# Intrinsic Formation Control Under Finite-Time Differential Game Framework

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**Abstract:** In this paper, the formation control problem of a multi-agent system is studied. The foraging behavior is modeled as a finite-horizon non-cooperative differential game under local information, and the existence and properties of Nash equilibria are studied. The formations are achieved in an intrinsic way in the sense that they are only attributed to the inter-agent interaction and geometric properties of the network, where the desired formations are not designated beforehand. Through the design of individual costs and network topology, regular polygons, antipodal formations and Platonic solids are achieved as Nash equilibria while inter-agent collision is avoided. While the focus is on the finite horizon case, it is also studied how the formation patterns would change as the length of the time interval tends to infinity. Finally, numerical simulations are provided in both two-dimensional and three-dimensional Euclidean space to demonstrate the effectiveness and feasibility of the proposed methods.

**Key Words:** Multi-agent system, Formation control, Differential games

## 1 Introduction

Multi-agent system is popular due to its advantages of better robustness as well as lower communication and computation burden. Inspired by biological modeling [5], there are numerous applications in motion planning of multi-robots system, where the agents control their own dynamics to achieve a cooperative task by exchanging information with neighbors [8].

In recent years, game theory, in particular evolutionary game theory, has been applied to multi-agent systems such as [6, 7]. Although there are numerous results on situations in which agents cooperative to achieve a common task, there are more practical scenarios where agents have individual and partially conflicting goals, thus leading to a non-cooperative setting. Differential games focus on multi-player decision making problems over time, where each agent aims to optimize its own, individual cost subject to the common state dynamics [2]. Compared to formation control methods based on navigation functions, game theoretical approach has the advantage of an optimization perspective as well as better performance such as robustness.

However, among the existing results of formation control problem based on differential game theory, most papers focus on the consensus problem [3, 11, 12]. For the non-consensus formation problems, most existing methodologies are based on the pre-defined formation pattern by regulating formation errors, which is then solved by transforming to a consensus problem, such as [4, 6]. The novelty of our paper lies in that the formation is achieved in an "intrinsic" way in the sense that it is only attributed to the inter-agent interaction and geometric properties of the network, where the desired formations are not designated directly in the controller [13, 15]. To the best of our knowledge, only a few papers in this field have considered the case outside the consensus framework. For example, Özgüler [9, 14] considers a foraging swarm behavior in one dimension. The equilibrium dynamics is reduced to a linear system thus explicit expressions can be derived for several features. However, how to achieve more general formation patterns in higher-dimensional spaces still remains to be solved.

In this paper, the formation control problem of a multi-agent system is studied in a finite-horizon non-cooperative differential game framework. Various formation patterns are achieved by Nash equilibrium strategies via designing network topologies and weight functions. The existence of Nash equilibrium for the nonlinear game is proved. Regular polygons, antipodal formations and Platonic solids are achieved in $\mathbb{R}^2$ and $\mathbb{R}^3$ while inter-agent collision is avoided. The main novelty of the proposed method is three-folded:

1) The formations are achieved in a distributed manner. Each agent only has access to the relative displacement of its neighbors and the foraging point, and the inter-agent collision can be naturally avoided.
2) The formations are realized in an intrinsic manner in two aspects. Firstly, the desired formation pattern is not designated in the individual cost beforehand. And different patterns are only attributed to the inter-agent interaction and geometric properties of the network. Secondly, the form of individual costs is identical to all agents and is also invariant with the number of agents, which makes the methodology robust and scalable.
3) The proposed framework is also tutorial since a novel systematic approach for formation control is provided, where the desired formation can be obtained by only intrinsically adjusting the network topology. Furthermore, the results not only lead to a better understanding of the natural phenomenon where a collaborative swarming behavior can result from non-cooperative individual actions, but also bring new inspiration in the construction of other formations.

The rest of the paper is organized as follows. In Section 2, the intrinsic formation control problem is formulated as a non-cooperative differential game. The existence and properties of Nash equilibrium are studied in Section 3. In Section 4, different formation patterns are designed in the space of $\mathbb{R}^2$ and $\mathbb{R}^3$ respectively. Numerical simulations are given in Section 5 and concluding remarks are drawn in Section 6.

## 2 Problem Formulation

### 2.1 Non-Cooperative Differential Games

We consider the formation control problem of $N$ agents and the dynamics of each agent is characterized by a single integrator

$$\dot{x_i} = u_i, \quad i = 1, 2, ..., N, \tag{1}$$

where $x_i \in \mathbb{R}^n$ is the position of the $i$-th agent and $u_i \in U$ is its control input. Here $U \subset \mathbb{R}^n$ is a compact set. For instance, for problems with no control constraints, we can choose $U$ as $U = \{u \in \mathbb{R}^n : \|u\| \leq M\}$ and $M < +\infty$ denotes a bounded constant that is arbitrarily large.

