Time-varying formation feasibility analysis for linear multi-agent systems with time delays and switching graphs

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Abstract: Time-varying formation control, formation tracking control, and formation-containment control for multi-agent systems have attracted significant attention recently, where the formation feasibility is a crucial common problem. For a given linear multi-agent system, not all the time-varying formations can be realized due to the dynamic restriction of each agent. The formation feasibility constraint reveals the requirement on the desired time-varying formation to be compatible with the agent dynamics. Formation reference is a representation for the macroscopic movement of the whole multi-agent system. Novel features of the formation feasibility constraint and the formation reference are the main focus of the current paper. Firstly, a time-delayed formation control protocol with switching directed topologies is constructed using local neighboring information. Then time-varying formation feasibility constraint is derived based on nonsingular transformations. It is proven that the time-varying formation feasibility constraint is independent of the time-varying delays and the switching directed topologies. Moreover, an explicit expression of the formation reference function is proposed. It is shown that neither the time-varying delays nor the switching directed topologies have influence on the obtained formation reference function.

Key Words: Time-varying formation control; formation feasibility; linear multi-agent system; time-varying delay; switching directed topology.

1 Introduction

As a fundamental branch of cooperative control for multi-agent systems, formation control has attracted significant interest in recent two decades [1]. The control strategies in the vast majority of the published studies can be roughly categorized into leader follower based, virtual structure based, behavior based, and consensus based ones. The first three strategies have been studied a lot in the robotics community [2]. It has been shown by Ren [3] that consensus based formation control strategy is more general, robust and scalable than the first three ones. One typical feature of the consensus based formation control is that only local neighbor-to-neighbor information exchange is required.

In many practical applications, such as target enclosing and obstacle avoidance, the desired formation for the multi-agent system may be time-varying. Note that time-invariant formation is just a special case of the time-varying one. Therefore, time-varying formation for linear multi-agent systems is more practical and general [4, 5]. Dong and Hu [6] studied the time-varying formation analysis and design problems for general multi-agent systems with switching directed topologies. Time delays were considered in [7], where necessary and sufficient conditions for multi-agent systems to achieve a given time-varying formation were presented. In [8], only output information of each agent was available, and a time-varying output formation controller was designed by using dynamic output feedback control. Outdoor experiments for formation control of a group of unmanned aerial vehicles (UAVs) on fixed/switching graphs were performed in [9] and [10], respectively. Necessary and sufficient conditions for linear multi-agent systems with multiple leaders to realize time-varying formation tracking were presented in [11]. Considering a leader of unknown control input, distributed time-varying formation tracking protocols were presented for homogeneous/heterogeneous multi-agent systems in [12] and [13]. The proposed formation tracking approaches were applied to solve the target enclosing problems for UAV swarm systems in [14] and [15], where formation flying experiments were respectively carried out under the influences of switching undirected/directed graphs. In the case with more than one leader, a more complicated formation-containment control problem were considered in [16–18], where the leaders need to accomplish a desired formation and the followers are required to converge to the convex hull spanned by the leaders simultaneously.

For time-varying formation control, formation tracking control, and formation-containment control of multi-agent systems, the formation feasibility is a crucial common problem since it determines whether the predefined formation can be realized or not. In [6], time-varying formation feasibility constraints and explicit expressions of the formation reference function for linear multi-agent systems with switching directed topologies were proposed. Formation feasibility problems for linear multi-agent systems subject to constant time delays were studied in [7]. For practical networked multi-agent systems, the communication time delay and topology switching may exist simultaneously. The time delay may be time-varying and the topology can be directed. To the best of our knowledge, time-varying formation control problems for linear multi-agent systems with both time-varying delays and switching directed topologies are still open. Moreover, for a given multi-agent system, whether a predefined formation can be realized is of essential impor-

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tance before designing formation protocols, and how to describe the macroscopic movement of a multi-agent system is also an interesting topic.

Motivated by the facts stated above, this paper focuses on analyzing the effects of time-varying delays and switching directed topologies on the time-varying formation control of linear multi-agent systems. More specifically, the relationships among time delays, switching topologies, formation feasibility constraint and formation reference function are investigated. Compared with the previous results on formation control, the contributions of this paper are threefold. Firstly, the formation can be time-varying, and the time-varying delays and switching directed topologies are considered simultaneously. In [19], the formation is time-invariant. Although the topology in [6] can be switching, it is required that there exist no time delays. In [7], only constant time delays were considered. Secondly, the time-varying formation feasibility constraint is presented and is proven to be independent of the time-varying delays and the switching directed topologies. The formation feasibility constraints in [7, 10, 19] can be treated as special cases of the one in the current paper. Thirdly, an explicit expression of the formation reference function is proposed. It is verified that neither the time-varying delays nor the switching directed topologies have effects on the formation reference function.

