Exact Computation of Delay Margin by PID Control: It Suffices to Solve a Unimodal Problem!

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Abstract: In this paper we study delay robustness of PID controllers in stabilizing systems containing uncertain, variable delays. We consider second-order unstable systems and seek analytical characterization and exact computation of the PID delay margin, where by PID delay margin we mean the maximal range of delay values within which the system can be robustly stabilized by a PID controller. Our contribution is threefold. First, we show that the delay margin achieved by PID control coincides with that by PD controllers. Second, we show that other than helping stabilize the delay-free part of a plant, the proportional control contributes no action to increase the delay margin. Finally, we show that the PID delay margin can be computed efficiently by solving a unimodal problem, that is, a univariate optimization problem that admits a unique maximum and hence is a convex optimization problem in one variable. This unimodal problem is one of pseudo-concave optimization and hence can be solved using standard convex optimization or gradient-based methods. As such, from a computational perspective, *the PID delay margin problem is completely resolved in this paper!* The results not only insure that the PID delay margin problem be readily solvable, but also provide fundamental conceptual insights into the PID control of delay systems, and analytical justifications to long-held engineering intuitions and heuristics, thus lending useful guidelines in the tuning and analytical design of PID controllers.

Key Words: Delay margin, robust stabilization, PID controller, uncertain time delay, nonlinear programming, pseudo-concavity.

1 Introduction

A central problem in the stabilization of time-delay systems is that of *delay margin*, which defines the largest range of delay values so that a delay system can be robustly stabilized, despite that the delay is unknown and may vary within that range. Determination of the delay margin is of fundamental interest and can aid control design in several ways. First and foremost, it provides a fundamental limit of robustness against uncertainties in the time delay, establishing a definitive boundary beyond which the delay system cannot be robustly stabilized via feedback. Secondly, it furnishes a performance benchmark in feedback design, that can help guide the tuning and redesign of the controller parameters. As such, knowledge on the delay margin is much desired and at times may be necessary, especially for circumstances where the system delay is uncertain or difficult to estimate. The traditional industrial processes and the modern interconnected systems such as distributed networks and cyber-physical systems belong to this category, in which long, variable delays in mass and information exchanges are commonplace.

It should be pointed out at the outset, nonetheless, that for a stable plant the delay margin problem is itself a moot issue; one recognizes readily that in this case the delay margin is infinite. It is also known that when employing certain sophisticated control laws, the delay margin can be made as large as desirable. To this effect, various nonlinear, time-varying controllers have been constructed, such as linear periodic controllers [19], nonlinear periodic controllers [8], and nonlinear adaptive controllers [3], which can be used to stabilize an unstable linear time-invariant (LTI) plant with arbitrarily long delays. On the other hand, when confined to LTI controllers, the problem remains open except for simple, isolated cases. There has been considerable, long-held research interest devoted to this problem, seeking to stabilize robustly LTI delay plants using LTI controllers. By far it is generally known that LTI controllers can only achieve a finite delay margin for unstable plants, and various bounds have been derived to estimate the margin (see, e.g., [4, 6, 9, 16, 18, 21, 22, 26, 27, 31]. Specifically, upper bounds on the delay margin are obtained in [9, 18, 21], while lower bounds are made available in [13, 16, 22, 25–27, 31], based on methods ranging from frequency-domain analysis to predictor feedback design, with controller complexity ranging from that of PID control to infinite-dimensional controllers.

Despite the considerable advances in the study of time-delay systems, the delay margin problem remains to be a formidable challenge in general, due to the fact that it in essence requires solving an infinite-dimensional optimal control synthesis problem which in turn poses an infinite-dimensional best approximation problem of analytic functions. Indeed, as of now the exact determination of the delay margin is possible only for systems containing one single unstable pole [18, 21], while the aforementioned bounds generally suffer from a varying degree of conservatism. It is worth noting that conventional feedback design methods such as LQR and $\mathcal{H}_\infty$ optimal control methods (see, e.g., [20, 29] and the references therein), predictor feedback design [10, 30], and LMI-based solutions [7, 17] are generally restricted to synthesis problems with a *fixed* delay and thus are not readily amenable to addressing delay robustness problems, lest that of the delay margin.

Inspired by the recent developments in [14, 24], in this paper we study the delay margin achievable by PID controllers.

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Our work is motivated both by the fundamental quest for the exact delay margin, and by the broad practical applications of PID control. In the latter regard, it is reassuring that among a multitude of advanced control design methods, PID control remains to stand out as the most favored method for its simplicity, robustness, ease of implementation and cost-effectiveness; indeed, a recent survey serves an awe-inspiring testimony of its widespread acceptance by the industrial control community [23]. Unsurprisingly, PID controllers have been widely adopted in industrial processes, which are typically subject to time delays. Earlier work on the delay margin achievable by PID control [16, 24] shows that for first-order unstable plants, PID controllers and more general LTI controllers result in the same delay margin. Inadvertently, this reveals that for first-order delay plants, PID controllers are in fact among the optimal LTI controllers in maximizing the delay margin. It was further shown in [14] that this result holds more generally for systems with one unstable pole and one nonminimum phase zero, and that a PD controller suffices to achieve the maximum.

In the present paper we focus on second-order unstable systems. Unlike in the previous work [14, 24], the analytical characterization of the PID delay margin now appears to be highly nontrivial a task and requires a significant leap of faith, both conceptually and analytically. Indeed, as we noted above, no analytical characterization nor exact computation of the delay margin has been found for second-order systems. Apart from the aforementioned bounds, the computation of the delay margin generally requires a brute-force search in three parameters, that is, the three PID controller coefficients constrained on a highly intricate manifold. Analytical results of this kind will thus help advance the PID control of delay systems by a major step. In this vein, it is useful to note that in industrial applications, it is typical and often adequate to model process dynamics by first- and second-order systems. In spite of its successes, however, with only three control parameters, PID control is also known to be essentially limited to first- and second-order systems [11, 28]. One is readily convinced (see, e.g., [14]) that in general PID controllers cannot stabilize a third- and higher-order system free of delay, lest that the system may contain delays.

In a distinctively different approach, we reformulate the delay margin problem as a *constrained nonlinear programming* problem over a parameter space of the PID controller coefficients. This problem is then tackled by using the *Fritz John conditions* [2]. It is interesting to note that while nonlinear programming is generally used as a numerical tool, in the present work we employ it to arrive at an analytical solution, which leads to a number of important findings, with fundamental conceptual insights and significant computational implications. First, we show that the delay margin achieved by PID control coincides with that by PD controllers; in other words, the integral control coefficient in the PID controller can be made vanishingly small. A highly nontrivial yet rather pleasant surprise from both a conceptual and technical standpoint, this result is consistent with and indeed provides an analytical proof to the long-held intuition that the integral control does no more than to achieve asymptotic tracking but plays no role in feedback stabilization.

