Distributed Output Feedback Consensus of Uncertain Nonlinear Multi-Agent Systems With Limited Data Rate

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Abstract: This paper considers the distributed output feedback consensus problem of uncertain nonlinear multi-agent systems with limited data rate. Each agent is modeled by an nth order integrator with unmeasurable states and unknown nonlinear dynamics. An $(n+2)$th order extended state observer (ESO) is first designed to estimate the unmeasurable agent states and the unknown nonlinear dynamics. Based on the output of the ESO, a distributed consensus protocol based on dynamic encoding and decoding is proposed. It is shown that, for a connected undirected network with nth order uncertain nonlinear agents, consensus can be guaranteed with merely one bit information exchange between each pair of adjacent agents at each time step.

Key Words: Multi-agent systems, uncertain nonlinear systems, distributed consensus, data rate, output feedback, extended state observer (ESO)

1 Introduction

There has been a great deal of interest in cooperative control of multi-agent systems [1, 2]. One of the most important issues in multi-agent systems is the distributed consensus, which is a process that a group of agents with different initial states reach an agreement by local communications [3]. Most of the early efforts are based on the assumption that every agent can obtain the exact information of its neighbours through communication networks (see, *e.g.*, [4, 5]). However, this assumption is not realistic in the general case since the capability of the digital networks that transmit information between agents is limited in practice [6]. Thus it is important and meaningful to study the minimum data rate required for distributed consensus.

In recent years, distributed consensus over capability-limited digital networks has become an active research area. In digital networks, for every agent, its information is first quantized, and then encoded and sent out to its neighbours. Thus, quantization is essential in information exchange between agents [7, 8]. In [9], an infinite-level logarithmic quantizer was developed to guarantee average consensus. In [10], a static uniform quantizer with an exponentially decaying scaling function was proposed. It was shown in [10] that for first-order integrator multi-agent systems, one bit information exchange between each pair of adjacent agents is enough to achieve average consensus. The approach in [10] was further generalized to the cases with link failures [11] and time-delay [12]. For multi-agent systems with second-order integrator dynamics, an integrative scheme for observer and encoder-decoder design was proposed in [13], and it was concluded that two bits information exchange suffice to guarantee consensus. Consensus of higher-order multi-agent systems with capability-limited digital networks was considered in [14–16]. Specifically, Qiu *et al.* [14, 15] showed that n bits of information exchange are necessary for nth order integrator multi-agent systems to achieve consensus.

The aforementioned works [9–16] concentrate on linear multi-agent systems. However, most practical control systems are inherently nonlinear and uncertain [17]. The distributed consensus of uncertain nonlinear multi-agent systems over capability-limited digital networks is very difficult and complicated due to the coupling of the uncertain nonlinearity, the quantization, and the constraints on network capability. In a recent paper [18], this problem was first visited under the assumption that the states of the agents are fully available for feedback. If the system exists uncertainty, a multi-loop control structure was proposed to ensure consensus. The approach in [18] relies on some prior knowledge or online adaptive approximation of the system uncertainties.

In this paper, we consider the distributed output feedback consensus problem of uncertain nonlinear multi-agent systems with limited data rate. This study is motivated by the work in [19, 20], where the extended state observer (ESO) based output feedback control approach was proposed. The essence of this approach is to view the unknown nonlinear dynamics as an extended state of the system, and then utilize an ESO to estimate it, and finally compensate for it in the control action, in real time. This philosophy is applied in the current paper. The main contributions of this paper are twofold: *First, the problem of distributed output feedback consensus of uncertain nonlinear multi-agent systems with limited data rate is first considered, and a novel higher-order ESO based encoding-decoding scheme is proposed.* Compared with [18], the proposed approach is output feedback, and the control structure is much simpler and also easier to be implemented in practice, especially for systems with general nonlinear uncertainties. *Second, we show that with the proposed distributed output feedback protocol, only one bit of information exchange between agents suffices to guarantee consensus for nth order uncertain nonlinear multi-agent systems.* From a theoretical viewpoint, the proposed approach achieves the lowest communicate data rate via output feedback (*i.e.*, one bit).

Notation: $\mathbf{1}_N$ is the N-dimensional column vector with all components being 1, and $J_N = (1/N)\mathbf{1}_N\mathbf{1}_N^{\mathrm{T}}$. I is the identity matrix with an appropriate dimension. For a given vector or matrix A, $\|A\|_\infty$ denotes its ∞-norm, and $\|A\|$ denotes its Euclidean norm. For a given positive number a, $\lfloor a \rfloor$ and $\lceil a \rceil$ represent the largest integer not greater than a and the smallest integer not less than a, respectively. Big O-notation in terms of ν is denoted as $O(\nu)$ and it is assumed that this holds for ν positive and sufficiently small.

2 Problem Statement and Preliminaries

Consider the following uncertain nonlinear multi-agent systems:

$$\begin{cases} \dot{x}_i = Ax_i + B[f_i(x_i) + u_i], \\ y_i = Cx_i,\ i = 1, \ldots, N, \end{cases} \tag{1}$$

where $x_i = [x_{i1}, x_{i2}, \ldots, x_{in}]^{\mathrm{T}} \in \mathbb{R}^n$, $u_i \in \mathbb{R}$, and $y_i \in \mathbb{R}$ are, respectively, the state, input, and output of the ith agent, $f_i(\cdot) : \mathbb{R}^n \to \mathbb{R}$ is an unknown second-order continuously differentiable function, and matrices $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times 1}$, and $C \in \mathbb{R}^{1\times n}$ represent a chain of integrators.