Throughout this paper, for simplicity of notation, 0 is used to denote zero matrices of appropriate size with zero vectors and zero number as special cases. $\mathbf{1}$ is used to represent a column vector of appropriate size with 1 as its elements. I stands for an identity matrix with appropriate dimension. $\otimes$ denotes the Kronecker product.

2 Preliminaries and problem formulation

2.1 Preliminaries on graph theory

Let a triplet $G = \{S, E, W\}$ denote a weighted directed graph of order N with $S = \{s_1, s_2, \cdots, s_N\}$ the node set, $E \subseteq \{(s_i, s_j) : s_i, s_j \in S, i \neq j\}$ the edge set and $W = [w_{ij}] \in \mathbb{R}^{N\times N}$ the weighted adjacency matrix. An edge of G is denoted by $e_{ij} = (s_i, s_j)$. The adjacency elements $w_{ij} > 0$ if and only if $e_{ji} \in E$, and $w_{ij} = 0$ otherwise. The neighbor set of node s_i is denoted by $N_i = \{s_j \in S : (s_j, s_i) \in E\}$. The in-degree and out-degree of node s_i are defined, respectively, as $\deg_{in}(s_i) = \sum_{j=1}^{N} w_{ij}$ and $\deg_{out}(s_i) = \sum_{j=1}^{N} w_{ji}$. The node s_i is balanced if and only if its in-degree and out-degree are equal, i.e., $\deg_{in}(s_i) = \deg_{out}(s_i)$. Denote by $D = \operatorname{diag}\{\deg_{in}(s_i), i = 1, 2, \cdots, N\}$ the degree matrix associated with G. The Laplacian matrix of G is defined as $L = D - W$. A directed path from node s_{i_1} to s_{i_l} is a sequence of ordered edges with the form of $(s_{i_k}, s_{i_{k+1}})$ with $s_{i_k} \in S$ $(k = 1, 2, \cdots, l-1)$. A directed graph G is called balanced if and only if all of its nodes are balanced. A directed graph is said to have a spanning tree if there exists at least one node having a directed path to all the other nodes.

The directed graph in the current paper can be switching. Assume that there exists an infinite sequence of uniformly bounded non-overlapping time intervals $[t_k, t_{k+1})$ $(k \in \mathbb{N})$, where $t_0 = 0$, $0 < \tau_0 \leqslant t_{k+1} - t_k$, and $\mathbb{N}$ is the set of natural numbers. The time sequence t_k $(k \in \mathbb{N})$ is named as the switching sequence, at which the directed graph changes. τ_0 is called the dwell time, during which the graph keeps fixed. Let $\sigma(t) : [0, +\infty) \to \{1, 2, \cdots, p\}$ be a switching signal whose value at time t is the index of the graph with $p \in \mathbb{N}$ and $p \geqslant 1$. Let $G_{\sigma(t)}$, $L_{\sigma(t)}$ and $w_{ij}^{\sigma(t)}$ denote the directed graph, the associated Laplacian matrix and the adjacency element at $\sigma(t)$, respectively. Let $N^i_{\sigma(t)}$ represent the neighbor set of the ith agent at $\sigma(t)$. For any given $\sigma(t) \in \{1, 2, \cdots, p\}$, the Laplacian matrix $L_{\sigma(t)}$ has the following properties.

Lemma 1 ([20]) *If $G_{\sigma(t)}$ is balanced, then $L_{\sigma(t)}$ has at least one zero eigenvalue, and $\mathbf{1}$ is both the left and right eigenvectors of $L_{\sigma(t)}$ associated with the zero eigenvalue, i.e., $\mathbf{1}^T L_{\sigma(t)} = 0$ and $L_{\sigma(t)}\mathbf{1} = 0$.*

Lemma 2 ([21]) *If $G_{\sigma(t)}$ has a spanning tree, then $L_{\sigma(t)}$ has a simple zero eigenvalue with the associated right eigenvector $\mathbf{1}$, and all the other $N-1$ eigenvalues have positive real parts.*

2.2 Problem description

Consider a general linear multi-agent system described by

$$\dot{x}_i(t) = Ax_i(t) + Bu_i(t), \tag{1}$$

where $i = 1, 2, \cdots, N$, $x_i(t) \in \mathbb{R}^n$ are the states, $u_i(t) \in \mathbb{R}^m$ are the control inputs, $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$ with $\operatorname{rank}(B) = m$. The interaction topology among the N agents in multi-agent system (1) is described by the switching directed graph $G_{\sigma(t)}$ with node s_i representing the agent i and e_{ij} denoting the interaction channel from agent i to agent j. The agents are required to form the predefined time-varying formation specified by the vector $h(t) = [h_1^T(t), h_2^T(t), \cdots, h_N^T(t)]^T \in \mathbb{R}^{nN}$, where $h_i(t)$ and $\dot{h}_i(t)$ $(i = 1, 2, \cdots, N)$ are uniformly continuous.