Secondly, we show that other than helping stabilize the delay-free part of the plant, the proportional control contributes no action to increase the delay margin; mathematically, this means that in maximizing the delay margin, the proportional control coefficient will lie on the boundary of the PID design parameter space defined by the task of stabilizing the delay-free plant. Finally, having determined the integral and proportional coefficients as alluded to above, the delay margin problem is effectively reduced to a univariate optimization problem, defined by the derivative control coefficient alone, thus pointing to the observation that the derivative control action constitutes the sole force in countering the destabilizing effect of the time delay. This too unveils a deep implication: with its phase lead, the derivative control attempts to balance the phase lag resulted from time delay. In yet another surprising discovery, we show that this univariate optimization problem is in fact one on a *pseudo-concave function*. Stated alternatively, the function admits a unique maximum which can be computed using standard convex optimization methods, or any gradient-based and bisection methods. In conclusion, the PID delay margin problem for second-order systems can be solved efficiently and in high precision as a *unimodal problem*. In other words, the problem is completely resolved from a numerical perspective.

The remainder of this paper is organized as follows. In Section 2, we introduce the definition of delay margin, along with a summary of its analytical solution for first-order systems. Section 3 gives a concise exposure to the mathematical tools required in the sequel, including the Fritz John conditions for nonlinear programming problems and the notion of pseudo-concavity. The main proper of the paper then begins with Section 4, where we present our results for second-order systems with real unstable poles. Section 5 addresses systems with complex conjugate poles. In each case, an analytical characterization of the delay margin is provided, together with explicit *a priori* bounds on the delay margin. Section 6 extends the analysis to nonminimum phase plants, followed by an illustrative example given in Section 7. The paper concludes in Section 8.

2 The Delay Margin Problem

We consider the feedback system depicted in Fig. 1, in which $P_\tau(s)$ denotes the plant subject to an uncertain delay τ, whose transfer function is given by

$$P_\tau(s) = P_0(s)e^{-\tau s}, \quad \tau \geq 0, \tag{1}$$

where $P_0(s)$ represents a finite dimensional delay-free plant. Assume that $P_0(s)$ can be stabilized by a certain finite-dimensional LTI controller $K(s)$. The delay margin [18, 22] of the system achievable by a LTI controller is defined as

$$\bar{\tau} = \sup\{\mu \geq 0 : \text{There exists some } K(s) \text{ stabilizing } P_\tau(s), \forall \tau \in [0, \mu)\}.$$

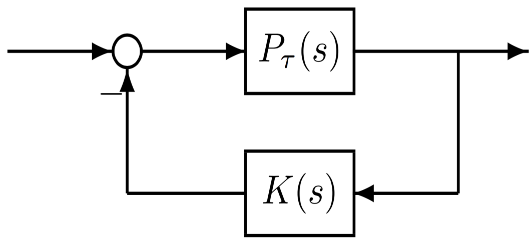

Fig. 1. Feedback control of a time-delay system.

Of particular interest in this paper is the delay margin achievable by PID controllers $K(s) = K_{PID}(s)$,

$$K_{PID}(s) = k_p + \frac{k_i}{s} + k_d s, \tag{2}$$

defined by

$$\bar{\tau}_{PID} = \sup\{\mu \ge 0 : \text{There exists some } K_{PID}(s) \text{ stabilizing } P_\tau(s),\ \forall \tau \in [0,\ \mu)\}.$$

Let

$$\tau(k_p,\ k_i,\ k_d) = \sup\{\mu \ge 0 : K_{PID}(s) \text{ stabilizes } P_\tau(s),\ \forall \tau \in [0,\ \mu)\}.$$

It follows that

$$\bar{\tau}_{PID} = \sup\{\tau(k_p,\ k_i,\ k_d) : K_{PID}(s) \text{ stabilizes } P_\tau(s),\ \forall \tau \in [0,\ \mu)\}.$$

For purpose of comparison, it is also of interest to consider the class of PD controllers

$$K_{PD}(s) = k_p + k_d s, \tag{3}$$

and accordingly, the delay margin achievable by PD control:

$$\bar{\tau}_{PD} = \sup\{\mu \ge 0 : \text{There exists some } K_{PD}(s) \text{ stabilizing } P_\tau(s),\ \forall \tau \in [0,\ \mu)\}.$$

Consider the open-loop transfer function

$$L_0(s) = P_0(s) K_{PID}(s). \tag{4}$$

From a practical design perspective, we impose the following assumption throughout the paper.

Assumption 1
(i) $|L_0(0)| > 1$,
(ii) $|L_0(\infty)| < 1$.

The assumptions (i) is necessary to insure the system's disturbance attenuation capability at low frequencies, while the assumption (ii) is required for noise reduction at high frequencies, both of which are standard requirements in feedback design.

Before proceeding, it is instructive to examine the delay margin of first-order unstable system

$$P_\tau(s) = \frac{1}{s-p} e^{-\tau s}, \quad p \ge 0. \tag{5}$$

Note that for the PID controller $K_{PID}(s)$ to stabilize this plant free of delay and to satisfy Assumption 1, it is necessary and sufficient that $k_p > p,\ k_i > 0$, and $|k_d| < 1$. It was shown in [18] that $\bar{\tau} = 2/p$, and further in [14, 24] that

$$\bar{\tau} = \bar{\tau}_{PID} = \bar{\tau}_{PD} = \frac{2}{p}. \tag{6}$$

This shows that for a first-order unstable plant, PID controllers are in fact among the optimal to achieve the largest possible delay margin and that the integral control has no effect on the delay margin. The optimal PID controller that asymptotically attains the delay margin is constructed in [14]:

$$k_p = p + \varepsilon^2,\ \ k_i = \varepsilon^3,\ \ k_d = 1 - \varepsilon, \tag{7}$$

for sufficiently small $\varepsilon > 0$. From this construction, it is clear that the integral control coefficient k_i is immaterial to the robust stabilization of $P_\tau(s)$. This is intuitively plausible, since the integral control is generally implemented for achieving such performance objectives as asymptotic tracking, with no regard to feedback stabilization. It is also of interest to see that the optimal PID coefficients all lie on their respective boundaries required to stabilize the delay-free plant.