The communications between different agents are modeled as an undirected graph $\mathcal{G} = \{\mathcal{V}, \mathcal{E}, \mathcal{A}\}$, where $\mathcal{V} = \{1, 2, \ldots, N\}$ is the node set, $\mathcal{E}$ is the edge set with element (i, j) describing the communication from node i to node j, and $\mathcal{A} = [a_{ij}] \in \mathbb{R}^{N\times N}$ is the weighted adjacency matrix. The pair $(i, j) \in \mathcal{E}$ means that there is information flow from node i to node j, and $(i, j) \in \mathcal{E}$ if and only if $(j, i) \in \mathcal{E}$. Node j is said to be a neighbour of node i if $(i, j) \in \mathcal{E}$. The neighborhood of the ith node is denoted by $\mathcal{N}_i = \{j \in \mathcal{V} | (i, j) \in \mathcal{E}\}$. The graph $\mathcal{G}$ is called connected if for any two nodes there exist a series of edges which connect the two nodes. $\deg_i = \sum_{j=1}^{N} a_{ij}$ is called the degree of i, and $d^* = \max_i \deg_i$ is called the degree of $\mathcal{G}$. The Laplacian matrix of $\mathcal{G}$ is denoted by $\mathcal{L} = \mathcal{D} - \mathcal{A}$, where $\mathcal{D} = \mathrm{diag}\{\deg_1, \ldots, \deg_N\}$. For a connected graph, its Laplacian matrix $\mathcal{L}$ is a symmetric positive semidefinite matrix and its eigenvalues in an ascending order are denoted by $0 = \lambda_1(\mathcal{L}) < \lambda_2(\mathcal{L}) \leq \cdots \leq \lambda_N(\mathcal{L})$.

In this paper, we assume that the information is exchanged through digital networks. Due to finite bit rates in digital networks, signals to be communicated between agents are to be encoded at transmitters and then decoded at receivers. The finite-time uniform quantizer $q(\cdot)$ is given by

$$q(\nu) = \begin{cases} 0, & -1/2 < \nu < 1/2, \\ i, & \frac{2i-1}{2} \leq \nu < \frac{2i+1}{2}, 1 \leq i \leq K-1, \\ K, & \nu \geq \frac{2K-1}{2}, \\ -q(-\nu), & \nu \leq -1/2, \end{cases}$$

where K is a positive integer. Similar to [10], it is assumed that the agents send out no signal when the output is zero. Thus, it is enough to use $\lceil \log_2(2K) \rceil$ bits to represent the output of the quantizer.

The problem we intend to solve is to design a distributed output feedback protocol, based on a dynamic encoding and decoding scheme, such that the states of the agents achieve the following consensus:

$$\lim_{t\to\infty} (x_i - x_j) = 0,\ 1 \leq i \neq j \leq N. \tag{2}$$

To end this section, we introduce two lemmas, which are useful in this paper.

Lemma 1 [10]: If $\mathcal{G}$ is connected and $T \in (0, 2/\lambda_N(\mathcal{L}))$, then $\rho_h \triangleq \max_{2\leq i\leq N} |1 - T\lambda_i(\mathcal{L})| < 1$. Furthermore, if $T \in (0, 2/(\lambda_2(\mathcal{L}) + \lambda_N(\mathcal{L})))$, then $\rho_h = 1 - T\lambda_2(\mathcal{L})$.

Lemma 2 [18]: Let

$$s_i = k_1 x_{i1} + k_2 x_{i2} + \cdots + k_{n-1} x_{i,n-1} + x_{in}, \tag{3}$$

where $k_1, k_2, \ldots, k_{n-1}$ are selected such that the polynomial $k_1 + k_2\nu + \cdots + k_{n-1}\nu^{n-1} + \nu^n$ is Hurwitz. If s_i is bounded and

$$\lim_{t\to\infty} s_i = \Lambda,\ 1 \leq i \leq N, \tag{4}$$

for some constant Λ, then (2) is satisfied. Furthermore, if the convergence rate in (4) is exponential, the convergence rate of (2) is also exponential.

3 Distributed Output Feedback Consensus

In this section, the higher-order ESO based encoding-decoding scheme is presented, and the convergence analysis of the closed-loop system is conducted.

3.1 Protocol Design

Let $x_{i,n+1} = f_i(x_i)$ and $x_{i,n+2} = \dot{x}_{i,n+1}$ be the extended states of agent i. The following $(n + 2)$th order ESO is designed for each agent:

$$\begin{cases} \dot{\widehat{x}}_{ij} = \widehat{x}_{i,j+1} + \dfrac{l_{ij}}{\varepsilon^j}(y_i - \widehat{x}_{i1}),\ 1 \leq j \leq n-1, \\ \dot{\widehat{x}}_{in} = \widehat{x}_{i,n+1} + \dfrac{l_{in}}{\varepsilon^n}(y_i - \widehat{x}_{i1}) + u_i, \\ \dot{\widehat{x}}_{i,n+1} = \widehat{x}_{i,n+2} + \dfrac{l_{i,n+1}}{\varepsilon^{n+1}}(y_i - \widehat{x}_{i1}), \\ \dot{\widehat{x}}_{i,n+2} = \dfrac{l_{i,n+2}}{\varepsilon^{n+2}}(y_i - \widehat{x}_{i1}), \end{cases} \tag{5}$$

where $\widehat{x}_i = [\widehat{x}_{i1}, \widehat{x}_{i2}, \ldots, \widehat{x}_{i,n+2}]^{\mathrm{T}} \in \mathbb{R}^{n+2}$ is the observer state with initial condition $\widehat{x}_i(0) = 0$, $\varepsilon < 1$ is a small positive constant, $L_i = [l_{i1}, l_{i2}, \ldots, l_{i,n+2}]^{\mathrm{T}} \in \mathbb{R}^{n+2}$ is selected such that the following matrix is Hurwitz:

$$E_i = \begin{bmatrix} -l_{i1} & 1 & 0 & \cdots & 0 \\ -l_{i2} & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -l_{i,n+1} & 0 & 0 & \cdots & 1 \\ -l_{i,n+2} & 0 & 0 & \cdots & 0 \end{bmatrix}. \tag{6}$$

Based on the output of the ESO (5), the information to be exchanged between agents is given by

$$\theta_i = M_\theta \mathrm{sat}_\varepsilon \left(\frac{1}{M_\theta} \left(k_1 \widehat{x}_{i1} + \cdots + k_{n-1}\widehat{x}_{i,n-1} + \widehat{x}_{in} \right) \right), \tag{7}$$

where M_θ is the bound selected such that the saturation will not be activated after the short transient period of the observer [21, 22], and $\mathrm{sat}_\varepsilon(\cdot)$ is an odd function defined by [20]

$$\mathrm{sat}_\varepsilon(\nu) = \begin{cases} \nu, & 0 \leq \nu \leq 1, \\ \nu + \dfrac{\nu - 1}{\varepsilon} - \dfrac{\nu^2 - 1}{2\varepsilon}, & 1 < \nu \leq 1 + \varepsilon, \\ 1 + \dfrac{\varepsilon}{2}, & \nu > 1 + \varepsilon. \end{cases}$$

Note that $\mathrm{sat}_\varepsilon(\cdot)$ is continuously differentiable, and $\left| \frac{\mathrm{dsat}_\varepsilon(\nu)}{\mathrm{d}\nu} \right| \leq 1$ and $|\mathrm{sat}_\varepsilon(\nu) - \mathrm{sat}(\nu)| \leq \frac{\varepsilon}{2}$, $\forall \nu \in \mathbb{R}$, where $\mathrm{sat}(\cdot)$ is the standard unity saturation function defined by $\mathrm{sat}(\nu) = \mathrm{sign}(\nu) \cdot \min\{1, |\nu|\}$.