Definition 1 ([5]) *Multi-agent system (1) is said to achieve the time-varying formation specified by $h(t)$ if for any given bounded initial states, there exists a vector-valued function $r(t) \in \mathbb{R}^n$ such that*

$$\lim_{t\to\infty} (x_i(t) - h_i(t) - r(t)) = 0 \ (i = 1, 2, \cdots, N),$$

where $r(t)$ is called the formation reference function.

Remark 1 *In Definition 1, $h_i(t)$ $(i = 1, 2, \cdots, N)$ are used to characterize the desired time-varying formation shape and $r(t)$ is a representation of the macroscopic movement of the whole formation. Moreover, $h_i(t)$ can be regarded as the relative offset vector of $x_i(t)$ with respect to $r(t)$ in the state space.*

Assumption 1 *Each possible interaction topology $G_{\sigma(t)}$ is balanced and has a spanning tree.*

Since $\operatorname{rank}(B) = m$, there exists a nonsingular matrix $T = [\tilde{B}^T, \bar{B}^T]^T$ with $\tilde{B} \in \mathbb{R}^{m\times n}$ and $\bar{B} \in \mathbb{R}^{(n-m)\times n}$ such that $\tilde{B}B = I_m$ and $\bar{B}B = 0$. Consider the following time-varying formation control protocol with both time-varying delays and switching topologies

$$\begin{aligned} u_i(t) = K \sum_{j\in N^i_{\sigma(t)}} w_{ij}^{\sigma(t)} [&(x_i(t-\tau(t)) - h_i(t-\tau(t))) \\ &- (x_j(t-\tau(t)) - h_j(t-\tau(t)))] \\ &- \tilde{B}(Ah_i(t) - \dot{h}_i(t)), \end{aligned} \tag{2}$$

where $i = 1, 2, \cdots, N$, $K \in \mathbb{R}^{m\times n}$ is a constant gain matrix, $\tau(t)$ is the time-varying delay satisfying that $0 \leq \tau(t) \leq \bar{\tau}$ and $|\dot{\tau}(t)| \leq \delta < 1$ with $\bar{\tau}$ and δ known constants.

Remark 2 *It should be pointed out that the proposed protocol (2) presents a general framework for consensus-based formation control approaches. Besides the neighboring relative information feedback term, a formation compensation signal dependent on $h_i(t)$ (i.e., $-\tilde{B}(Ah_i(t) - \dot{h}_i(t))$) is also given to expand the feasible time-varying formation set. Many existing protocols, such as protocols presented in [3, 16, 18, 19] can be viewed as special cases of protocol (2). Choosing $h_i(t) \equiv 0$ $(i = 1, 2, \cdots, N)$, formation control protocol (2) becomes the well-known consensus protocol with both time-varying delays and switching topologies.*

Definition 2 *A formation specified by $h(t)$ is said to be feasible for multi-agent system (1) under control protocol (2), if there exists a gain matrix K such that the desired formation can be realized. The constraint on $h(t)$, which should be satisfied by the feasible formation, is called the formation feasibility constraint.*

Let $x(t) = [x_1^T(t), x_2^T(t), \cdots, x_N^T(t)]^T$. Under protocol (2), the closed-loop dynamics of multi-agent system (1) can be written in a compact form as follows

$$\begin{cases} \dot{x}(t) = (I_N \otimes A)\, x(t) + \left(L_{\sigma(t)} \otimes BK\right) x(t-\tau(t)) \\ \qquad - \left(L_{\sigma(t)} \otimes BK\right) h(t-\tau(t)) \\ \qquad - (I_N \otimes B\tilde{B}A) h(t) \\ \qquad + (I_N \otimes B\tilde{B})\dot{h}(t),\ t > 0, \\ x(t) = \varphi(t),\ t \in [-\tau(t), 0], \end{cases} \tag{3}$$

where $\varphi(t)$ is a continuous vector-valued function on $t \in [-\tau(t), 0]$.