3 Mathematical Background

3.1 Fritz John Condition

The Fritz John condition [2] concerns general constrained nonlinear programming problems whose objective function and constraints are differentiable, and the constraints may be equalities or inequalities. A general description of this class of problems can be stated as

$$\begin{aligned} &\min \quad f(x) \\ \text{s.t.} \quad & g_i(x) \le 0, \quad i \in \mathcal{I}_1 = \{1, 2, \cdots, m\}, \\ & h_j(x) = 0, \quad j \in \mathcal{I}_2 = \{m+1, m+2, \cdots, k\}, \end{aligned} \tag{8}$$

where $f : \mathbb{R}^n \to \mathbb{R}$, $g_i : \mathbb{R}^n \to \mathbb{R}$ and $h_j : \mathbb{R}^n \to \mathbb{R}$ all have continuous first-order partial derivatives on $\mathbb{R}^n$. The Fritz John condition provides a first-order necessary condition of optimality to the programming problem above, which is given in the following lemma.

Lemma 1 *The Fritz John Condition: If x^* is an optimal solution of $f(x)$ in* (8)*, then there exists a row vector $\lambda = [\lambda_0, \lambda_1, \lambda_2, \ldots, \lambda_k]$ such that:*

$$\lambda_0 \nabla f(x^*) + \sum_{i \in \mathcal{I}_1} \lambda_i \nabla g_i(x^*) + \sum_{j \in \mathcal{I}_2} \lambda_j \nabla h_j(x^*) = 0 \tag{9}$$

$$\sum_{i \in \mathcal{I}_1} \lambda_i g_i(x^*) = 0 \tag{10}$$

$$\lambda_i \ge 0, i \in \{0\} \cup \mathcal{I}_1 \tag{11}$$

$$\lambda \ne 0, \tag{12}$$

where $\nabla\theta(x) = [\partial\theta(x)/\partial x_1,\ \cdots,\ \partial\theta(x)/\partial x_n]^\top$ denotes the gradient of $\theta(x)$.

It is useful to note that the Fritz John condition is applicable to problems where the constraints may not satisfy certain regularity conditions, specifically when the gradient vectors $\nabla g_i(x^*)$ $(i = 1, \cdots, m)$ and $\nabla h_j(x^*)$ $(j = m+1, \cdots, k)$ are linearly dependent, to which the seemingly better-known Karush-Kuhn-Tucker condition [12] may fail to apply. For a comprehensive development of Fritz John condition, we refer to [2, 15].

3.2 Concavity, Quasi-Concavity and Pseudo-Concavity

We next give a brief exposure of concavity and its generalized notions.

Definition 1 *A function $\theta : \mathbb{R}^n \to \mathbb{R}$ is said to be concave over a convex set $\Gamma \in \mathbb{R}^n$ if for any $x,\ y \in \Gamma$,*

$$\theta((1-\alpha)x + \alpha y) \ge (1-\alpha)\theta(x) + \alpha\theta(y), \quad \alpha \in [0,\ 1], \tag{13}$$

and quasi-concave if

$$\theta((1-\alpha)x + \alpha y) \ge \min\{\theta(x),\ \theta(y)\}, \qquad \alpha \in [0,\ 1]. \tag{14}$$

Apparently, every concave function is quasi-concave. It is well-known that concave and quasi-concave functions admit a unique maximum, which can be found using convex optimization methods. In particular, for a univariate concave or quasi-concave optimization problem, known alternatively as a unimodal optimization problem, the optimal solution can be solved efficiently and in high precision by gradient-based and bisection methods.

For a first-order differentiable function, pseudo-concave functions possess properties essentially similar to those of concave functions, and for that reason, can be computed in a similar manner.

Definition 2 *A differentiable function $\theta : \mathbb{R}^n \to \mathbb{R}$ is said to be pseudo-concave over a convex set $\Gamma \in \mathbb{R}^n$ if for any $x,\, y \in \Gamma$,*

$$\nabla\theta(x)^\top (y-x) \le 0 \Rightarrow \theta(y) - \theta(x) \le 0. \tag{15}$$

The pseudo-concave function also guarantees the existence of a unique maximum.

4 Second-Order Systems with Real Unstable Poles

The delay margin problem for second-order unstable systems appears considerably more difficult. At present, no analytical characterization of the delay margin is available. The exact computation of the PID delay margin generally requires a brute-force search in the three PID coefficients $k_p,\ k_i,\ k_d$.

Consider the following plant with two real unstable poles:

$$P_\tau(s) = \frac{1}{(s-p_1)(s-p_2)} e^{-\tau s},\ p_1 > 0,\ p_2 > 0. \tag{16}$$

Our main result of this section is given below.

Theorem 1 *Let $P_\tau(s)$ be given by (16). Define the set*

$$\Omega = \{(k_p,\ k_i,\ k_d) : k_i > 0,\ k_d > p_1 + p_2, \\ (k_p - p_1p_2)(k_d - (p_1+p_2)) - k_i > 0\}. \tag{17}$$

Then the following statements hold:
(i)

$$\bar{\tau}_{PID} = \bar{\tau}_{PD}. \tag{18}$$

(ii)

$$\bar{\tau}_{PD} = \sup_{k_d > p_1+p_2} \hat{\tau}(k_d), \tag{19}$$

where

$$\hat{\tau}(k_d) = \tau(p_1p_2,\ 0,\ k_d), \tag{20}$$

$$\tau(k_p,\ k_i,\ k_d) = \min_l \tau_l(k_p,\ k_i,\ k_d), \tag{21}$$

with $\tau_l(k_p,\ k_i,\ k_d)$ defined on Ω as

$$\tau_l(k_p,\ k_i,\ k_d) = \frac{\tan^{-1}\frac{\omega_l}{p_1}}{\omega_l} + \frac{\tan^{-1}\frac{\omega_l}{p_2}}{\omega_l} + \frac{\tan^{-1}\frac{k_d\omega_l - \frac{k_i}{\omega_l}}{k_p}}{\omega_l} - \frac{\pi}{\omega_l}, \tag{22}$$

and $\omega_l > 0$ being the solutions of the polynomial equation

$$\omega^6 - (k_d^2 - (p_1^2+p_2^2))\omega^4 - (k_p^2 - p_1^2p_2^2 - 2k_dk_i)\omega^2 - k_i^2 = 0. \tag{23}$$

(iii) $\hat{\tau}(k_d)$ is pseudo-concave for $k_d > p_1 + p_2$.
(iv)

$$\bar{\tau}_{PD} \le \min\left\{ \left(\frac{\tan^{-1}\sqrt{\frac{2p_2}{p_1}}}{\sqrt{\frac{2p_2}{p_1}}}\right)\frac{1}{p_1},\ \left(\frac{\tan^{-1}\sqrt{\frac{2p_1}{p_2}}}{\sqrt{\frac{2p_1}{p_2}}}\right)\frac{1}{p_2} \right\}. \tag{24}$$

Proof. To streamline the proof, we choose to highlight each key step with a subtitle.