Since the communication network is a digital network, data can only be exchanged in discrete-time. Data to be transmitted at time $t = kT$ is denoted by $\theta_i(kT)$, where $k = 0, 1, \ldots$, and T is the constant time interval. For agent j, the encoder Υ_j is given by

$$\begin{cases} \xi_j(0) = 0, \\ \xi_j((k+1)T) = g(kT)\Delta_j((k+1)T) + \xi_j(kT), \\ \Delta_j((k+1)T) = q\left(\dfrac{\theta_j((k+1)T) - \xi_j(kT)}{g(kT)}\right), \\ k = 0, 1, \ldots \end{cases} \tag{8}$$

where $\xi_j(kT)$ and $\Delta_j(kT)$ are, respectively, the internal state and output of Υ_j, and $g(\cdot)$ is a scaling function to be specified later. The agent j broadcasts $\Delta_j(kT)$ to its neighbors. If agent i is a neighbor of agent j, agent i receives $\Delta_j(kT)$ and then uses the following decoder Ψ_{ji} to estimate $\theta_j(kT)$:

$$\begin{cases} \widehat{\theta}_{ji}(0) = 0, \\ \widehat{\theta}_{ji}((k+1)T) = g(kT)\Delta_j((k+1)T) + \widehat{\theta}_{ji}(kT), \\ k = 0, 1, \ldots \end{cases} \tag{9}$$

where $\widehat{\theta}_{ji}(kT)$ is the output of Ψ_{ji}.

Based on the observer (5), encoder (8), and decoder (9), we propose the following distributed protocol:

$$\begin{cases} u_i(t) = -\vartheta_i(t) - \lambda\theta_i(t),\ t \in [0, T), \\ u_i(t) = -\vartheta_i(t) + \displaystyle\sum_{j \in \mathcal{N}_i} a_{ij}\left(\widehat{\theta}_{ji}(kT) - \xi_i(kT)\right), \\ \quad t \in [kT, (k+1)T),\ k = 1, 2, \ldots \end{cases} \tag{10}$$

where $\lambda > 0$ and ϑ_i is given by

$$\vartheta_i = M_\vartheta \mathrm{sat}_\varepsilon\left(\frac{1}{M_\vartheta}\left(k_1\widehat{x}_{i2} + \cdots + k_{n-1}\widehat{x}_{in} + \widehat{x}_{i,n+1}\right)\right), \tag{11}$$

with M_ϑ the saturation bound.

3.2 Convergence Analysis

To obtain the convergence of the closed-loop system, we need the following standard assumptions [10, 13, 15]:

Assumption A1: The communication graph $\mathcal{G}$ is undirected and connected.

Assumption A2: There exists a known nonnegative constant C_x such that $\max_{1 \le i \le N, 1 \le j \le n} |x_{ij}(0)| \le C_x$.

Assumption A2 is to guarantee that the quantizer will not be saturated at initial steps. By Assumption A2 and (3), one has $\max_{1 \le i \le N} |s_i(0)| \le C_s$, where $C_s = C_x\left(\sum_{j=1}^{n-1} k_j + 1\right)$.

For subsequent use, some notations are defined as

$$\begin{aligned} S(kT) &= [s_1(kT), s_2(kT), \ldots, s_N(kT)]^{\mathrm{T}}, \\ \Theta(kT) &= [\theta_1(kT), \theta_2(kT), \ldots, \theta_N(kT)]^{\mathrm{T}}, \\ \widehat{\Theta}(kT) &= [\xi_1(kT), \xi_2(kT), \ldots, \xi_N(kT)]^{\mathrm{T}}, \\ e(kT) &= \Theta(kT) - \widehat{\Theta}(kT), \\ \delta(kT) &= S(kT) - J_N S(kT). \end{aligned}$$

Under the control law (10), one has

$$\begin{aligned} \dot{s}_i(t) &= \sum_{j=1}^{n-1} k_j x_{i,j+1}(t) + x_{i,n+1}(t) + u_i(t) \\ &= \sum_{j \in \mathcal{N}_i} a_{ij}\left(\widehat{\theta}_{ji}(kT) - \xi_i(kT)\right) \\ &\quad + \sum_{j=1}^{n-1} k_j x_{i,j+1}(t) + x_{i,n+1}(t) - \vartheta_i(t), \\ &\quad t \in [kT, (k+1)T),\ k = 1, 2, \ldots \end{aligned} \tag{12}$$

It follows from (12) that

$$\begin{aligned} s_i((k+1)T) &= s_i(kT) + T\sum_{j \in \mathcal{N}_i} a_{ij}\left(\widehat{\theta}_{ji}(kT) - \xi_i(kT)\right) \\ &+ \int_{kT}^{(k+1)T}\left(\sum_{j=1}^{n-1} k_j x_{i,j+1}(t) + x_{i,n+1}(t) - \vartheta_i(t)\right)\mathrm{d}t. \end{aligned} \tag{13}$$

By the structure of the encoder Υ_j and decoder Ψ_{ji}, one has

$$\widehat{\theta}_{ji}(kT) = \xi_j(kT),\ i \in \mathcal{N}_j,\ j = 1, 2, \ldots, N. \tag{14}$$