The interaction between agents is modeled by a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, where the vertex set $\mathcal{V} = \{1, 2, ..., N\}$ denotes the agents in the network and $\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$ is the edge set. We say that agent $j$ is a neighbor of agent $i$ if $(j, i) \in \mathcal{E}$, and the set of neighbors of agent $i$ is denoted by $\mathcal{N}_i = \{j : (j, i) \in \mathcal{E}\}$. The graph is undirected if $(j, i) \in \mathcal{E}$ means $(i, j) \in \mathcal{E}$.

The multi-agent system aims at moving towards a foraging point while realizing some specific formation patterns. Without loss of generality, the foraging point is assumed to coincide with the origin. Let $x = [x_1^T, ..., x_N^T]^T$. Each agent is associated with an individual cost function to minimize based on local information

$$J_i(x(0), u_i, u_{-i}) = \int_0^T l_i(x, u_i)dt, \tag{2}$$

where $u_{-i}$ is the strategy profile of all players except for player $i$ and $l_i(x, u_i)$ is given by

$$l_i(x, u_i) = \frac{1}{2}\|u_i\|^2 + \frac{k_1}{2}\|x_i\|^2 + g_i(\|x_i - x_j\|_{j \in \mathcal{N}_i}). \tag{3}$$

Here $k_1$ is some positive constant and the terminal time $T$ is given. The first and second terms penalize the energy consumption and the distance to foraging location respectively. The third term denotes inter-agent interaction, where only relative distances of neighbors are available.

In order to avoid inter-agent collision, some assumptions are made on the interaction cost $g_i$ for each agent.

**Assumption 1.** *For each agent i, function $g_i$ is positive, Lipschitz continuous, and strictly decreasing with respect to any $\|x_i - x_j\|$ with $j \in \mathcal{N}_i$. In addition, $g_i(\cdot) \to \infty$ as any $\|x_i - x_j\| \to 0$ while $g_i(\cdot) \to 0$ as any $\|x_i - x_j\| \to \infty$.*

**Assumption 2.** *The interaction graph is undirected and each pair of neighbors has identical weights for their relative distance, i.e. $\frac{\partial g_i}{\partial \|x_i - x_j\|} = \frac{\partial g_j}{\partial \|x_i - x_j\|}$ if $(i, j) \in \mathcal{E}$.*

Note that the differential game played by each agent $i$ can be considered as an optimal control problem with $u_j$ $(j \neq i)$ given. Although the cost function is minimized w.r.t. $u_i$, a common state should be taken into consideration.

$$\begin{aligned} &\min_{u_i} \;\; J_i(x(0), u_i, u_{-i}) \\ &s.t. \;\; \dot{x} = \sum_{j=1}^{N} B_j u_j, \end{aligned} \tag{4}$$

where the column-wise block matrix $B_j \in \mathbb{R}^{nN \times n}$ consists of $N$ blocks of size $n \times n$, where only the $j - th$ block is an identity matrix while the others are all zeros, and $u_j$ $(j \neq i)$ are given by the optimal strategies of others.

A continuous function $u_i(t)$ defined on $[0, T]$ is called an *admissible control* if $u_i(t) \in U$ for any $t \in [0, T]$..

**Definition 1.** *The set of admissible control functions $(u_1^*, u_2^*, ..., u_N^*)$ is called Nash Equilibrium Strategy if*

$$J_i(x(0), u_i^*, u_{-i}^*) \leq J_i(x(0), u_i, u_{-i}^*), \tag{5}$$

*for all $u_i \neq u_i^*$, $i = 1, 2, ..., N$.*

### 2.2 Intrinsic Formation Control

For many applications such as sensor networks with homogeneous agents, we are only interested in the relative formation pattern of agents, without designating either the orientation of the whole configuration or the order of the agents. Then instead of the specific position of each agent, a manifold of relative configuration is investigated. Here we use an undirected graph $\mathcal{P}$ to represent the desired relative configuration with its skeleton as the edges. Denote $S_p = \{\sigma_1, ..., \sigma_{o_p}\}$ as its permutation group. Each permutation $\sigma_i \in S_p$ can be described by its permutation matrix $P_\sigma = [e_{\sigma_i(1)}, ..., e_{\sigma_i(N)}]^T$, where $e_{\sigma_i(k)} \in \mathbb{R}^N$ has 1 at element $\sigma_i(k)$ and 0 elsewhere. Denote $x_p^*$ as the vertex coordinates of an arbitrary displacement of $\mathcal{P}$ centered at the origin. Then through vertex permutation and rotation of the body, the whole vertices set can be obtained as the union of a finite number of disjoint closed manifolds, i.e., $\mathcal{M}_\mathcal{P} = \bigcup_{k=1}^{o_p} \mathcal{M}_k$ with

$$\begin{aligned} \mathcal{M}_k = \{x \in \mathbb{R}^{nN} : x = (I_N \otimes R)(P_{\sigma_k} \otimes I_n)x_p^*, \\ \forall R \in SO(n)\}, \quad k = 1, ..., o_p, \end{aligned} \tag{6}$$

where $SO(n)$ denotes the set of $n \times n$ real orthogonal matrices with determinant 1, $I_N$ is the identity matrix of dimension $N \times N$, and $\otimes$ denotes the Kronecker product.