The set Ω of the triple (k_p, k_i, k_d). Consider first the delay-free system. The closed-loop characteristic polynomial is given by

$$s^3 + (k_d - (p_1+p_2))s^2 + (k_p + p_1p_2)s + k_i = 0.$$

It follows form the Routh-Hurwitz criterion that $P_0(s)$ can be stabilized by the PID controller if and only if $(k_p + p_1p_2)(k_d - (p_1+p_2)) - k_i > 0$, $k_d > p_1 + p_2$ and $k_i > 0$. On the other hand, to satisfy *Assumption 1*, it is necessary that $|k_p| > p_1p_2$. Thus, the set Ω defines the set of all feasible coefficients that enable the PID controller $K_{PID}(s)$ to stabilize $P_0(s)$.

The delay margin $\tau(k_p, k_i, k_d)$ with a fixed (k_p, k_i, k_d). Consider the open-loop frequency response

$$L_0(j\omega) = \frac{1}{(-p_1 + j\omega)(-p_2 + j\omega)}\left(k_p + \frac{k_i}{j\omega} + jk_d\omega\right).$$

We examine its magnitude

$$|L_0(j\omega)|^2 = \frac{(k_p^2 - 2k_dk_i) + k_d^2\omega^2 + \frac{k_i^2}{\omega^2}}{(\omega^2+p_1^2)(\omega^2+p_2^2)}$$

and solve all crossover frequencies ω_l such that $|L_0(j\omega_l)| = 1$. This gives rise to the crossover frequencies $\omega_l > 0$ as the solutions to the equation in (23). At these frequencies,

$$\measuredangle L_0(j\omega_l) = 2\pi + \tan^{-1}\frac{\omega_l}{p_1} + \tan^{-1}\frac{\omega_l}{p_2} + \tan^{-1}\frac{k_d\omega_l - \frac{k_i}{\omega_l}}{k_p}.$$

Since $L_0(j\omega_l) = e^{j\measuredangle L_0(j\omega_l)}$, we can match the phase of $L_0(j\omega_l)$ with that of the delay, by setting

$$\tau_l\omega_l = \tan^{-1}\frac{\omega_l}{p_1} + \tan^{-1}\frac{\omega_l}{p_2} + \tan^{-1}\frac{k_d\omega_l - \frac{k_i}{\omega_l}}{k_p} - \pi$$

for some $\tau_l \ge 0$. Evidently,

$$1 + P_0(j\omega_l)K_{PID}(j\omega_l)e^{-j\tau_l\omega_l} = 0.$$

In other words, the system becomes unstable at τ_l. On the other hand, for any $\tau < \min_l \tau_l$,

$$1 + P_0(j\omega)K_{PID}(j\omega)e^{-j\tau\omega} \ne 0, \quad \forall\omega \ge 0;$$

that is, the system is stable for all $\tau < \min_l \tau_l$. Write $\tau_l = \tau_l(k_p, k_i, k_d)$. It follows at once that $\tau(k_p, k_i, k_d) = \min_l \tau_l(k_p, k_i, k_d)$.

Optimization of $\tau(k_p,\ k_i,\ k_d)$. A key idea in this paper, one that differs distinctively from the previous work, is to recast the maximization of $\tau(k_p, k_i, k_d)$ as a constrained nonlinear programming problem. Toward this end, we note that

$$\begin{aligned}\bar{\tau}_{PID} &= \sup_{(k_p,k_i,k_d)\in\Omega} \tau(k_p,k_i,k_d) \\ &= \sup_{(k_p,k_i,k_d)\in\Omega} \min_l \tau_l(k_p,k_i,k_d) \\ &\le \min_l \sup_{(k_p,k_i,k_d)\in\Omega} \tau_l(k_p,k_i,k_d).\end{aligned} \tag{25}$$

Consider then the constrained problem

$$\begin{aligned} &\min \quad f(k_p,k_i,k_d,\omega_l) \\ \text{s.t.}\quad & g_1 = p_1 + p_2 - k_d \le 0 \\ & g_2 = -k_i \le 0 \\ & g_3 = k_i - (k_p - p_1p_2)(k_d - (p_1+p_2)) \le 0 \\ & h\ = \omega_l^6 - (k_d^2 - (p_1^2+p_2^2))\omega_l^4 \\ & \qquad - (k_p^2 - p_1^2p_2^2 - 2k_dk_i)\omega_l^2 - k_i^2 = 0,\end{aligned}$$

where

$$f(k_p,k_i,k_d,\omega_l) = -\sum_{i=1}^{2}\frac{\tan^{-1}\frac{\omega_l}{p_i}}{\omega_l} - \frac{\tan^{-1}\frac{k_d\omega_l-\frac{k_i}{\omega_l}}{k_p}}{\omega_l} + \frac{\pi}{\omega_l}.$$

Clearly,

$$\sup_{(k_p,k_i,k_d)\in\Omega} \tau_l(k_p,k_i,k_d) = -\min_{g_1,g_2,g_3,h} f(k_p,k_i,k_d,\omega_l).$$

Note that in this formulation, the first three inequality constraints represent the set Ω, where the less than or equal sign ($\leq$) is brought in to enable the equivalence between the infimum and minimum. The last equality constraint characterizes the crossover frequency ω_l.

Next, we invoke the Fritz-John condition and examine the first-order conditions

$$\begin{aligned}
&\lambda_0 \bigtriangledown f(k_p^*,k_i^*,k_d^*,\omega_l^*) + \sum_{i=1}^{3}\lambda_i \bigtriangledown g_i(k_p^*,k_i^*,k_d^*,\omega_l^*) \\
&\qquad\qquad + \lambda_4 \bigtriangledown h(k_p^*,k_i^*,k_d^*,\omega_l^*) = 0 \\
&\sum_{i=1}^{3}\lambda_i g_i(k_p^*,k_i^*,k_d^*,\omega_l^*) = 0 \\
&\lambda_i \geq 0, i\in\{0,1,2,3\} \\
&\lambda = [\lambda_0,\ \lambda_1,\ \lambda_2,\ \lambda_3,\ \lambda_4] \neq 0.
\end{aligned}$$