The current paper mainly focuses on studying the explicit expressions and features of the formation feasibility constraint and the formation reference function for time-delayed multi-agent system (3) under switching topologies.

3 Main results

In this section, explicit expressions of the formation feasibility constraint and the formation reference function are derived. It is proven that both the formation feasibility constraint and the formation reference function are independent of the switching topologies and the time-varying delays.

Theorem 1 *If multi-agent system (1) under the time-delayed formation protocol (2) with switching directed topologies achieves time-varying formation specified by $h(t)$, then the formation feasibility constraint is independent of the time-varying delays or the switching topologies, and satisfies*

$$\lim_{t\to\infty} \bar{B}\left(A(h_i(t) - h_j(t)) - (\dot{h}_i(t) - \dot{h}_j(t))\right) = 0, \tag{4}$$

where $i, j \in \{1, 2, \cdots, N\}$.

Proof: Let $\theta_i(t) = x_i(t) - h_i(t)$ and $\theta(t) = [\theta_1^T(t), \theta_2^T(t), \cdots, \theta_N^T(t)]^T$. Then the time-delayed multi-agent system (3) with time-varying topologies can be rewritten as

$$\begin{aligned} \dot{\theta}(t) = (I_N \otimes A)\theta(t) + (L_{\sigma(t)} \otimes BK)\theta(t-\tau(t)) \\ + (I_N \otimes (A - B\tilde{B}A))h(t) \\ + (I_N \otimes (B\tilde{B} - I_n))\dot{h}(t). \end{aligned} \tag{5}$$

Let $U = [\mathbf{1}_N/\sqrt{N}, \bar{U}]$ be an orthogonal constant matrix. If Assumption 1 holds, it follows from Lemma 1 that $(\mathbf{1}_N^T/\sqrt{N})L_{\sigma(t)} = 0$, $L_{\sigma(t)}(\mathbf{1}_N/\sqrt{N}) = 0$ and

$$U^T L_{\sigma(t)} U = \begin{bmatrix} 0 & 0 \\ 0 & \bar{U}^T L_{\sigma(t)} \bar{U} \end{bmatrix}. \tag{6}$$

Let $\xi(t) = (U^T \otimes I_n)\theta(t) = [\xi_1^T, \xi_2^T, \cdots, \xi_N^T]^T$ and $\varsigma(t) = [\xi_2^T, \xi_3^T, \cdots, \xi_N^T]^T$. From (6), multi-agent system (5) can be transformed into

$$\begin{aligned} \dot{\xi}_1(t) = A\xi_1(t) + \left(\tfrac{1}{\sqrt{N}}\mathbf{1}_N^T \otimes (A - B\tilde{B}A)\right) h(t) \\ + \left(\tfrac{1}{\sqrt{N}}\mathbf{1}_N^T \otimes (B\tilde{B} - I_n)\right) \dot{h}(t), \end{aligned} \tag{7}$$

and

$$\begin{aligned} \dot{\varsigma}(t) = (I_{N-1} \otimes A)\,\varsigma(t) + \left((\bar{U}^T L_{\sigma(t)} \bar{U}) \otimes BK\right) \varsigma(t-\tau(t)) \\ + (\bar{U}^T \otimes (A - B\tilde{B}A))h(t) + (\bar{U}^T \otimes (B\tilde{B} - I_n))\dot{h}(t). \end{aligned} \tag{8}$$

Let

$$\theta_F(t) = (U \otimes I_n)[\xi_1^T(t), 0]^T. \tag{9}$$

Note that $[\xi_1^T(t), 0]^T = e_1 \otimes \xi_1(t)$ with e_1 an N-dimensional column vector having 1 as its first component and 0 elsewhere. It follows from (9) that

$$\theta_F(t) = (U \otimes I_n)\left(e_1 \otimes \xi_1(t)\right) = \frac{1}{\sqrt{N}}\mathbf{1}_N \otimes \xi_1(t). \tag{10}$$

Recalling that $\xi(t) = [\xi_1^T(t), \varsigma^T(t)]^T$ and $\theta(t) = (U \otimes I_n)\xi(t)$, one gets

$$\theta(t) - \theta_F(t) = (U \otimes I_n)[0, \varsigma^T(t)]^T. \tag{11}$$

Because $U \otimes I_n$ is nonsingular, from (11), one has that $\lim_{t\to\infty}(\theta(t) - \theta_F(t)) = 0$ if and only if

$$\lim_{t\to\infty} \varsigma(t) = 0, \tag{12}$$

that is, for any $i \in \{1, 2, \cdots, N\}$,

$$\lim_{t\to\infty}\left(x_i(t) - h_i(t) - \frac{1}{\sqrt{N}}\xi_1(t)\right) = 0 \tag{13}$$

if and only if $\lim_{t\to\infty}\varsigma(t) = 0$. Therefore, $\varsigma(t)$ represents the time-varying formation error.