Note that $\mathcal{R} = \{I_N \otimes R : R \in SO(n)\}$ is a closed group under matrix multiplication. Here we introduce the following definition to characterize the invariance of a mapping under such group action.

**Definition 2.** *A vector field $f(x) : \mathbb{R}^{nN} \to \mathbb{R}^{nN}$ is called $\mathcal{R}$-invariant if $f(\bar{R}x) = \bar{R}f(x)$ for any $\bar{R} \in \mathcal{R}$. A value function $V(x) : \mathbb{R}^{nN} \to \mathbb{R}$ is called $\mathcal{R}$-invariant if $V(\bar{R}x) = V(x)$ for any $\bar{R} \in \mathcal{R}$.*

Since we have no requirement for the absolute attitude of the whole configuration, such rotation freedom should also be reflected in the design of differential games.

**Assumption 3.** *$g_i(\|x_i - x_j\|_{j \in \mathcal{N}_i})$ is $\mathcal{R}$-invariant for $i = 1, ..., N$.*

Finally, the intrinsic formation problem is formulated.

**Problem 1.** *Consider a multi-agent system playing a non-cooperative game defined in (4), design the penalty function $g_i$ and interaction topology $\mathcal{N}_i$ for each agent such that the obtained Nash equilibrium trajectories converge to some point on the relative formation manifold, namely the Nash equilibrium strategy $(u_1^*, u_2^*, ..., u_N^*)$ satisfies*

$$J_i(x(0), u_i^*, u_{-i}^*) \leq J_i(x(0), u_i, u_{-i}^*),$$

*for all $u_i \neq u_i^*$, $i = 1, 2, ..., N$, and $x^*(T)$ converges to $\mathcal{M}_{\mathcal{P}}$ as $T \to \infty$.*

*The design should be "intrinsic" namely the desired configuration is not designated directly in $\{g_i\}$, and different formations are only attributed to the geometric properties of network topologies. The Nash equilibrium trajectories should be free from inter-agent collision between neighbors.*

Note that in this paper a finite time game is considered, where $T \to \infty$ indicates $T$ is large enough but still finite.

## 3 Nash Equilibrium for Differential Game

In this section, the Nash equilibrium for the designed non-cooperative differential game is investigated. The existence of Nash equilibrium strategy is proved under some assumptions, and the corresponding optimal strategies are analyzed.

### 3.1 Existence of Nash Equilibrium Strategy

In this part, the existence of Nash equilibrium strategies for the non-cooperative dynamic game (4) is investigated. In order to guarantee the existence of the Nash equilibrium strategies, the optimal controller of each individual game is required to be achievable. The differential game played by agent $i$ can be regarded as an optimal control problem w.r.t. $u_i$ with the strategies of other agents $u_{-i}^*$ considered as known functions. However, for general nonlinear games which are not convex, the existence of optimal solutions is not trivial and closeness of the solution space remains to be analyzed.

Firstly a general assumption is made on the initial configuration of the agents.

**Assumption 4.** *The initial positions of the agents do not coincide, i.e. $x_i(0) \neq x_j(0)$ for all $i, j \in \mathcal{V}$ and $i \neq j$.*

Here we define an additional state $x_0^i$ as

$$\dot{x}_0^i = \frac{1}{2}\|u_i\|^2 + \frac{k_1}{2}\|x_i\|^2 + g_i(\|x_i - x_j^*\|_{j\in\mathcal{N}_i}),$$

with $x_0^i(0) = 0$ and $x_j^*(t)$ considered as known functions.

Then the differential game played by agent $i$ in (4) is equivalent to

$$\begin{aligned} \min_{u_i} \quad & q(\tilde{x}^i(T)) \\ s.t. \quad & \dot{\tilde{x}}^i(t) = \tilde{f}\left(t, \tilde{x}^i, u_i\right), \quad \tilde{x}^i(0) = \tilde{x}^{i(0)} \\ & u_i(t) \in U, \quad \forall t \in [0, T] \end{aligned} \tag{7}$$

where $\tilde{x}^i = [x_0^i, x_i^T]^T \in \mathbb{R}^{n+1}$ is the augmented state vector, $\tilde{x}^{i(0)} = [0, x_i(0)^T]^T$, $q(\tilde{x}^i(T)) = x_0^i(T)$ and $f\left(t, \tilde{x}^i, u_i\right)$ is defined by

$$\tilde{f}\left(t, \tilde{x}^i, u_i\right) = \begin{bmatrix} l_i(x_i, u_i, x_{-i}^*) \\ u_i \end{bmatrix},$$

where $x_{-i}^*(t)$ are regarded as known functions on $[0, T]$.