Thus (13) can be written into the following compact form:

$$\begin{aligned} S((k+1)T) =& S(kT) - T\mathcal{L}\widehat{\Theta}(kT) + \Pi(kT) \\ =& (I - T\mathcal{L})\Theta(kT) + T\mathcal{L}e(kT) \\ &+ S(kT) - \Theta(kT) + \Pi(kT), \end{aligned} \tag{15}$$

where $\Pi(kT) = [\pi_1(kT), \pi_2(kT), \ldots, \pi_N(kT)]^{\mathrm{T}}$, with

$$\pi_i(kT) = \int_{kT}^{(k+1)T}\left(\sum_{j=1}^{n-1} k_j x_{i,j+1}(t) + x_{i,n+1}(t) - \vartheta_i(t)\right)\mathrm{d}t.$$

Since $\mathcal{L}J_N = J_N\mathcal{L} = 0$, it follows from (15) that

$$\begin{aligned} &\Theta((k+1)T) - \widehat{\Theta}(kT) \\ =& S((k+1)T) - \widehat{\Theta}(kT) + \Theta((k+1)T) - S((k+1)T) \\ =& (I + T\mathcal{L})e(kT) - T\mathcal{L}\delta(kT) \\ &+ (I + T\mathcal{L})(S(kT) - \Theta(kT)) \\ &+ \Pi(kT) + \Theta((k+1)T) - S((k+1)T). \end{aligned} \tag{16}$$

Thus by (15) and (16), we have

$$\begin{cases} \delta((k+1)T) = (I - T\mathcal{L})\delta(kT) + T\mathcal{L}e(kT) \\ \qquad + (I - J_N)(S(kT) - \Theta(kT) + \Pi(kT)), \\ e((k+1)T) = \left(\Theta((k+1)T) - \widehat{\Theta}(kT)\right) \\ \qquad - g(kT)Q\left(\dfrac{\Theta((k+1)T) - \widehat{\Theta}(kT)}{g(kT)}\right), \end{cases} \tag{17}$$

where $Q([\nu_1, \ldots, \nu_N]^{\mathrm{T}}) = [q(\nu_1), \ldots, q(\nu_N)]^{\mathrm{T}}$. Let

$$\omega(kT) = \frac{1}{g(kT)}\delta(kT), \tag{18}$$

$$z(kT) = \frac{1}{g(kT)}e(kT). \tag{19}$$

Then we have the following lemma, which shows the boundedness of $\omega(kT)$.

Lemma 3: Consider the closed-loop system formed by (1), (5), (8), (9), and (10). Suppose Assumptions A1 and A2 hold. Let

$$T \in (0, 2/\lambda_N(\mathcal{L}))\,, \tag{20}$$

$$\gamma \in (\rho_h, 1), \tag{21}$$

$$K \geq \left\lfloor K_1(T,\gamma) - \frac{1}{2} \right\rfloor + 1, \tag{22}$$

$$K_1(T,\gamma) = \frac{\sqrt{N} T^2 \lambda_N^2(\mathcal{L})}{2\gamma(\gamma - \rho_h)} + \frac{1 + 2Td^*}{2\gamma} + \frac{1}{2\gamma} T\lambda_N(\mathcal{L}), \tag{23}$$

$$g(kT) = \max\left\{g_o\gamma^k, \varepsilon\right\}, \tag{24}$$

$$g_0 \geq \max\left\{\frac{2M_\theta}{2K+1}, \frac{8\sqrt{N}(\gamma-\rho_h)(C_s+1)}{2T\sqrt{N}\lambda_N(\mathcal{L}) + \gamma - \rho_h}\right\}, \tag{25}$$

$$\Phi = \max\left\{\frac{2\sqrt{N}(C_s+1)}{g_0\gamma}, \frac{T\sqrt{N}\lambda_N(\mathcal{L})}{2\gamma(\gamma-\rho_h)} + \frac{1}{4\gamma}\right\}. \tag{26}$$

Then there exists $\varepsilon^* > 0$ such that for any $\varepsilon \in (0, \varepsilon^*)$ and $t = 0, T, 2T, \ldots$, $\|\omega(t)\| \leq \Phi$, and the quantizer will never be saturated.

Proof: When $t = 0$, by Assumption A2, one has

$$\|\omega(0)\| = \frac{\|\delta(0)\|}{g_0} \leq \frac{\sqrt{N}\|\delta(0)\|_\infty}{g_0} \leq \frac{2\sqrt{N}C_s}{g_0} < \Phi. \tag{27}$$

When $t = T$, it follows from (10) that

$$\begin{aligned}\dot{s}_i(t) =& k_1 x_{i2}(t) + \cdots + k_{n-1}x_{in}(t) + x_{i,n+1}(t) \\ &- \vartheta_i(t) - \lambda\theta_i(t),\ t \in [0,T).\end{aligned} \tag{28}$$

Now we claim that there exists sufficiently small ε such that $\max_{1\leq i\leq N}|s_i(t)| \leq C_s + 1, \forall t \in [0,T]$. This claim will be proved by contradiction. By the boundedness of the control $u_i(t)$, and $|s_i(0)| \leq C_s$, there exists ε-independent $t_0 > 0$ such that $|s_i(t)| \leq C_s + 1/2, \forall t \in [0, t_0]$. If the claim is not true, $t_0 \in (0,T)$, and there exist $t_0 \leq t_1 < t_2 < T$ such that

$$\begin{cases}|s_i(t_1)| = C_s + 1/2, \\ |s_i(t_2)| = C_s + 1, \\ C_s + 1/2 \leq |s_i(t)| \leq C_s + 1,\ t \in [t_1, t_2].\end{cases} \tag{29}$$

Consider the scaled ESO estimation error $\eta_i = [\eta_{i1}, \eta_{i2}, \ldots, \eta_{i,n+2}]^{\mathrm{T}}$, with

$$\eta_{ij} = \frac{x_{ij} - \hat{x}_{ij}}{\varepsilon^{n+2-j}},\ 1 \leq j \leq n+2. \tag{30}$$

By (1) and (5), the dynamics of η_i can be written as

$$\begin{cases}\varepsilon\dot{\eta}_{ij} = \eta_{i,j+1} - l_{ij}\eta_{i1},\ 1 \leq j \leq n+1, \\ \varepsilon\dot{\eta}_{i,n+2} = \varepsilon\dot{x}_{i,n+2} - l_{i,n+2}\eta_{i1}.\end{cases} \tag{31}$$