Since the time-delayed multi-agent system (3) with time-varying directed interaction topologies achieves the time-varying formation specified by $h(t)$, one gets that $\lim_{t\to\infty}\varsigma(t) = 0$. Let $F_{\sigma(t)}\left(\varsigma(t), \varsigma(t-\tau(t))\right) = (I_{N-1} \otimes A)\,\varsigma(t) + \left((\bar{U}^T L_{\sigma(t)} \bar{U}) \otimes BK\right) \varsigma(t-\tau(t))$. Since $\lim_{t\to\infty}\varsigma(t) = 0$, it follows that $\lim_{t\to\infty} F_{\sigma(t)}\left(\varsigma(t), \varsigma(t-\tau(t))\right) = 0$. Let $H(t) = (\bar{U}^T \otimes (A - B\tilde{B}A))h(t) + (\bar{U}^T \otimes (B\tilde{B} - I_n))\dot{h}(t)$. Note that $h(t)$ and $\dot{h}(t)$ are uniformly continuous, which implies that $H(t)$ is uniformly continuous. Thus, (8) can be rewritten as

$$\dot{\varsigma}(t) = F_{\sigma(t)}\left(\varsigma(t), \varsigma(t-\tau(t))\right) + H(t). \tag{14}$$

In what follows, one will prove that $\lim_{t\to\infty} H(t) = 0$ by contradiction. Assume that $\lim_{t\to\infty} H(t) = c \neq 0$ with c denoting a nonzero constant or $H(t)$ does not have a limit.

i) If $\lim_{t\to\infty} H(t) = c \neq 0$, then it holds from (14) that $\lim_{t\to\infty} \dot{\varsigma}(t) = c \neq 0$, which contradicts $\lim_{t\to\infty} \varsigma(t) = 0$.

ii) If $H(t)$ does not have a limit, then there exist a positive constant ε and an infinite time sub-sequence $\{t_i, i \in \mathbb{N}\}$ with $\lim_{i\to\infty} t_i = \infty$ such that $\|H(t_i)\| > \varepsilon > 0$. Since $H(t)$ is uniformly continuous, there is a positive constant δ such that the following inequality holds for $t \in [t_i - \delta, t_i + \delta]$

$$\|H(t)\| \geqslant \frac{1}{2}\|H(t_i)\| > \frac{1}{2}\varepsilon. \tag{15}$$

Note that $\lim_{t\to\infty} F_{\sigma(t)}(\varsigma(t), \varsigma(t-\tau(t))) = 0$. Then, there exists a time instant $t_\varepsilon > \bar{\tau} > 0$ such that when $t > t_\varepsilon$ it holds that

$$\left\|F_{\sigma(t)}(\varsigma(t), \varsigma(t-\tau(t)))\right\| < \frac{1}{4}\varepsilon. \tag{16}$$

Thus, when $t_i > t_\varepsilon + 2\delta$, for any $t \in [t_i - \delta, t_i + \delta]$, one can obtain from (15) and (16) that

$$\|\dot{\varsigma}(t)\| = \left\|F_{\sigma(t)}(\varsigma(t), \varsigma(t-\tau(t))) + H(t)\right\| > \frac{1}{4}\varepsilon. \tag{17}$$

For $\dot{\varsigma}(t_i) > 0$, calculating the integral of (17) in $[t_i - \delta, t_i + \delta]$ gives

$$\int_0^{t_i+\delta} \dot{\varsigma}(s)\mathrm{d}s \geqslant \int_0^{t_i-\delta} \dot{\varsigma}(s)\mathrm{d}s + \frac{1}{2}\varepsilon\delta. \tag{18}$$

Since $\{t_i\}$ is an infinite sequence, by taking the limit of $t_i \to \infty$ on both sides of (18), one can obtain that $\lim_{t\to\infty} \varsigma(t) = \lim_{t\to\infty} \int_0^t \dot{\varsigma}(s)\mathrm{d}s$ cannot have a finite limit, which is contradictory with $\lim_{t\to\infty} \varsigma(t) = 0$. For $\dot{\varsigma}(t_i) < 0$, a contradiction can be obtained in the same way.