**Definition 3.** *Under the dynamics (7), a point $\tilde{x}_1^i \in \mathbb{R}^{n+1}$ is called reachable from $\tilde{x}^{i(0)}$ if there exists an admissible control $u_i(t)$ defined on $[0, T]$ such that the derived trajectory satisfies $\tilde{x}^i(T) = \tilde{x}_1^i$. The set of all reachable points from $\tilde{x}^{i(0)}$ is called the reachable set, which is denoted by $\mathcal{R}_T$.*

Then the existence of optimal control to (7) is studied.

**Theorem 1.** *If $U$ is compact, then for any given trajectories $x_j(t)$ $(j \neq i)$ satisfying Assumption 4, there exists an admissible control function $u_i^*(t)$ that solves the optimal control problem in (7) (equivalently in (4)), i.e.*

$$\begin{aligned} & q(\tilde{x}_s^i(T)) = \min_{u_i} J_i, \\ & \tilde{x}_s^i(T) \in \mathcal{R}_T, \end{aligned} \tag{8}$$

*where $\tilde{x}_s^i$ is the optimal trajectory corresponding to $u_i^*$ and inter-agent collision can be naturally avoided.*

*Proof.* It is obvious that in $\mathbb{R}^n (n \geq 2)$ space, for any continuous trajectories of $x_j^*(t)$ with $t \in [0, T]$ and $j \neq i$, we can always find a smooth and bounded trajectory $\bar{x}_i(t)$ for agent $i$ on $[0, T]$ such that it does not collide with $x_j^*(t)$ for any $t \in [0, T]$ and $j \neq i$. Thus it means that the corresponding cost $J(\bar{u}_i)$ is bounded by some $\eta < +\infty$.

Since $l_i(x_i, u_i, x_{-i}^*(t)) > 0$ is bounded below, we know that the infimum must be obtained by

$$0 \leq inf\ J_i(x(0), u_i, u_{-i}^*) \leq \eta,$$

where the infimum is over all admissible control.

Therefore, it is enough to only consider the trajectories satisfying $J_i(x(0), u_i, u_{-i}^*) = g(\tilde{x}^i(T)) \leq \eta$. On that subset of reachable set (denoted by $\bar{\mathcal{R}}_T$), $l_i(x_i, u_i, x_{-i}^*(t))$ is continuous and uniformly bounded, where the property of collision-free is also guaranteed since $g_i(\|x_i - x_j\|_{j\in\mathcal{N}_i})$ is bounded. Hence $\tilde{f}(t, \tilde{x}^i, u_i)$ is continuous and uniformly bounded on $[0, T]$ for the considered subset, making $\tilde{x}^i(t)$ also Lipschitz and continuously differentiable.

Now consider a sequence of convergent reachable points $\{\xi_k\}_{k=1}^{+\infty} \in \bar{\mathcal{R}}_T$ such that $\lim_{k\to\infty} q(\xi_k) = inf J(u_i)$. Since $\xi_k \in \bar{\mathcal{R}}_T$, there exist admissible controls $u_k^i(t)$ such that the corresponding trajectories $\tilde{x}_k^i(t)$ with $\tilde{x}_k^i(0) = \tilde{x}^{i(0)}$ satisfy

$$\begin{aligned} & \tilde{x}_k^i(t) = \int_0^t \tilde{f}(\tau, \tilde{x}_k^i(\tau), u_k^i(\tau))d\tau, \quad 0 \leq t \leq T, \\ & \tilde{x}_k^i(T) = \xi_k. \end{aligned} \tag{9}$$

Here we denote $\tilde{f}_k(t) = \tilde{f}(t, \tilde{x}_k^i(t), u_k^i(t))$. Due to the fact that $\dot{\tilde{x}}_k^i(t) = \tilde{f}_k(t)$ is uniformly bounded, we know that $\tilde{x}_k^i(t)$ is uniformly equicontinuous. Then by Ascoli-Arzelà Theorem, the uniformly bounded function sequence $\{\tilde{x}_k^i(t)\}$ must converge uniformly to a continuous function $\tilde{x}_s^i(t)$ with $\tilde{x}_s^i(0) = \tilde{x}^{i(0)}$ along some subsequence:

$$\lim_{k\to\infty} \tilde{x}_k^i(t) = \tilde{x}_s^i(t), \quad \forall t \in [0, T],$$

where we abuse the indices for the subsequence and thus

$$\lim_{k\to\infty} \tilde{x}_k^i(T) = \lim_{k\to\infty} \xi_k = \tilde{x}_s^i(T).$$

Furthermore, since $g(\cdot)$ is continuous, it holds that

$$q(\tilde{x}_s^i(T)) = \lim_{k\to\infty} q(\tilde{x}_k^i(T)) = inf J(u_i).$$

Then it remains to show that $\tilde{x}_s^i(T) \in \bar{\mathcal{R}}_T$, namely $\tilde{x}_s^i(t)$ is a trajectory corresponding to some admissible control.