In this vein, we first evaluate the partial derivatives of $f(k_p,k_i,k_d,\omega_l)$, $h(k_p,k_i,k_d,\omega_l)$ and $g_i(k_p,k_i,k_d,\omega_l)$, $i=1,2,3$ with respect to k_d, k_i and k_p, which results in the equations

$$\frac{\partial}{\partial k_d}: \quad \lambda_0\frac{1}{1+\Pi^2}\frac{1}{k_p^*} + \lambda_1 + \lambda_3(k_p^*-p_1p_2) + 2\lambda_4(k_d^*\omega_l^{*2}-k_i^*)\omega_l^{*2} = 0 \quad (26)$$

$$\frac{\partial}{\partial k_i}: \quad \lambda_0\frac{1}{1+\Pi^2}\frac{1}{k_p^*} - \lambda_2\omega_l^{*2} + \lambda_3\omega_l^{*2} + 2\lambda_4(k_d^*\omega_l^{*2}-k_i^*)\omega_l^{*2} = 0 \quad (27)$$

$$\frac{\partial}{\partial k_p}: \quad \lambda_0\frac{1}{1+\Pi^2}(k_d^*-\frac{k_i^*}{\omega_l^{*2}})\frac{1}{k_p^{*2}} + \lambda_3(-k_d^*+p_1+p_2) + 2\lambda_4(-k_p^*)\omega_l^{*2} = 0, \quad (28)$$

where $\Pi = \frac{k_d^*\omega_l^* - \frac{k_i^*}{\omega_l^*}}{k_p^*}$. Solving the equation (28) gives rise to

$$\lambda_4 = \frac{\lambda_0\frac{1}{1+\Pi^2}(k_d^*-\frac{k_i^*}{\omega_l^{*2}})\frac{1}{k_p^{*2}} + \lambda_3(-k_d^*+p_1+p_2)}{2k_p^*\omega_l^{*2}}. \quad (29)$$

By substituting (29) into (26) and (27), we obtain that

$$\begin{aligned}
&\lambda_0\omega_l^{*2}\frac{1}{1+\Pi^2} + \lambda_1 k_p^*\omega_l^{*2} + \lambda_3(k_p^*-p_1p_2)k_p^*\omega_l^{*2} \\
&\quad + \lambda_0\frac{1}{1+\Pi^2}(k_d^*\omega_l^{*2}-k_i^*)^2\frac{1}{k_p^{*2}} \qquad (30)\\
&\quad + \lambda_3(-k_d^*+p_1+p_2)(k_d^*\omega_l^{*2}-k_i^*)\omega_l^{*2} = 0
\end{aligned}$$

and

$$\begin{aligned}
&\lambda_0\omega_l^{*2}\frac{1}{1+\Pi^2} + (\lambda_3-\lambda_2)k_p^*\omega_l^{*4} \\
&\quad + \lambda_0\frac{1}{1+\Pi^2}(k_d^*\omega_l^{*2}-k_i^*)^2\frac{1}{k_p^{*2}} \qquad (31)\\
&\quad + \lambda_3(-k_d^*+p_1+p_2)(k_d^*\omega_l^{*2}-k_i^*)\omega_l^{*2} = 0.
\end{aligned}$$

In light of the condition $\sum_{i=1}^{3}\lambda_i g_i(k_p^*,k_i^*,k_d^*,\omega_l^*) = 0$, it follows that $g_i(k_p^*,k_i^*,k_d^*,\omega_l^*) = 0$ whenever $\lambda_i > 0$, which means that the constraint $g_i \leq 0$ is active.

Assume that $\lambda_3 = 0$. Since $k_p^* \geq p_1p_2 > 0$ and $\omega_l^* > 0$, it follows from (30) that $\lambda_0 = \lambda_1 = 0$. In turn, we find that $\lambda_2 = 0$ by (31). Since $\lambda_0 = \lambda_1 = \lambda_2 = \lambda_3 = 0$, $\lambda_4 \neq 0$ by the fact that $\lambda \neq 0$. Hence, from (28), it is necessary that $k_p^* = 0$ or $\omega_l^* = 0$, which leads to contradiction. Therefore, we assert that $\lambda_3 > 0$ and

$$g_3 = k_i^* - (k_p^*-p_1p_2)(k_d^*-(p_1+p_2)) = 0.$$

Assume then that $\lambda_1 > 0$. It follows that $k_d^* = p_1+p_2$ and $k_i^* = (k_p^*-p_1p_2)(k_d^*-(p_1+p_2)) = 0$, that is, the controller reduces to a PD controller. From [14], however, $\tau(k_p,\ 0,\ p_1+p_2) = 0$. Thus, it suffices to consider $\lambda_1 = 0$ only. Assume that $\lambda_2 = 0$. We now have $\lambda_1 = \lambda_2 = 0$ and $\lambda_3 > 0$. From (26) and (27), it is necessary that $\omega_l^{*2} = k_p^* - p_1p_2$. Substituting $\omega_l^{*2} = k_p^*-p_1p_2$ and $k_i^* = (k_p^*-p_1p_2)(k_d^*-(p_1+p_2))$ into the equality constraint h, we obtain that

$$0 = h(k_p^*,k_i^*,k_d^*,\omega_l^{*2}) = -4p_1p_2\omega_l^{*2},$$

which results in contradiction again since $p_i > 0$, $i=1,2$ and $\omega_l^* > 0$. Consequently, we assert that $\lambda_2 > 0$, and hence $k_i^* = 0$. Since $k_i^* = (k_d^*-(p_1+p_2))(k_p^*-p_1p_2) = 0$ and $k_d^* > p_1+p_2$, it follows that $k_p^* = p_1p_2$. From the inequality (25), we now have $\bar{\tau}_{PID} \leq \bar{\tau}_{PD}$ and

$$\sup_{(k_p,k_d)\in\Omega} \tau_l(k_p,0,k_d) = \sup_{k_d>p_1+p_2} \tau_l(p_1p_2,0,k_d).$$

The proof of (i) and (ii) is completed by noting that $\bar{\tau}_{PID} \geq \bar{\tau}_{PD}$.