Therefore, if $\lim_{t\to\infty} \varsigma(t) = 0$, one can prove that $\lim_{t\to\infty} H(t) = 0$, that is

$$\begin{aligned}\lim_{t\to\infty} \Big(&\left(\bar{U}^T \otimes (A - B\tilde{B}A)\right) h(t) \\ &+ \left(\bar{U}^T \otimes (B\tilde{B} - I_n)\right) \dot{h}(t)\Big) = 0,\end{aligned} \tag{19}$$

which means that the formation feasibility constraint on $h(t)$ is determined by (19). In what follows, it will be proven that condition (4) is equivalent to condition (19).

On the one hand, if condition (4) holds, then for any $i, j \in \{1, 2, \cdots, N\}$, it can be verified that

$$\begin{aligned}\lim_{t\to\infty} \Big(&\bar{B}\left(A - B\tilde{B}A\right)(h_i(t) - h_j(t)) \\ &+ \bar{B}\left(B\tilde{B} - I_n\right)\left(\dot{h}_i(t) - \dot{h}_j(t)\right)\Big) = 0,\end{aligned} \tag{20}$$

and

$$\begin{aligned}\lim_{t\to\infty} \Big(&\tilde{B}\left(A - B\tilde{B}A\right)(h_i(t) - h_j(t)) \\ &+ \tilde{B}\left(B\tilde{B} - I_n\right)\left(\dot{h}_i(t) - \dot{h}_j(t)\right)\Big) = 0.\end{aligned} \tag{21}$$

It follows from (20) and (21) that

$$\begin{aligned}\lim_{t\to\infty} \Big(&T\left(A - B\tilde{B}A\right)(h_i(t) - h_j(t)) \\ &+ T\left(B\tilde{B} - I_n\right)\left(\dot{h}_i(t) - \dot{h}_j(t)\right)\Big) = 0.\end{aligned} \tag{22}$$

For any $i, j \in \{1, 2, \cdots, N\}$, pre-multiplying the both sides of (22) by T^{-1} gives

$$\begin{aligned}\lim_{t\to\infty} \Big(&\left(A - B\tilde{B}A\right)(h_i(t) - h_j(t)) \\ &+ \left(B\tilde{B} - I_n\right)\left(\dot{h}_i(t) - \dot{h}_j(t)\right)\Big) = 0.\end{aligned} \tag{23}$$

Note that $\left(L_{\sigma(t)} \otimes I_n\right) h(t) = [f_1^T(t), f_2^T(t), \cdots, f_N^T(t)]^T$, where $f_i(t) = \sum_{j=1}^{N} w_{ij}^{\sigma(t)}(h_i(t) - h_j(t))$ $(i = 1, 2, \cdots, N)$. From (23), one has

$$\begin{aligned}\lim_{t\to\infty} \Big(&(L_{\sigma(t)} \otimes (A - B\tilde{B}A))h(t) \\ &+ (L_{\sigma(t)} \otimes (B\tilde{B} - I_n))\dot{h}(t)\Big) = 0.\end{aligned} \tag{24}$$

Substituting

$$L_{\sigma(t)}(t) = U \begin{bmatrix} 0 & 0 \\ 0 & \bar{U}^T L_{\sigma(t)} \bar{U} \end{bmatrix} U^T$$

into (24) and pre-multiplying the both sides of (24) by $U^T \otimes I_n$ yields

$$\begin{aligned}\lim_{t\to\infty} \Big(&\left(\bar{U}^T L_{\sigma(t)} \bar{U} \bar{U}^T \otimes (A - B\tilde{B}A)\right) h(t) \\ &+ \left(\bar{U}^T L_{\sigma(t)} \bar{U} \bar{U}^T \otimes (B\tilde{B} - I_n)\right) \dot{h}(t)\Big) = 0.\end{aligned} \tag{25}$$

It follows from Lemma 2 that all the eigenvalues of $\bar{U}^T L_{\sigma(t)} \bar{U}$ have positive real parts, which means that $\bar{U}^T L_{\sigma(t)} \bar{U}$ is invertible. Pre-multiply the both sides of (25) by $(\bar{U}^T L_{\sigma(t)} \bar{U})^{-1} \otimes I_n$ gives

$$\lim_{t\to\infty} \Big((\bar{U}^T \otimes (A - B\tilde{B}A))h(t) + (\bar{U}^T \otimes (B\tilde{B} - I_n))\dot{h}(t)\Big) = 0,$$

that is, condition (19) holds.