Since $\tilde{x}_k^i(t) = \int_0^t \tilde{f}_k(\tau)d\tau$ and $\tilde{f}_k(t)$ is continuous and bounded, we can see that $\tilde{x}_s^i(t)$ is continuously differentiable. Furthermore, by Moore-Osgood theorem, it can be shown that $\dot{\tilde{x}}_k^i(t) \to \dot{\tilde{x}}_s^i(t)$ for any $t \in [0,T]$.

Here we denote $\dot{\tilde{x}}_s^i(t)$ as $\tilde{f}_s(t)$, which is partitioned as

$$\tilde{f}_s(t) = \begin{bmatrix} l_i^*(t) \\ u_i^*(t) \end{bmatrix},$$

where $l_i^*(t) \in \mathbb{R}$ and $u_i^*(t) \in \mathbb{R}^n$ for any $t \in [0,T]$.

Then it holds that $u_k^i(t) \to u_i^*(t)$. Since $U$ is compact, we have that $u_i^*(t) \in U$ for any $t \in [0,T]$, which implies that $u_i^*(t)$ is an admissible control function.

In addition, since $u_k^i \to u_i^*$, $\tilde{x}_k^i \to \tilde{x}_s^i$ and $l_i(\cdot)$ is continuous, we have

$$\lim_{k\to\infty} l_i\left(\tilde{x}_k^i, u_k^i\right) = l_i\left(\tilde{x}_s^i, u_i^*\right) = l_i^*(t), \quad \forall t \in [0,T].$$

Therefore, we have proved that $\tilde{f}_s(t) = \tilde{f}(t, \tilde{x}_s^i, u_i^*)$, which means that

$$\begin{aligned} &\dot{\tilde{x}}_s^i = \tilde{f}(t, \tilde{x}_s^i, u_i^*), \\ &q(\tilde{x}_s^i(T)) = \min_{u_i \in U} J_i(x(0), u_i, u_{-i}^*). \end{aligned}$$

Hence it holds that $u_i^*$ is the optimal solution to (7) and $\tilde{x}_s^i$ is the corresponding optimal trajectory. □

### 3.2 Description of Nash Equilibrium

For the $N$-tuple differential game in (4), the necessary condition for $\{u_i^*(t)\}_{i=1}^N$ to be an open-loop Nash equilibrium strategy is that there exists $N$ costate functions $p_i(t) : [0,T] \to \mathbb{R}^{Nn}$ such that [2]:

$$\begin{aligned} u_i^* &= \underset{u_i \in U}{argmin}\, H_i(t, p_i, x^*, u_i, u_{-i}^*), \\ \dot{x}^* &= \sum_{j=1}^N B_j u_j^*, \quad x^*(0) = x_0, \\ \dot{p}_i &= -\frac{\partial H_i(t, p_i, x^*, u_1^*, ..., u_N^*)}{\partial x}, \quad p_i(T) = 0, \end{aligned} \tag{10}$$

where $H_i(t, p_i, x, u_1, ...u_N) = l_i(x, u_i) + {p_i}^T (\sum\limits_{j=1}^N B_j u_j)$.

Since we have proved the existence of Nash equilibrium in Section 3.1, the above two point boundary value problem (TPBVP) must have solutions for any $x_0$. For general nonlinear TPBVP, the uniqueness and analytic expression of the solution are difficult to obtain.

We will show that under certain assumptions, there exists a unique local solution to (10) near some invariant manifold to be defined later. Thus the above adjoint system also provides a sufficient condition for a local Nash equilibrium. Furthermore, the Nash equilibrium trajectories would converge to the manifold of required formation patterns as $T \to \infty$.

**Lemma 2.** *Consider the following adjoint system with two-point boundary conditions:*

$$\begin{bmatrix} \dot{x}(t) \\ \dot{\lambda}(t) \end{bmatrix} = \begin{bmatrix} 0 & -I \\ -A & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ \lambda(t) \end{bmatrix}, \quad \begin{bmatrix} x(0) \\ \lambda(T) \end{bmatrix} = \begin{bmatrix} x_0 \\ 0 \end{bmatrix} \tag{11}$$

*where $x(t), \lambda(t) \in \mathbb{R}^n$ for any $t$, and $x_0$ is given.*

*If $A \in \mathbb{S}_+$, then for any $x_0$, the solution for the above two point boundary value problem is unique. Furthermore, it also holds that $\lim\limits_{T\to\infty} Ax(T) = 0$.*

Due to the limitation of space, the proof of Lemma 2 is omitted.

We can deduce that if $x_0$ is located in a neighborhood of the subspace $KerA$, then for $T$ that is large enough, the whole trajectory of $x$ will also be bounded near the subspace and approaches the invariant subspace "steadily" as $\dot{x}(T), \ddot{x}(T) \to 0$.