Pseudo-concavity of $\hat{\tau}(k_d)$. For $k_p^* = p_1p_2$ and $k_i^* = 0$, it is immediately seen that the polynomial equation (23) admits one unique positive solution $\omega_l > 0$, that is, $\omega_l = \sqrt{k_d^2-(p_1^2+p_2^2)}$. Denote $\omega_0 = \sqrt{k_d^2-(p_1^2+p_2^2)}$. It follows that

$$\hat{\tau}(k_d) = \frac{\tan^{-1}\frac{\omega_0}{p_1}}{\omega_0} + \frac{\tan^{-1}\frac{\omega_0}{p_2}}{\omega_0} + \frac{\tan^{-1}\frac{k_d\omega_0}{p_1p_2}}{\omega_0} - \frac{\pi}{\omega_0}. \quad (32)$$

We examine the first-order derivative of $\hat{\tau}(k_d)$, denoted as $\hat{\tau}'(k_d)$. Then $\hat{\tau}(k_d)$ is a pseudo-concave function if and only if there exists a unique $\bar{k}_d$ such that $\hat{\tau}'(k_d) > 0$ for $p_1+p_2 < k_d < \bar{k}_d$ and $\hat{\tau}'(k_d) < 0$ for $k_d > \bar{k}_d$. To facilitate the proof, we rewrite $\hat{\tau}(k_d) = n(g(k_d))$, where

$$n(\omega_0) = \sum_{i=1}^{2}\frac{\tan^{-1}\frac{\omega_0}{p_i}}{\omega_0} + \frac{\tan^{-1}\frac{\omega_0\sqrt{\omega_0^2+(p_1^2+p_2^2)}}{p_1p_2}}{\omega_0} - \frac{\pi}{\omega_0}$$

with $\omega_0 \geq \sqrt{2p_1p_2}$ and $g(k_d) = \sqrt{k_d^2-(p_1^2+p_2^2)}$. Taking the derivative of $\hat{\tau}(k_d)$, we have $\hat{\tau}'(k_d) = n'(\omega_0)g'(k_d)$, where $g'(k_d) = \frac{k_d}{\sqrt{k_d^2-(p_1^2+p_2^2)}} > 0$. Furthermore,

$$n'(\omega_0) = \left(\sum_{i=1}^{2}\left(\frac{\frac{\omega_0}{p_i}}{1+(\frac{\omega_0}{p_i})^2} - \tan^{-1}\frac{\omega_0}{p_i}\right) + \frac{m_1(\omega_0)}{1+m_1(\omega_0)^2} - \tan^{-1}m_1(\omega_0) + \frac{m_2(\omega_0)}{1+m_1(\omega_0)^2} + \pi\right)\frac{1}{\omega_0^2},$$

where

$m_1(\omega_0) = \frac{\omega_0\sqrt{\omega_0^2+(p_1^2+p_2^2)}}{p_1p_2}$ and $m_2(\omega_0) = \frac{\omega_0^3}{p_1p_2\sqrt{\omega_0^2+(p_1^2+p_2^2)}}$.

We first prove the existence of $\bar{k}_d$ such that $\hat{\tau}'(\bar{k}_d) = 0$. Consider $s(\omega_0) = \omega_0^2 n'(\omega_0)$. We have $s(\omega_0) \to -\frac{\pi}{2} < 0$

when $\omega_0 \to \infty$. On the other hand, in light of the fact [14] that $\tau(k_p, 0, p_1+p_2) = 0$ and $\tau(k_p, 0, k_d) > 0$, $\forall k_d > p_1+p_2$, we have $\hat{\tau}'(p_1+p_2) > 0$. Hence, when $\omega_0 \to \sqrt{2p_1p_2}$, we have

$$s(\omega_0) = \frac{\omega_0^2}{g'(p_1+p_2)}\hat{\tau}'(p_1+p_2) > 0.$$

By the continuity of $s(\omega_0)$, there must exist at least one $\bar{\omega}_0 \geq \sqrt{2p_1p_2}$, such that $s(\bar{\omega}_0) = 0$. Correspondingly, there exists some $\bar{k}_d > p_1+p_2$ such that $\hat{\tau}'(\bar{k}_d) = 0$.

We next verify the uniqueness of $\bar{k}_d$ by proving $s(\omega_0)$ is a monotonically decreasing function. Taking the derivative of $s(\omega_0)$ yields

$$s'(\omega_0) = -\sum_{i=1}^{2}\frac{2}{p_i}\frac{(\frac{\omega_0}{p_i})^2}{(1+(\frac{\omega_0}{p_i})^2)^2} - \frac{\Phi(\omega_0)}{(1+(m_1(\omega_0))^2)^2},$$

where

$$\begin{aligned}\Phi(\omega_0) &= \frac{2}{p_1p_2}m_1(\omega_0)^2\left(\frac{p_1p_2m_1(\omega_0)}{\omega_0} + \frac{\omega_0^3}{p_1p_2m_1(\omega_0)}\right) \\ &\quad - m_2'(\omega_0)\left(1+m_1(\omega_0)^2\right) + 2m_2(\omega_0)m_1(\omega_0)m_1'(\omega_0) \\ &= \frac{\omega_0^2\gamma(\omega_0)}{p_1^3p_2^3(\omega_0^2+(p_1^2+p_2^2))^{3/2}},\end{aligned}$$

and

$$\begin{aligned}\gamma(\omega_0) &= 6\omega_0^6 + 11(p_1^2+p_2^2)\omega_0^4 + (7p_1^4+7p_2^4+12p_1^2p_2^2)\omega_0^2 \\ &\quad + (2p_1^4+2p_2^4+p_1^2p_2^2)(p_1^2+p_2^2).\end{aligned}$$

Obviously, $\gamma(\omega_0) > 0$, and so $\Phi(\omega_0) > 0$. As such, $s'(\omega_0) < 0$ and $s(\omega_0)$ is monotonically decreasing with ω_0. Moreover, since $g'(k_d) > 0$ and $\hat{\tau}'(k_d) = s(\omega_0)g'(k_d)/\omega_0^2$, the uniqueness of $\bar{k}_d$ is confirmed, which also insures that $\hat{\tau}'(k_d) > 0$ for $p_1+p_2 < k_d < \bar{k}_d$ and $\hat{\tau}'(k_d) < 0$ for $k_d > \bar{k}_d$. This establishes that fact that $\hat{\tau}(k_d)$ is pseudo-concave.

Upper bound of $\bar{\tau}_{PD}$. In (32), since $\tan^{-1}\frac{k_d\omega_0}{p_1p_2} \leq \pi/2$ and $\tan^{-1}\frac{\omega_0}{p_i} \leq \pi/2,\ i = 1,\ 2$ for any $\omega_0 > 0$, it follows that

$$\hat{\tau}(k_d) \leq \min\left\{\frac{\tan^{-1}\frac{\omega_0}{p_1}}{\omega_0}, \frac{\tan^{-1}\frac{\omega_0}{p_2}}{\omega_0}\right\}.$$

Since $\frac{\tan^{-1}\frac{\omega_0}{p_i}}{\omega_0}$ is monotonically decreasing with ω_0 and ω_0 is monotonically increasing with k_d, the maximum of this upper bound is achieved at $k_d = p_1+p_2$, which leads to

$$\bar{\tau}_{PD} \leq \min\left\{\left(\frac{\tan^{-1}\sqrt{\frac{2p_2}{p_1}}}{\sqrt{\frac{2p_2}{p_1}}}\right)\frac{1}{p_1}, \left(\frac{\tan^{-1}\sqrt{\frac{2p_1}{p_2}}}{\sqrt{\frac{2p_1}{p_2}}}\right)\frac{1}{p_2}\right\}.$$