On the other hand, since $\mathrm{rank}(\bar{U}^T) = N - 1$, without loss of generality, let $\bar{U}^T = [\hat{u}, \hat{U}]$ with $\hat{u} \in \mathbb{R}^{(N-1)\times 1}$ and $\hat{U} \in \mathbb{R}^{(N-1)\times(N-1)}$ being of full rank. If condition (19) holds, one gets

$$\begin{aligned}\lim_{t\to\infty} \Big(&([\hat{u}, \hat{U}] \otimes (A - B\tilde{B}A))h(t) \\ &+ ([\hat{u}, \hat{U}] \otimes (B\tilde{B} - I_n))\dot{h}(t)\Big) = 0.\end{aligned} \tag{26}$$

Note that $\bar{U}^T \mathbf{1} \big/ \sqrt{N} = 0$. One has $\hat{u} = -\hat{U}\mathbf{1}$. Let $\hat{h}(t) = [h_2^T(t), h_3^T(t), \cdots, h_N^T(t)]^T$. Then it follows from (26) that

$$\lim_{t\to\infty} \left(\hat{U} \otimes I_n\right)\left(\hat{\Psi} - \Psi\right) = 0, \tag{27}$$

where

$$\hat{\Psi} = (I_{N-1} \otimes (A - B\tilde{B}A))\hat{h}(t) + (I_{N-1} \otimes (B\tilde{B} - I_n))\dot{\hat{h}}(t),$$

$$\Psi = (\mathbf{1} \otimes (A - B\tilde{B}A))h_1(t) + (\mathbf{1} \otimes (B\tilde{B} - I_n))\dot{h}_1(t).$$

Because $\hat{U}$ is invertible, one can pre-multiply the both sides of (27) by $\hat{U}^{-1} \otimes I_n$ and obtain that for any $i \in \{2, 3, \cdots, N\}$

$$\begin{aligned}\lim_{t\to\infty} \Big(&(A - B\tilde{B}A)(h_i(t) - h_1(t)) \\ &+ (B\tilde{B} - I_n)(\dot{h}_i(t) - \dot{h}_1(t))\Big) = 0.\end{aligned} \tag{28}$$

From (28), it can be obtained that for any $i, j \in \{1, 2, \cdots, N\}$

$$\lim_{t\to\infty}\Big((A-B\tilde{B}A)(h_i(t)-h_j(t)) \\ +(B\tilde{B}-I_n)(\dot{h}_i(t)-\dot{h}_j(t))\Big)=0. \tag{29}$$

Pre-multiplying the both sides of (29) by T, one gets $\forall i, j \in \{1, 2, \cdots, N\}$

$$\lim_{t\to\infty}\left(\bar{B}A\left(h_i(t)-h_j(t)\right)-\bar{B}\left(\dot{h}_i(t)-\dot{h}_j(t)\right)\right)=0, \tag{30}$$

which means that condition (19) is sufficient for condition (4). Therefore, condition (4) is equivalent to condition (19) which describes the formation feasibility constraint for multi-agent system (1) under time-delayed formation protocol (2) with switching directed topologies. Moreover, from (4), one sees that the formation feasibility constraint is independent of the time-varying delays or the switching topologies. This completes the proof of Theorem 1. □

Remark 3 *From Theorem 1, one sees that formation feasibility constraint (4) is a necessary condition for multi-agent system (1) under protocol (2) to achieve the desired formation specified by $h(t)$. The physical meaning behind the formation feasibility constraint is that the predefined time-varying formation specified by $h(t)$ should be compatible with the dynamics of the multi-agent systems. By utilizing the properties of the Laplacian matrix and applying nonsingular transformations, the time-varying formation feasibility constraint (4) is derived. The formation feasibility analysis is a novel problem after a time-varying vector $h(t)$ is applied to specify the desired formation shape, which has not been considered in the consensus control.*

In the following, an explicit expression of the formation reference function is derived to describe the macroscopic movement of the time-varying formation under the influence of the switching topologies and time-varying delays.