Then by investigating the linearized model on some invariant manifold of dynamics in (10), the local convergence is also studied. We firstly define a potential function

$$W(x) = \frac{1}{2} \sum_{i=1}^N g_i(\|x_i - x_j\|_{j \in \mathcal{N}_i}). \tag{12}$$

**Theorem 3.** *If $U$ is compact and Assumption 4 is satisfied, there exists Nash equilibrium strategies to the non-cooperative differential game defined by (4). And the Nash equilibrium trajectories $x^*(t)$ together with $N$ functions $\lambda_i(t) : [0,T] \to \mathbb{R}^n$ satisfy the following TPBVP*

$$\begin{cases} \dot{x}_i^* = -\lambda_i, \\ \dot{\lambda}_i = -k_1 x_i^* - \frac{\partial g_i}{\partial x_i}^T |_{x=x^*}, \end{cases} \tag{13a}$$

$$\begin{cases} x_i^*(0) \; given, \\ \lambda_i(T) = 0, \end{cases} \tag{13b}$$

*where $i = 1, 2, ..., N$.*

*The Nash equilibrium strategies and the corresponding trajectories have the properties*

1) *$x_i^*(t)$ and $u_i^*(t)$ are bounded and Lipschitz continuous on $[0,t]$,*
2) *It can be guaranteed that the optimal trajectories are collision-free, i.e. $x_i^*(t) \neq x_j^*(t)$ for any $j \in \mathcal{N}_i$ and $t \in [0,T]$.*

*Furthermore, if the manifold*

$$\mathcal{M}_T = \{x, \lambda \in \mathbb{R}^{Nn} : x \in \mathcal{M}_\mathcal{P}, \lambda = 0\},$$

*is invariant under (13a) and $H_W|_{x \in \mathcal{M}_\mathcal{P}} \succeq -k_1 I$, then $x^*(T)$ converges locally to $\mathcal{M}_\mathcal{P}$ as $T \to \infty$.*

*Proof.* The existence of Nash equilibrium and its properties can be proved by Theorem 1. We derive (13a) directly from (10) after straight computation, where $\lambda_i(t) = B_i^T p_i(t)$.

From the expression of $\mathcal{M}_\mathcal{P}$, it is easy to see that each subset $\mathcal{M}_k$ in (6) is a closed manifold and is invariant under (13a) with $\lambda = 0$. Since $\mathcal{M}_\mathcal{P}$ is composed of a finite number of disjoint manifolds, the local convergence to $\mathcal{M}_\mathcal{P}$ can be considered on different subset manifold independently and it is enough to only check the neighborhood of any one manifold $\mathcal{M}_k$. Based on Assumption 2, it is easy to see that $\frac{\partial W(x)}{\partial x_i} = \frac{\partial g_i}{\partial x_i}$ for any $i = 1, ..., N$, and the linearized dynamics of (13a) at $x_T^* = proj_{\mathcal{M}_k} x^*(T) \in \mathcal{M}_k$ is given by

$$\begin{bmatrix} \dot{\bar{x}}(t) \\ \dot{\lambda}(t) \end{bmatrix} = \begin{bmatrix} 0 & -I \\ -k_1 I - H_W(x_T^*) & 0 \end{bmatrix} \begin{bmatrix} \bar{x} \\ \lambda(t) \end{bmatrix}, \tag{14}$$

with $\bar{x} = x - x_T^*$.

Then the local convergence to $\mathcal{M}_k$ can be shown by Lemma 2 when $k_1 I + H_W|_{x \in \mathcal{M}_\mathcal{P}} \succeq 0$. □

## 4 Graph Design for Different Formation Patterns

In this part, we will show that for the designed differential games, different formation patterns can be realized by adjusting network topology, and the scale of the desired pattern is determined by the parameters in individual costs. The novelty of the proposed method lies in the fact that the formation pattern is only attributed to the geometric properties of the inter-agent topology.

Here we consider local results in a neighborhood of the largest invariant manifold contained in the equilibrium set

$$\mathcal{M}_e = \{x_i, \lambda_i \in \mathbb{R}^n : k_1 x_i = \frac{\partial g_i(\|x_i - x_j\|_{j\in\mathcal{N}_i})}{\partial x_i}^T, \\ \lambda_i = 0, i = 1, \cdots N\}.$$

The goal is to design the interaction functions $g_i$ under Assumptions 1-3 and the inter-agent topology such that

1) $\mathcal{M}_T$ is invariant under (13a), i.e., $\mathcal{M}_T \subseteq \mathcal{M}_e$;
2) $H_W|_{x\in\mathcal{M}_\mathcal{P}} \succeq -k_1 I$.