Note that by (29), x_i is bounded in the time interval $[0, t_2]$. By the boundedness of x_i and u_i, one can verify that there exist ε-independent positive constants N_{i1} and N_{i2} such that $\forall t \in [0, t_2]$,

$$|\dot{x}_{i,n+2}| \leq N_{i1} + N_{i2}\|\eta_i\|,\ 1 \leq i \leq N. \tag{32}$$

Since the observer gain L_i is selected such that the matrix E_i is Hurwitz, there exists an unique positive definite matrix solution $P_i \in \mathbb{R}^{(n+2)\times(n+2)}$ satisfying the equation $P_iE_i + E_i^{\mathrm{T}}P_i = -I$. Consider the Lyapunov function candidate $V_i(\eta_i) = \eta_i^{\mathrm{T}}P_i\eta_i$, and let σ_{i1} and σ_{i2} be the minimal and maximal eigenvalues of the matrix P_i, respectively. Then one has

$$\sigma_{i1}\|\eta_i\|^2 \leq V_i(\eta_i) \leq \sigma_{i2}\|\eta_i\|^2,\ \left|\frac{\partial V_i(\eta_i)}{\partial \eta_{i,n+2}}\right| \leq 2\sigma_{i2}\|\eta_i\|. \tag{33}$$

It follows from (31)-(33) that the derivative of $V_i(\eta_i)$ can be computed as

$$\begin{aligned}\frac{\mathrm{d}V_i(\eta_i)}{\mathrm{d}t} =& \frac{1}{\varepsilon}\left(\sum_{j=1}^{n+1}(\eta_{i,j+1} - l_{ij}\eta_{i1})\frac{\partial V_i(\eta_i)}{\partial\eta_{ij}}\right. \\ &\left.-l_{i,n+2}\eta_{i1}\frac{\partial V_i(\eta_i)}{\partial\eta_{i,n+2}}\right) + \dot{x}_{i,n+2}\frac{\partial V_i(\eta_i)}{\partial\eta_{i,n+2}} \\ \leq& -\frac{1}{\varepsilon}\|\eta_i\|^2 + 2\sigma_{i2}\left(N_{i1} + N_{i2}\|\eta_i\|\right)\|\eta_i\| \\ \leq& -\left(\frac{\sigma_{i1} - 2\sigma_{i2}^2N_{i2}\varepsilon}{\sigma_{i1}\sigma_{i2}\varepsilon}\right)V_i(\eta_i) \\ &+ \frac{2\sigma_{i2}N_{i1}}{\sqrt{\sigma_{i1}}}\sqrt{V_i(\eta_i)}.\end{aligned} \tag{34}$$

Considering $\frac{\mathrm{d}V_i(\eta_i)}{\mathrm{d}t} = 2\sqrt{V_i(\eta_i)}\frac{\mathrm{d}\sqrt{V_i(\eta_i)}}{\mathrm{d}t}$, the inequality above can be further written as

$$\frac{\mathrm{d}\sqrt{V_i(\eta_i)}}{\mathrm{d}t} \leq -\left(\frac{\sigma_{i1} - 2\sigma_{i2}^2N_{i2}\varepsilon}{2\sigma_{i1}\sigma_{i2}\varepsilon}\right)\sqrt{V_i(\eta_i)} + \frac{\sigma_{i2}N_{i1}}{\sqrt{\sigma_{i1}}}. \tag{35}$$

Let $\varepsilon < \frac{\sigma_{i1}}{2\sigma_{i2}^2N_{i2}}$. It follows from (33) and (35) that

$$\begin{aligned}\|\eta_i\| \leq& \frac{\sqrt{V_i(\eta_i)}}{\sqrt{\sigma_{i1}}} \\ \leq& \left(\frac{\sqrt{V_i(\eta_i(0))}}{\sqrt{\sigma_{i1}}} - \frac{2\sigma_{i1}\sigma_{i2}^2N_{i1}\varepsilon}{\sigma_{i1}\left(\sigma_{i1} - 2\sigma_{i2}^2N_{i2}\varepsilon\right)}\right) \\ &\times e^{-\frac{\sigma_{i1} - 2\sigma_{i2}^2N_{i2}\varepsilon}{2\sigma_{i1}\sigma_{i2}\varepsilon}t} + \frac{2\sigma_{i1}\sigma_{i2}^2N_{i1}\varepsilon}{\sigma_{i1}\left(\sigma_{i1} - 2\sigma_{i2}^2N_{i2}\varepsilon\right)}.\end{aligned} \tag{36}$$

Noting that $\sqrt{V_i(\eta_i(0))} \propto \frac{1}{\varepsilon^{n+1}}$, the righthand side of the inequality above converges to $O(\varepsilon)$ for any $t \geq -\varepsilon\ln\varepsilon^{n+2}$. Let ε_0 be sufficiently small such that for any $\varepsilon \in (0, \varepsilon_0)$, $\|\eta_i(t)\| = O(\varepsilon), \forall t \in [t_1, t_2]$. Thus one can select appropriate saturation bounds M_θ and M_ϑ such that θ_i and ϑ_i are out of saturation in the time interval $[t_1, t_2]$.

Let $W_i(s_i) = \frac{1}{2}s_i^2$. It follows from (28) that $\forall t \in [t_1, t_2]$,

$$\begin{aligned}\frac{\mathrm{d}W_i(s_i)}{\mathrm{d}t} =& s_i\left(\sum_{j=1}^{n-1}\varepsilon^{n-j+1}k_j\eta_{i,j+1} + \varepsilon\eta_{i,n+1}\right. \\ &\left.-\lambda s_i + \sum_{j=1}^{n-1}\lambda k_j\varepsilon^{n-j+2}\eta_{ij} + \lambda\varepsilon^2\eta_{in}\right) \\ \triangleq& -\lambda s_i^2 + \varpi(\eta_i, \varepsilon).\end{aligned} \tag{37}$$

Since $C_s + 1/2 \le |s_i(t)| \le C_s + 1$, $t \in [t_1, t_2]$, and $|\varpi(\eta_i, \varepsilon)| \to 0$ as $\varepsilon \to 0$, one can select sufficiently small $\varepsilon_1 \in (0, \varepsilon_0]$ such that for any $\varepsilon \in (0, \varepsilon_1)$, $\frac{\mathrm{d}W_i(s_i)}{\mathrm{d}t} < 0$, $\forall t \in [t_1, t_2]$. This contradicts (29). Thus there exists sufficiently small ε such that $\max_{1\le i\le N} |s_i(t)| \le C_s + 1$, $\forall t \in [0, T]$. It follows that

$$\|\omega(T)\| \le \frac{\sqrt{N}\|\delta(T)\|_\infty}{g(T)} \le \frac{2\sqrt{N}(C_s+1)}{g_0\gamma} \le \Phi. \quad (38)$$

What is more, by (25), one has

$$\left\| \frac{\Theta(T) - \widehat{\Theta}(0)}{g(0)} \right\|_\infty \le \frac{M_\theta}{g_o} < K + \frac{1}{2}. \quad (39)$$

Thus when $t = T$, $\|\omega(T)\| \le \Phi$, and the quantizer is unsaturated.