Theorem 2 *If multi-agent system (1) under the time-delayed formation protocol (2) with switching directed topologies achieves time-varying formation specified by $h(t)$, then the formation reference function $r(t)$ is independent of the switching topologies and the time-varying delays, and satisfies*

$$\lim_{t\to\infty}\left(r(t)-r_0(t)-r_h(t)\right)=0, \tag{31}$$

where

$$r_0(t)=e^{At}\left(\frac{1}{N}\sum_{i=1}^{N}x_i(0)\right),$$

$$r_h(t)=\tfrac{1}{N}\sum_{i=1}^{N}(B\tilde{B}-I_n)h_i(t)-e^{At}\left(\tfrac{1}{N}\sum_{i=1}^{N}B\tilde{B}h_i(0)\right) \\ +\int_0^t e^{A(t-s)}\left(\tfrac{1}{N}\sum_{i=1}^{N}(AB\tilde{B}-B\tilde{B}A)h_i(s)\right)ds.$$

Proof: If multi-agent system (1) under the time-delayed formation protocol (2) with switching directed topologies achieves time-varying formation specified by $h(t)$, then (13) holds. From Definition 1 and (13), one gets

$$\lim_{t\to\infty}\left(r(t)-\frac{1}{\sqrt{N}}\xi_1(t)\right)=0. \tag{32}$$

Therefore, the formation reference function is determined by the state of subsystem (7). Note that $\xi(t)=(U^T\otimes I_n)\theta(t)$, $\theta(t)=x(t)-h(t)$ and $U=[\mathbf{1}_N/\sqrt{N},\bar{U}]$. It can be obtained that

$$\xi_1(0)=\frac{1}{\sqrt{N}}\left(\mathbf{1}_N^T\otimes I_n\right)\left(x(0)-h(0)\right). \tag{33}$$

It can be verified that

$$\begin{aligned}&\tfrac{1}{\sqrt{N}}\int_0^t e^{A(t-s)}(\mathbf{1}_N^T\otimes(B\tilde{B}-I_n))\dot{h}(s)ds\\ &=\tfrac{1}{\sqrt{N}}\,e^{A(t-s)}(\mathbf{1}_N^T\otimes(B\tilde{B}-I_n))h(s)\Big|_{s=0}^{s=t}\\ &\quad-\tfrac{1}{\sqrt{N}}\int_0^t \tfrac{d}{ds}e^{A(t-s)}(\mathbf{1}_N^T\otimes(B\tilde{B}-I_n))h(s)ds\\ &=\tfrac{1}{\sqrt{N}}(\mathbf{1}_N^T\otimes(B\tilde{B}-I_n))h(t)\\ &\quad-\tfrac{1}{\sqrt{N}}e^{At}(\mathbf{1}_N^T\otimes(B\tilde{B}-I_n))h(0)\\ &\quad-\tfrac{1}{\sqrt{N}}\int_0^t e^{A(t-s)}(-A)(\mathbf{1}_N^T\otimes(B\tilde{B}-I_n))h(s)ds,\end{aligned} \tag{34}$$

and

$$\begin{aligned}&\tfrac{1}{\sqrt{N}}A(\mathbf{1}_N^T\otimes(B\tilde{B}-I_n))h(s)\\ &=\tfrac{1}{\sqrt{N}}\left(\mathbf{1}_N^T\otimes(AB\tilde{B}-A)\right)h(s).\end{aligned} \tag{35}$$

From (7) and (32)-(35), one gets that the formation reference function $r(t)$ satisfies (31). Moreover, from (31), one sees that $r(t)$ is independent of the switching topologies and the time-varying delays. Therefore, the conclusion of Theorem 2 can be obtained. □

Remark 4 *In the case where $h(t)\equiv 0$, the time-varying formation control problem becomes a consensus control problem, and the formation reference function $r(t)$ becomes the consensus function $c(t)$ which determines the consensus value; that is,*

$$\lim_{t\to\infty}\left(c(t)-e^{At}\left(\frac{1}{N}\sum_{i=1}^{N}x_i(0)\right)\right)=0. \tag{36}$$

From (36), one can conclude that both the time-varying delays and switching topologies have no effect on the consensus function. More specifically, if $h(t)\equiv 0$ and $A=0$, the consensus function (36) becomes

$$\lim_{t\to\infty}\left(c(t)-\left(\frac{1}{N}\sum_{i=1}^{N}x_i(0)\right)\right)=0, \tag{37}$$

which is just the average consensus function for first-order multi-agent systems with constant/time-varying delays or switching topologies studied in [20, 25, 26].

4 Conclusions

Time-varying formation feasibility and reference function of linear multi-agent systems with time-varying delays and switching directed topologies were studied. A time-varying formation control protocol was constructed using local neighbor-to-neighbor information. The time-varying formation feasibility constraint was derived to reveal the compatible relationship between feasible formation and the dynamics of each agent. An explicit expression of the formation reference function was proposed to describe the macroscopic movement of the whole time-varying formation. It is proven that both the formation feasibility constraint and the formation reference function are independent of the time-varying delays and the switching directed topologies.

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