**Assumption 5.** *For each individual game, the penalty term for inter-agent interaction has the structure*

$$g_i(\|x_i - x_j\|_{j\in\mathcal{N}_i}) = k_2 \sum_{j\in\mathcal{N}_i} \frac{1}{\|x_i - x_j\|^m}, \tag{15}$$

*where $k_2, m > 0$ are some positive constants.*

Since only relative distance of neighbors is involved in $g_i$, we know that both $\{g_i\}_{i=1}^N$ and $W(x)$ are $\mathcal{R}$-invariant. We have the following properties about rotation transformations.

**Proposition 1.** *If a value function $V(x)$ is $\mathcal{R}$-invariant, then $\frac{\partial V}{\partial x}$ is also $\mathcal{R}$-invariant. The number of positive, negative and zero eigenvalues of $\frac{\partial^2 V}{\partial x^2}$ is also invariant under rotation transformations.*

Then the dynamics in (13a) is $\mathcal{R}$-invariant and the positive semi-definiteness of $H_W(x) + k_1 I$ on $\mathcal{M}_\mathcal{P}$ is also invariant. Therefore, the above two requirements on $\mathcal{M}_\mathcal{P}$ can be examined by only checking one point on the manifold, thus leading to the following topology design problem.

**Problem 2.** *For the desired pattern manifold $\mathcal{M}_\mathcal{P}$, considering an arbitrary coordinates $x_p^* \in \mathcal{M}_\mathcal{P}$, design the graph topology $\mathcal{E}$ such that $x_p^*$ satisfies*

*1)* $k_1 x_{pi}^* = k_2 \sum_{j\in\mathcal{N}_i} \frac{x_{pi}^* - x_{pj}^*}{\|x_{pi}^* - x_{pj}^*\|^{m+2}}, \quad i = 1, ..., N,$
*2)* $H_W(x_p^*) + k_1 I \succeq 0.$

Since the individual costs for the game are symmetric, we also expect to realize some symmetric formations. In this section, we focus on two types of symmetric patterns that are widely used in many applications: regular formation and antipodal formation.

### 4.1 Regular Formation Design

The symmetry of the formation often leads to many good performances in various promising applications, such as the maximal observability of sensor network [10]. In this part, the pattern with highest symmetry is studied, which means regular polygon in $\mathbb{R}^2$ and regular polyhedron in $\mathbb{R}^3$.

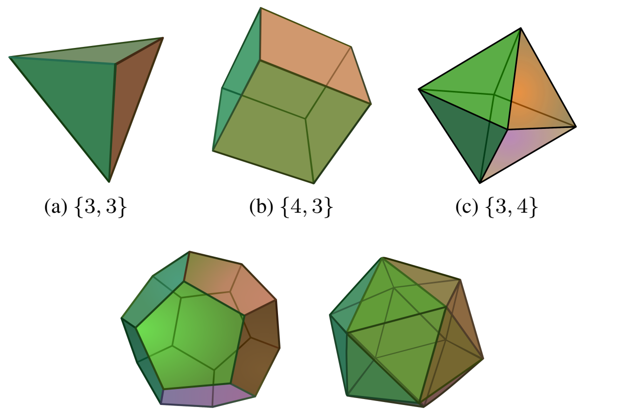
Firstly, the regular polygon in $\mathbb{R}^2$ space is studied with arbitrary $N \geq 2$.

**Proposition 2.** *Consider $N$ agents moving in $\mathbb{R}^2$ space with a complete graph $\mathcal{G}$ (i.e., $\mathcal{N}_i = \{j \in \mathcal{V} : j \neq i\}$). Take the vertex coordinates $x^*$ of an arbitrary configuration on the circle centered at the origin with radius*

$$r = \begin{cases} \left(\frac{k_2 m}{2^m k_1} \sum_{l=1}^{(n-1)/2} \frac{1}{sin(l\pi/n)^m}\right)^{\frac{1}{(m+2)}} & \text{if } n \text{ is odd} \\ \left(\frac{k_2 m}{2^m k_1}\left(\sum_{l=1}^{(n-2)/2} \frac{1}{sin(l\pi/n)^m} + \frac{1}{2}\right)\right)^{\frac{1}{(m+2)}} & \text{if } n \text{ is even} \end{cases}$$

*If $m$ and $N$ are chosen such that $H_W(x^*) + k_1 I \succeq 0$, Problem 2 is solved, which implies the regular polygon formation will be achieved by Nash equilibrium trajectories when $T \to \infty$.*

Unlike the regular polygons in $\mathbb{R}^2$ space, there exist only five types of regular polyhedra in the three-dimensional Euclidean space, which are known as *Platonic solids* as shown in Fig. 1. A regular polyhedron is identified by its Schläfli symbol of the form $\{p, q\}$, where $p$ is the number of sides of each face and $q$ the number of faces meeting at each vertex.



(a) $\{3,3\}$ (b) $\{4,3\}$ (c) $\{3,4\}$ (d) $\{5,3\}$ (e) $\{3,5\}$

Fig. 1: Regular Polyhedra.