For any given integer $\alpha \ge 1$, suppose that $\|\omega(kT)\| \le \Phi$ and the quantizer is unsaturated for any $k = 0, 1, \ldots, \alpha$. Next, we show that $\|\omega((\alpha+1)T)\| \le \Phi$, and the quantizer is unsaturated when $k = \alpha + 1$.

Since $\|\omega(kT)\| \le \Phi$ for $k = 0, 1, \ldots, \alpha$, and ϑ_i is bounded by an ε-independent positive constant, it can be readily verified that x_i is bounded by some ε-independent positive constant in the time interval $[0, (\alpha+1)T]$. Similar to (30)-(36), one can select sufficiently small ε and appropriate saturation bounds such that θ_i and ϑ_i are out of saturation, and $\|\eta_i\| = O(\varepsilon)$, $\forall t \in [\alpha T\ (\alpha+1)T]$.

Consider three cases:

Case 1): $g(\alpha T) = g_0\gamma^\alpha$, $g((\alpha+1)T) = g_0\gamma^{\alpha+1}$. In this case, by (16)-(19), one has

$$\begin{cases} \omega((\alpha+1)T) = \gamma^{-1}(I - T\mathcal{L})\omega(\alpha T) + \gamma^{-1}T\mathcal{L}z(\alpha T) \\ \qquad + \dfrac{(I - J_N)(S(\alpha T) - \Theta(\alpha T) + \Pi(\alpha T))}{g_0\gamma^{\alpha+1}}, \\ z((\alpha+1)T) = \gamma^{-1}\left(\Delta((\alpha+1)T) - Q(\Delta((\alpha+1)T))\right), \end{cases} \quad (40)$$

where

$$\begin{aligned} \Delta((\alpha+1)T) =& (I + T\mathcal{L})z(\alpha T) - T\mathcal{L}\omega(kT) + \widetilde{\Pi}(\alpha T), \\ \widetilde{\Pi}(\alpha T) =& \frac{1}{g(\alpha T)}\left[(I + T\mathcal{L})(S(\alpha T) - \Theta(\alpha T))\right. \\ & \left. + \Pi(\alpha T) + \Theta((\alpha+1)T) - S((\alpha+1)T)\right]. \end{aligned}$$

Note that $\forall t \in [\alpha T\ (\alpha+1)T]$,

$$s_i(kT) - \theta_i(kT) = \sum_{j=1}^{n-1} k_j \varepsilon^{n+2-j}\eta_{ij}(kT) + \varepsilon^2\eta_{in}(kT),$$

$$\pi_i(kT) = \int_{kT}^{(k+1)T} \left(\sum_{j=1}^{n-1} k_j\varepsilon^{n+1-j}\eta_{i,j+1} + \varepsilon\eta_{i,n+1} \right) \mathrm{d}t.$$

Thus $\|s(\alpha T) - \Theta(\alpha T)\| = O(\varepsilon^3)$, $\|s((\alpha+1)T) - \Theta((\alpha+1)T)\| = O(\varepsilon^3)$, and $\Pi(\alpha T) = O(\varepsilon^2)$, and as a consequence $\|\widetilde{\Pi}(\alpha T)\| = O(\varepsilon^2)$. Since $\|\omega(\alpha T)\| \le \Phi$, $\|z(\alpha T)\|_\infty \le \frac{1}{2\gamma}$, and $\|\mathcal{L}\| \le \lambda_N(\mathcal{L})$, it follows from (40) that

$$\begin{aligned} \|\omega((\alpha+1)T)\| \le& \left\|\gamma^{-1}(I - T\mathcal{L})\omega(\alpha T)\right\| \\ & + \left\|\gamma^{-1}T\mathcal{L}z(\alpha T)\right\| \\ & + \left\| \frac{(I - J_N)(S(\alpha T) - \Theta(\alpha T) + \Pi(\alpha T))}{g_0\gamma^{\alpha+1}} \right\| \\ \le& \frac{\rho_h}{\gamma}\max\left\{ \frac{2\sqrt{N}(C_s+1)}{g_0\gamma}, \frac{T\sqrt{N}\lambda_N(\mathcal{L})}{2\gamma(\gamma - \rho_h)} + \frac{1}{4\gamma} \right\} \\ & + \frac{T\sqrt{N}\lambda_N(\mathcal{L})}{2\gamma^2} + O(\varepsilon^2) \\ \le& \frac{\rho_h}{\gamma}\left(\frac{T\sqrt{N}\lambda_N(\mathcal{L})}{2\gamma(\gamma - \rho_h)} + \frac{1}{4\gamma} \right) \\ & + \frac{T\sqrt{N}\lambda_N(\mathcal{L})}{2\gamma^2} + O(\varepsilon^2) \\ =& \frac{T\sqrt{N}\lambda_N(\mathcal{L})}{2\gamma(\gamma - \rho_h)} + \frac{1}{4\gamma} + \frac{\rho_h - \gamma}{4\gamma^2} + O(\varepsilon^2). \end{aligned} \quad (41)$$

