v + \frac{f_t}{h_t}\left(\widehat{x_t^v y_t^v} - \hat{x}_t^v \hat{y}_t^v\right).$$

Optimal filtering of adjoint equation

Proposition 4.3

Let Assumptions 1, 2 and $O_t = 0$ hold. The optimal filtering of (p_t, q_t, k_t) depending on $\mathscr{F}_t^Y$ satisfies an FBSDE

$$\begin{cases} d\hat{p}_t = (B_t\hat{p}_t - o_t)dt + \left[C_t\hat{p}_t + \dfrac{f_t}{h_t}\left(\widehat{x_tp_t} - \hat{x}_t\hat{p}_t\right)\right]d\hat{w}_t, \\ -d\hat{q}_t = (a_t\hat{q}_t - A_t\hat{p}_t + L_t\hat{x}_t + l_t)dt - \hat{K}_t d\hat{w}_t, \\ \hat{p}_0 = -Ny_0 - n, \quad \hat{q}_T = M\hat{x}_T - F\hat{p}_T + m \end{cases} \tag{6}$$

with

$$\hat{K}_t = \hat{k}_t + \frac{f_t}{h_t}\left(\widehat{x_tq_t} - \hat{x}_t\hat{q}_t\right),$$

Optimal filtering of adjoint equation

Continuation of Proposition 4.3

where $(\hat{x}, \hat{y})$, $\hat{w}$ and $\widehat{x^{\mathbf{m}}p}$ satisfy (4) with $v = u$, (5), and

$$\begin{cases} d\widehat{x_t^{\mathbf{m}}p_t} = \Big[(\mathbf{m}a_t + B_t)\widehat{x_t^{\mathbf{m}}p_t} - o_t\widehat{x_t^{\mathbf{m}}} + \mathbf{m}\left(b_t u_t + \bar{b}_t + c_t C_t + \bar{c}_t \bar{C}_t\right)\widehat{x_t^{\mathbf{m}-1}p_t}\Big]dt \\ \qquad\qquad + \left[\mathbf{m}c_t\widehat{x_t^{\mathbf{m}-1}p_t} + C_t\widehat{x_t^{\mathbf{m}}p_t} + \dfrac{f_t}{h_t}\left(\widehat{x_t^{\mathbf{m}+1}p_t} - \hat{x}_t\widehat{x_t^{\mathbf{m}}p_t}\right)\right]d\hat{w}_t, \\ \widehat{x_0^{\mathbf{m}}p_0} = -e_0^{\mathbf{m}}(Ny_0 + n), \quad \mathbf{m} = 1, 2, 3, \cdots, \end{cases}$$

respectively.

An example

Example 5.1

$$\inf_{v \in \mathscr{U}_{ad}} J[v],$$

$$J[v] = \frac{1}{2}\mathbb{E}\left\{\int_0^T \left[O_t(y_t^v)^2 + R_t v_t^2\right] dt + N(y_0^v)^2 + 2ny_0^v\right\},$$

$$\text{(State)} \begin{cases} -dy_t^v = \left(B_t y_t^v + C_t z_t^v + \bar{C}_t \bar{z}_t^v + D_t v_t\right) dt - z_t^v dw_t - \bar{z}_t^v d\bar{w}_t, \\ y_T^v = G. \end{cases}$$

Suppose that w_t is observable at time t.

Is it easy to solve?

- Control and estimate cann't be separated in the sense of classical separation principle.
- The existence and uniqueness of optimal filtering for Hamiltonian isn't an immediate result.

Is it easy to solve?

- Control and estimate cann't be separated in the sense of classical separation principle.

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Assumption

Introduce

$$\begin{cases} \dot{\alpha}_t - \left(2B_t + C_t^2 + \bar{C}_t^2\right)\alpha_t - \dfrac{1}{R_t} D_t^2 \alpha_t^2 + O_t = 0, \\ \alpha_0 = -N, \end{cases} \tag{7}$$

$$\begin{cases} \dot{\beta}_t - \left(B_t + C_t^2 + \bar{C}_t^2 + \dfrac{1}{R_t} D_t^2 \alpha_t\right)\beta_t = 0, \\ \beta_0 = -n. \end{cases}$$

Assumption 4

The solution α of (7) satisfies

$$\frac{1}{\alpha_t}\bar{C}_t^2 + \frac{1}{R_t} D_t^2 \geq 0.$$

Optimal control

Proposition 5.1

Let Assumptions 1∼4 hold. The optimal control is uniquely denoted by

$$u_t = \frac{1}{R_t} D_t(\alpha_t \hat{y}_t + \beta_t),$$

where $\hat{y}$ is the unique solution of

$$\begin{cases} d\hat{p}_t = (B_t\hat{p}_t - O_t\hat{y}_t)dt + C_t\hat{p}_t dw_t, \\ -d\hat{y}_t = \Big(B_t\hat{y}_t + C_t\hat{z}_t + \bar{C}_t\hat{\bar{z}}_t + D_t u_t\Big)dt - \hat{z}_t dw_t, \\ \hat{p}_0 = -Ny_0 - n, \quad \hat{y}_T = \hat{G}. \end{cases}$$

- Theorems 4.1-4.3 and Propositions 4.1-4.3 are used to derive the proposition.

Summary

This talk introduces a backward separation method, which is also applicable to some stochastic differential games with noisy observation.

- Wang, Xiao and Xiong [Automatica, 2018] for nonzero sum game.
- Shi, Wang and Xiong [ESAIM COCV, 2020] for leader-follower game.

Summary

This talk introduces a backward separation method, which is also applicable to some stochastic differential games with noisy observation.

- Wang, Xiao and Xiong [Automatica, 2018] for nonzero sum game.
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Thank you for your attention!