The proof is now completed. ■

It is thus clear from Theorem 1 (i) that for a second-order unstable plant, the delay margin achieved by PID control coincides with that by PD controllers. Theorem 1 (ii) indicates that the proportional control coefficient lies on the boundary of its allowable range in Ω, pointing to the fact that integral control contributes no effect to enlarge the delay margin. The pseudo-concavity of $\hat{\tau}(k_d)$, established in Theorem 1 (iii), reveals that the delay margin can be effectively computed by solving a unimodal problem. Inadvertently, it also implies, since the maximum of $\hat{\tau}(k_d)$ is achieved in the interior of the range of k_d, that it is generally not possible to obtain an explicit expression of the delay margin, unlike in the case of first-order plants. At last, the *a priori* bound stated in Theorem 1 (iv) provides an intrinsic bound independent of PID controller design, and in turn an estimate that can be used to guide the numerical optimization.

5 Second-Order Systems with Complex Conjugate Unstable Poles

We now develop in parallel the delay margin for plants that contain a pair of complex conjugate unstable poles, which can be described by

$$P_\tau(s) = \frac{1}{(s-p)(s-p^*)}e^{-\tau s}, \tag{33}$$

where $p = \alpha + j\beta$, $\mathrm{Re}(p) = \alpha > 0$, and p^* denotes the complex conjugate of p. Second-order plants with complex conjugate poles are numerous in, e.g., mechanical and electronic systems, with such unstable behaviors exhibited by, e.g., oscillators. It should be noted that this case and that of the real unstable poles are mutually exclusive, except in limiting situations. For this reason, the plant (33) must be dealt with separately.

Likewise, we characterize the exact delay margin in the following theorem. Due to space constraint, from this point onward, we shall omit the proofs.

Theorem 2 *Let $P_\tau(s)$ be given by* (33). *Define the set*

$$\begin{aligned}\Lambda = \{&(k_p,\ k_i,\ k_d) : k_i > 0,\ k_d > 2\alpha, \\ &(k_p - |p|^2)(k_d - 2\alpha) - k_i > 0\}.\end{aligned} \tag{34}$$

Then the following statements hold:

(i)

$$\bar{\tau}_{PID} = \bar{\tau}_{PD}. \tag{35}$$

(ii)

$$\bar{\tau}_{PD} = \sup_{k_d > 2\alpha}\hat{\tau}(k_d), \tag{36}$$

where

$$\hat{\tau}(k_d) = \tau(|p|^2,\ 0,\ k_d), \tag{37}$$

$$\tau(k_p,\ k_i,\ k_d) = \min_l \tau_l(k_p,\ k_i,\ k_d), \tag{38}$$

with $\tau_l(k_p,\ k_i,\ k_d)$ defined on Λ as

$$\begin{aligned}\tau_l(k_p,\ k_i,\ k_d) &= \frac{\tan^{-1}\frac{\omega_l-\beta}{\alpha}}{\omega_l} + \frac{\tan^{-1}\frac{\omega_l+\beta}{\alpha}}{\omega_l} \\ &\quad + \frac{\tan^{-1}\frac{k_d\omega_l - \frac{k_i}{\omega_l}}{k_p}}{\omega_l} - \frac{\pi}{\omega_l},\end{aligned} \tag{39}$$

and $\omega_l > 0$ being the solutions of the polynomial equation

$$\begin{aligned}\omega^6 - (k_d^2 - 2(\alpha^2-\beta^2))\omega^4 - (k_p^2 - |p|^4 - 2k_dk_i)\omega^2 \\ - k_i^2 = 0.\end{aligned} \tag{40}$$

(iii) $\hat{\tau}(k_d)$ is pseudo-concave for $k_d > 2\alpha$.

(iv)

$$\bar{\tau}_{PD} \leq \frac{\tan^{-1}\frac{\sqrt{2}|p|+\beta}{\alpha}}{\sqrt{2}|p|}. \tag{41}$$

6 Effect of Nonminimum Phase Zero

In this section, we extend the preceding results to systems containing nonminimum phase zeros. Consider the plant

$$P_\tau(s) = \frac{s-z}{(s+z)(s-p)}e^{-\tau s}, \tag{42}$$

where $z > 0$ is a nonminimum phase zero, and $p > 0$ denotes an unstable pole. The following result shows that the presence of a nonminimum phase zero will invariably reduce the delay margin.

Theorem 3 *Let $P_\tau(s)$ be given by* (42). *Let $z > p$. Then,*

$$\bar{\tau}_{PID} = \bar{\tau}_{PD} = 2\left(\frac{1}{p} - \frac{1}{z}\right). \tag{43}$$

The exact margin (43) coincides with that of [18], which was shown to be achievable by a general LTI controller for plants with a single nonminimum phase zero and a single unstable pole. That a PD controller can result in the same delay margin was shown in [14]. Our present result reinforces this fact and shows that still, the delay margins achieved by PID and PD controllers coincide.

It can be readily verified that the feasible set of (k_p, k_i, k_d) defined by the inequalities resulted from the first column of the Routh array and Assumption 1: $0 < k_d < 1$, $k_i > 0$, $(1+k_d)z > k_p + p$, and $[(1+k_d)z - (k_p+p)](z(k_p - p) - k_i) > (1-k_d)zk_i$, is empty whenever $z \le p$. In other words, the delay-free plant $P_0(s)$ cannot be stabilized by any PID controller under this circumstance, lest the delay plant.

7 An Illustrative Example

We now use a numerical example to illustrate our results.

Example 1 In this example we consider the second-order system in the form of (16) and (33), with two real poles and a pair of complex conjugate poles, respectively. We analyze the example in these two cases.

Real unstable poles: In this case we take $p_1 = 0.2$, and let p_2 vary in the interval $[0, 3]$. Fig. 2 shows the exact delay margins achieved by PID and PD controllers, respectively, together with bounds obtained in this paper and elsewhere. Here the PID delay margin is computed by a brute-force search in the allowable range Ω of the triple $(k_p,\ k_i,\ k_d)$, while the delay margin achieved by a PD controller is computed by solving the unimodal optimization problem in (19). The two curves overlap, for all $p_2 \in [0,\ 3]$. It is worth noting that an appreciable gap exists between the bounds and the exact delay margin.