Thus one can select sufficiently small ε such that $\frac{\rho_h - \gamma}{4\gamma^2} + O(\varepsilon^2) \le 0$, and consequently $\|\omega(\alpha+1)T\| \le \Phi$. What is more,

$$\begin{aligned} \|\Delta((\alpha+1)T)\|_\infty \le& \|(I + T\mathcal{L})\|_\infty\|z(\alpha T)\|_\infty \\ & + T\|\mathcal{L}\|\|\omega(\alpha T)\| + \|\widetilde{\Pi}(kT)\| \\ \le& \frac{1 + 2Td^*}{2\gamma} + T\lambda_N(\mathcal{L}) \\ & \times \left(\frac{T\sqrt{N}\lambda_N(\mathcal{L})}{2\gamma(\gamma - \rho_h)} + \frac{1}{4\gamma} \right) + O(\varepsilon^2) \\ \le& \frac{1 + 2Td^*}{2\gamma} + \frac{\sqrt{N}T^2\lambda_N^2(\mathcal{L})}{2\gamma(\gamma - \rho_h)} \\ & + \frac{1}{2\gamma}T\lambda_N(\mathcal{L}) - \frac{1}{4\gamma}T\lambda_N(\mathcal{L}) + O(\varepsilon^2). \end{aligned} \quad (42)$$

Similarly, one can select sufficiently small ε such that $-\frac{1}{4\gamma}T\lambda_N(\mathcal{L}) + O(\varepsilon^2) \le 0$. It follows that

$$\|\Delta((\alpha+1)T)\|_\infty \le K_1(T, \gamma) < K + \frac{1}{2}. \quad (43)$$

Case 2): $g(\alpha T) = g_0\gamma^\alpha$, $g((\alpha+1)T) = \varepsilon$. In this case, one has

$$\begin{cases} \omega((\alpha+1)T) = \dfrac{g_0\gamma^\alpha}{\varepsilon}(I - T\mathcal{L})\omega(\alpha T) + \dfrac{g_0\gamma^\alpha}{\varepsilon}T\mathcal{L}z(\alpha T) \\ \qquad + \dfrac{(I - J_N)(S(\alpha T) - \Theta(\alpha T) + \Pi(\alpha T))}{\varepsilon}, \\ z((\alpha+1)T) = \dfrac{g_o\gamma^\alpha}{\varepsilon}\left(\Delta((\alpha+1)T) - Q(\Delta((\alpha+1)T))\right). \end{cases}$$

Case 3): $g(\alpha T) = g((\alpha+1)T) = \varepsilon$. In this case, $\varepsilon \ge g_0\gamma^\alpha$, and

$$\begin{cases} \omega((\alpha+1)T) = (I - T\mathcal{L})\omega(\alpha T) + T\mathcal{L}z(\alpha T) \\ \qquad + \dfrac{(I - J_N)(S(\alpha T) - \Theta(\alpha T) + \Pi(\alpha T))}{\varepsilon}, \\ z((\alpha+1)T) = \Delta((\alpha+1)T) - Q(\Delta((\alpha+1)T)). \end{cases}$$

Note that in Case 2), $\frac{g_0\gamma^\alpha}{\varepsilon} \le \frac{1}{\gamma}$, and in Case 3), $1 < \frac{1}{\gamma}$. Thus similar to Case 1), one can verify that in Cases 2) and 3), $\|\omega((\alpha+1)T)\| \le \Phi$, and the quantizer is unsaturated when $k = \alpha+1$. Thus for any $t = 0, T, 2T, \ldots$, $\|\omega(t)\| \le \Phi$, and the quantizer will never be saturated. This completes the proof of Lemma 3.

Based on Lemma 3, we are in a position to state our main result.

Theorem 1: Consider the closed-loop system formed by (1), (5), (8), (9), and (10). Suppose Assumptions A1 and A2 hold, and T, γ, K, $g(kT)$, and g_0 are taken as in Lemma 3. Then there exists $\varepsilon^\dagger > 0$ such that for any $\varepsilon \in (0, \varepsilon^\dagger)$ and $t_0 > 0$,

$$\lim_{\varepsilon\to 0} |x_{ij}(t) - \widehat{x}_{ij}(t)| = 0,\ 1 \le i \le N,\ 1 \le j \le n+2, \quad (44)$$

uniformly in $t \in [t_0, \infty)$, and

$$\lim_{\varepsilon\to 0, t\to\infty} (x_i - x_j) = 0. \quad (45)$$

Proof: By Lemma 3, for any $\varepsilon \in (0, \varepsilon^*)$ and $k = 0, 1, \ldots$, $\|\omega(kT)\| \le \Phi$. Then following the same line of (30)-(36), one can obtain that for any $t_0 > 0$, there exists $\varepsilon^\dagger \in (0, \varepsilon^*]$ such that for any $\varepsilon \in (0, \varepsilon^\dagger)$, (44) holds uniformly in $t \in [t_0, \infty)$. On the other hand, by (18), (24), and the boundedness of $\omega(kT)$, one has $\lim_{\varepsilon\to 0, k\to\infty} \|\delta(kT)\| = 0$, and as a result $\lim_{\varepsilon\to 0, k\to\infty} s_i(kT) = \Lambda$ for some constant Λ. What is more, for sufficiently large k and $t \in [kT, (k+1)T)$, one has

$$\begin{aligned} s_i(t) - s_i(kT) = (t - kT) \sum_{j\in\mathcal{N}_i} a_{ij} \left(\widehat{\theta}_{ji}(kT) - \xi_i(kT)\right) \\ + \int_{kT}^{t} \left(\sum_{j=1}^{n-1} k_j \varepsilon^{n+1-j} \eta_{i,j+1} + \varepsilon \eta_{i,n+1} \right) \mathrm{d}t. \end{aligned} \quad (46)$$

By (46), one has that $s_i(t) - s_i(kT) \to 0$ as $k \to \infty$ and $\varepsilon \to 0$, $\forall t \in [kT, (k+1)T)$. Thus $\lim_{\varepsilon\to 0, t\to\infty} s_i = \Lambda$. Finally, by Lemma 2, (45) holds. This completes the proof of Theorem 1.

From Theorem 1, one can notice that the number of the quantization levels, $2K+1$, increases to infinity as $N \to \infty$. However, in practice the bit rate is generally limited. Next theorem shows that no matter how many agents there are, the proposed ESO based protocol guarantees consensus with a fixed number of quantization levels.