It is quite inspiring to see that only polyhedra $\{p, q\}$ with $p = 3$ can be achieved by a complete graph, where each face is a triangle and thus structurally rigid. Since tetrahedron is the smallest rigid polyhedron in $\mathbb{R}^3$ space, we consider constructing cube and dodecahedron as polyhedral compounds of two and five tetrahedra respectively.

**Proposition 3.** *Assume $m = 1$ in (15). The Platonic solids of $\{3,3\}$, $\{3,4\}$ and $\{3,5\}$ are be formed by Nash equilibrium trajectories under a complete graph as $T \to \infty$, and the formations of cube and dodecahedron are obtained under following incomplete graphs respectively:*

*1) For the cube ($\{4,3\}$), we equally divide the eight agents into two distinct groups. The four agents in each group are completely connected. The graph has another four edges which provide an one-by-one undirected connection between two groups.*
*2) For the dodecahedron ($\{5,3\}$), twenty agents are divided into five distinct subsets with four agents in each group. Each agent has exactly four neighbors by connecting to the other three agents in its own group and an agent in another group. Furthermore, there exists at least one edge between any two groups.*

### 4.2 Antipodal Formation Design

In this part, we consider the antipodal formation pattern, which can be regarded as “consensus upon antagonism” [1].

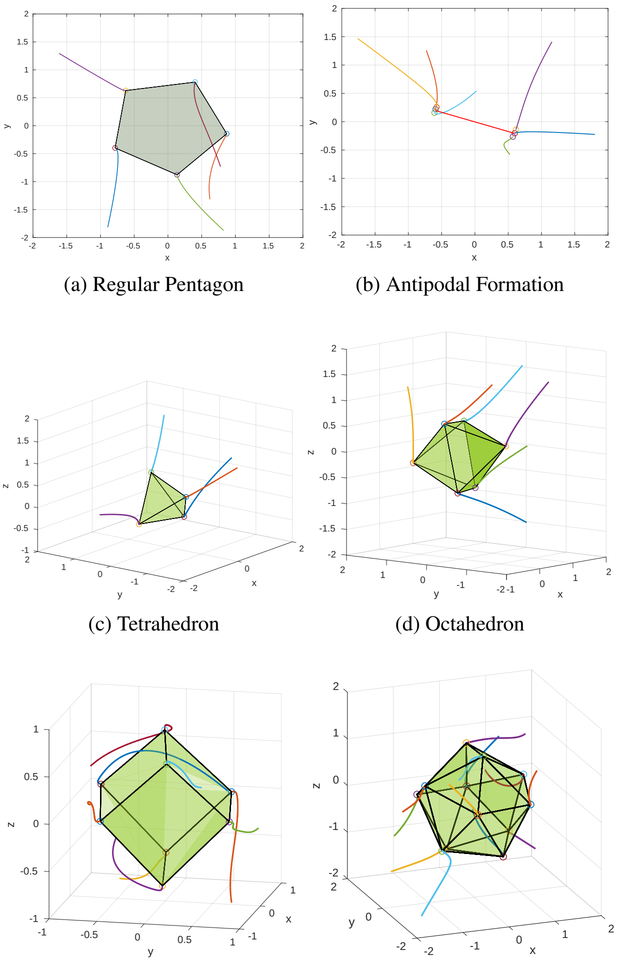
**Definition 4** (Antipodal Formation)**.** *Consider the multi-agent system with $N$ even. Antipodal formation refers to the phenomenon that agents converge to two distinct points that with opposite signs. In Euclidean space, it means that the two converging points are symmetric with the origin.*

**Proposition 4.** *Consider $m = 1$ in (15). For a system of $N$ agents ($N$ is even) connected by an undirected ring, Problem 2 is solved, which implies the antipodal formation will be achieved by Nash equilibrium trajectories when $T \to \infty$.*

Note that for the *Platonic solids* and antipodal patterns, there are limited number of distinct vertices. Then for a given $m$ in (15), a consistent result could be obtained for all choices of $N$. As the limitation of space, the proof of Proposition 2, 3 and 4 are omitted.

## 5 Simulation

In this section, numerical simulation results are provided to show the formation of several patterns under different inter-agent graphs. We choose $m = 1$ in (15). Under Assumption 4, random initial positions are generated for the multi-agent system. The Nash equilibrium trajectories with $T = 20$ are given as shown in Fig. 2.



(a) Regular Pentagon (b) Antipodal Formation

(c) Tetrahedron (d) Octahedron

(e) Cube (f) Icosahedron

Fig. 2: Nash Equilibrium Trajectories.

## 6 Conclusion and Future Work

In this paper, a novel game theoretic approach is proposed to solve the formation control problem for multi-agent systems. The problem is formulated as a non-cooperative differential game where only local information is available. The existence of Nash equilibrium is proved and collision-free formations of regular polygons, antipodal formations and Platonic solids are achieved by only intrinsically adjusting the inter-agent topology. Future work concerns extending the current results to infinite time-horizon games.

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