Complex conjugate unstable poles: In a similar vein, we fix $\beta = 1$, and allow α to vary in $[0, 3]$. Fig. 3 plots the exact delay margins of the system in (33), achieved by PID and PD controllers. Likewise, the two curves overlap, for all $\alpha \in [0,\ 3]$. Existing upper bounds are also plotted in Fig. 3 for purpose of comparison.

Finally, it is instructive to examine the 3-D surface manifold of $\tau(k_p,\ k_i,\ k_d)$ with $k_i = 0$, to see how the delay margin may vary with the parameters k_p and k_d. Fig. 4 shows the 3-D surfaces of $\tau(k_p,\ 0,\ k_d)$ in the two cases. In both cases, the surfaces exhibit features consistent with the analytical results, indicating that $\tau(k_p,\ 0,\ k_d)$ has a unique maximum, which lies at the terminal value of k_p, but generally in the interior of the allowable range of k_d. Fig. 5 exhibits the concave property of the function $\hat{\tau}(k_d)$, which further affirms the observation. From these plots, the maximum in the two cases is found to be 0.6009 and 0.2311, which are consistent with the exact delay margin for the plants corresponding to $p_1 = 0.2$, $p_2 = 0.8$ in Fig. 2, and $p = 1 + j$ in Fig. 3, respectively.

8 Conclusion

In this paper, we have investigated the delay margin of second-order systems under PID control, subject to unknown constant delays. An analytical characterization of the delay margin is obtained by optimizing PID controller parameters. We proved that the delay margin achieved by PID control coincides with that by PD controllers, which, while consistent with one's intuition, appears to be both surprising and appealing a result. We also showed that the exact delay margin can be determined by solving a unimodal problem, which is a univariate optimization problem admitting a unique maximum and constitutes one of convex optimization in one variable. The results thus insure that the PID delay margin problem can be solved readily and efficiently. The conceptual insights gained in solving this problem are appealing, which shed further light into the fundamental understanding of PID control, and to the tuning and analytical design of PID controllers.

Future extension of this work may be pursued in several directions. In this vein we come to recognize the following

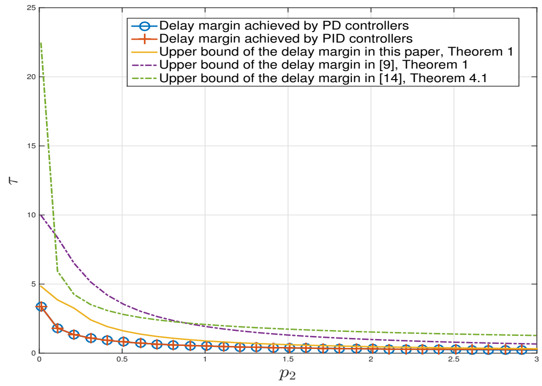

Fig. 2. Exact delay margins $\bar{\tau}_{PID}$, $\bar{\tau}_{PD}$ of system (16), and comparison to existing upper bounds.

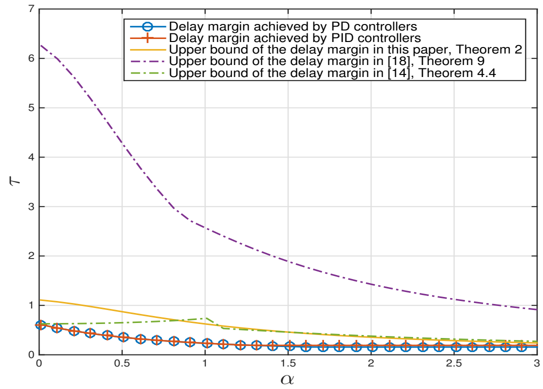

Fig. 3. Exact delay margins $\bar{\tau}_{PID}$, $\bar{\tau}_{PD}$ of system (33), and comparison to existing upper bounds.

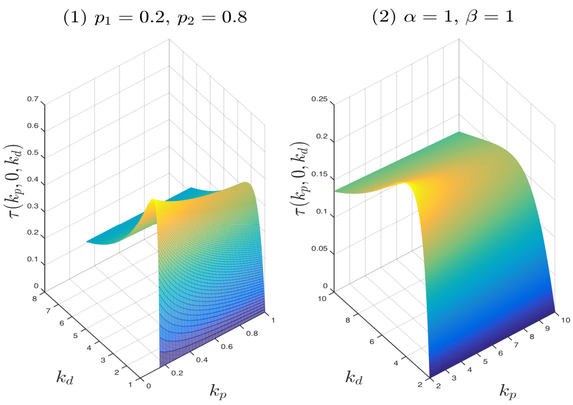

Fig. 4. Relationships between $\tau(k_p,\ 0,\ k_d)$ and (k_p, k_d) for systems (16) and (33).

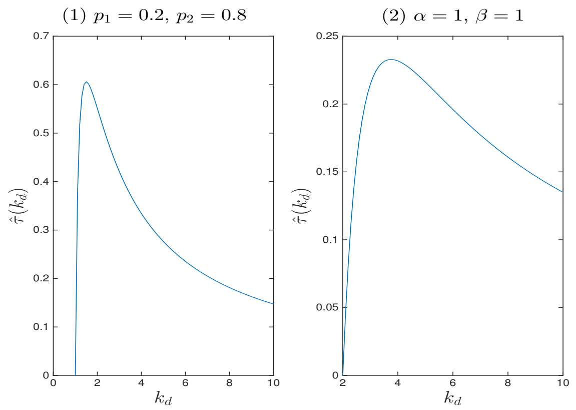

Fig. 5. Pseudo-concavities of $\hat{\tau}(k_d)$ for systems (16) and (33).

points.

- We note that it is generally difficult to stabilize high-order unstable delay systems using PID control due to insufficient degrees of design freedom. Notwithstanding this limitation of PID control, it remains plausible to consider certain augmented, PID-based controllers (see, e.g., [14]). It is also useful to delineate classes of high-order delay systems that can be robustly stabilized by PID control.
- In practice, it is mandatory to implement the derivative control in conjunction with a low-pass filter, so that the PID controller possesses the form [1] of

$$K_{PID}^{T_f}(s) = k_p + \frac{k_i}{s} + \frac{k_d s}{1 + T_f s},$$

  where $T_f > 0$. It is possible to determine the delay margin achievable by $K_{PID}^{T_f}(s)$ as well. In doing so, it will be useful to characterize analytically the tradeoff between the achievable delay margin and the filter bandwidth.
- The delay margin is closely related to the classical concepts of gain and phase margin, two fundamental measures of robustness that can be optimized analytically using the $\mathcal{H}_\infty$ control theory [5]. It will be of interest to seek their counterparts in the context of PID control.

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