Theorem 2: Consider the closed-loop system formed by (1), (5), (8), (9), and (10). Suppose Assumptions A1 and A2 hold. For any given $K \ge 1$, let $g(kT)$ and g_0 be taken, respectively, as in (24) and (25), and

$$T \in \left(0, \min\left\{2/\left(\lambda_2(\mathcal{L}) + \lambda_N(\mathcal{L})\right), T_m\right\}\right), \quad (47)$$

$$\begin{aligned} T_m =& 2K\epsilon_0\lambda_2\big(\sqrt{N}\lambda_N^2(\mathcal{L}) + 2\epsilon_0\lambda_2(\mathcal{L})d^* \\ &+ \epsilon_0\lambda_2(\mathcal{L})\lambda_N(\mathcal{L}) + (2K+1)(1-\epsilon_0)\epsilon_0\lambda_2^2(\mathcal{L})\big)^{-1}, \end{aligned} \quad (48)$$

$$\gamma = 1 - (1-\epsilon_0)T\lambda_2(\mathcal{L}), \quad (49)$$

where $\epsilon_0 \in (0,1)$. Then there exists $\varepsilon^\dagger > 0$ such that for any $\varepsilon \in (0, \varepsilon^\dagger)$ and $t_0 > 0$, (44) holds uniformly in $t \in [t_0, \infty)$ and (45) holds.

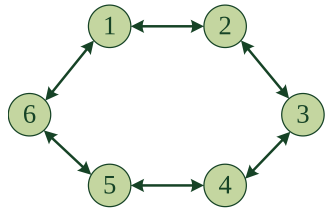

Fig. 1: Communication network $\mathcal{G}$.

Proof: By (47), (48), and (49), it can be verified that the conditions (20) and (21) are satisfied, and

$$\frac{1}{2} < K_1(T, \gamma) < K + \frac{1}{2}. \quad (50)$$

It follows from (50) that the condition (22) holds. This together with Theorem 1 yields the conclusion of Theorem 2.

Remark 1: In general, for an nth order system, the ESO is of the order of $n+1$ [19, 20]. In the proposed approach, we adopt a higher-order ESO to estimate the system state x_i and the uncertain nonlinear function $x_{i,n+1} = f_i(x_i)$. From the proof of Lemma 3, one can observe that the ESO estimation error is incorporated into the quantizer (see (40)). For the $(n+2)$th order ESO (5), the estimation error of $[x_i^{\mathrm{T}}, x_{i,n+1}]^{\mathrm{T}}$ is of the order of $O(\varepsilon^2)$, while for a standard ESO, the error is of the order of $O(\varepsilon)$. Note that the scaling function $g(kT)$ converges to ε as $k \to \infty$. Thus we use a higher-order ESO rather than a standard one to handle the stubborn effects of the estimation error in the quantization process.

Remark 2: Note that the results in Theorem 2 show that for a connected undirected network with nth order uncertain nonlinear agents, no matter how many agents there are, one can always design an ESO based distributed protocol to guarantee consensus with only one bit information exchange between agents. From a theoretical perspective, the proposed approach achieves the lowest data rate via output feedback. In the literature, the least data rate needed for first-order multi-agent systems is one bit [10], for second-order multi-agent systems is two bits [11], and for nth order multi-agent systems is n bits [14].

4 Example

Consider a group of uncertain second-order pendulum systems [24, 25]:

$$\begin{cases} \dot{x}_{i1} = x_{i2}, \\ \dot{x}_{i2} = -c_i \sin(x_{i1}) - d_i x_{i2} + u_i, \\ y_i = x_{i1},\ i = 1, \ldots, 6, \end{cases} \quad (51)$$

where $c_i = 2i - 1$ and $d_i = 0.2i$. The communication network among the six agents is described by Fig. 1. We point out that for these pendulum systems with limited communication data rate, no existing results are applicable to solve the distributed consensus problem. Specifically, the approaches in [10, 13–15] are for linear systems, and the approach in [18] is based on full state feedback.

The initial conditions of (51) are assumed to be randomly distributed in $[-5, 5]$. The ESO is designed with $L_i = [4\ 6\ 6\ 1]^{\mathrm{T}}$ and $\varepsilon = 0.01$. Here we adopt a one-bit quantizer (*i.e.*, $K = 1$). Let $C_x = 5$, and the parameters

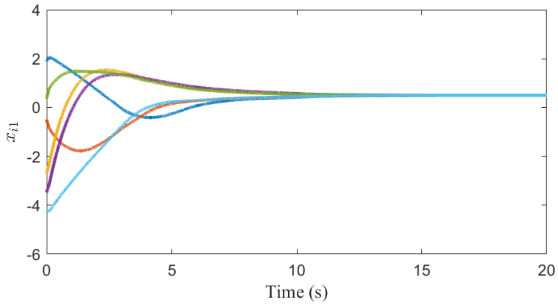

Fig. 2: Time response of x_{i1}.

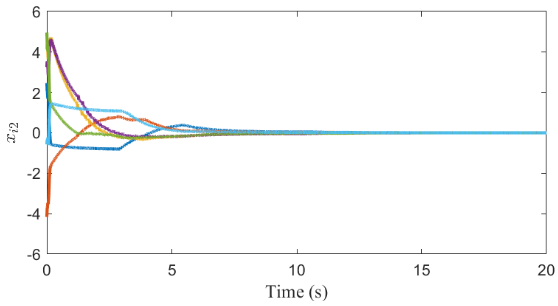

Fig. 3: Time response of x_{i2}.

$k_1 = 1$, $C_s = 10$, $M_\theta = 11$, $M_\vartheta = 15$, $\epsilon_0 = 0.5$, $\lambda = 1$, $\gamma = 0.99$, $T = 0.02$, and $g_0 = 8.41$. The time response of the states of the six agents are depicted in Figs. 2 and 3. It can be observed that the uncertain nonlinear multi-agent system (51) achieves consensus with a one-bit quantizer.

5 Conclusion

In this paper, we explored the data rate problem for distributed consensus of uncertain nonlinear multi-agent systems via output feedback. A higher-order ESO based encoding-decoding scheme was proposed, and a distributed protocol was given in terms of the states of the observers, encoders, and decoders. It was proved that, for a connected undirected network with nth order uncertain nonlinear agents, the proposed output feedback protocol guarantees consensus with only one bit information exchange between agents. An example has been presented to verify the obtained results.

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