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OF TECHNOLOGY

Attitude Control of Multi-rigid-body Systems: from Synchronization to Intrinsic Formation

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Joint work with Zhixin Liu, Johan Thunberg, Wenjun Song, Silun Zhang, and Yiguang Hong

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Outline

- Multi-agent systems
- Rigid body systems
- Leader-follower synchronization of orientation
- Full attitude synchronization
- Intrinsic formation control for reduced attitude

Multi-agent systems

A number of different entities ("agents") equipped with homogeneous or heterogeneous dynamics

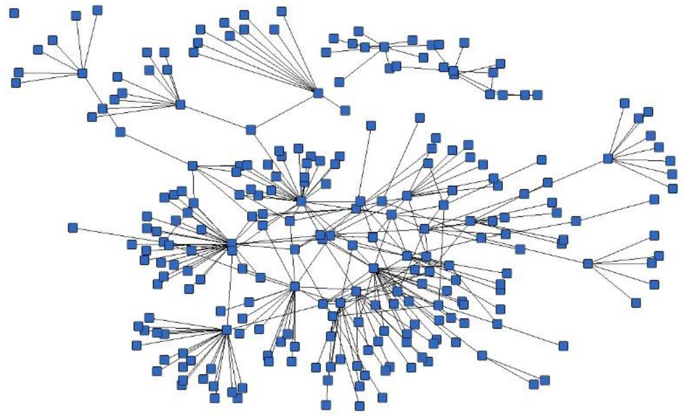
$$\dot{x}_i = f_i(x_i, u_i), \mathrm{x_i} \in R^{n_i}, i = 1, \cdots, N$$
$$y_i = h_i(x_i, x_j | j \in N_i)$$

Information exchange is limited to specific neighboring agents according to a "neighboring" graph

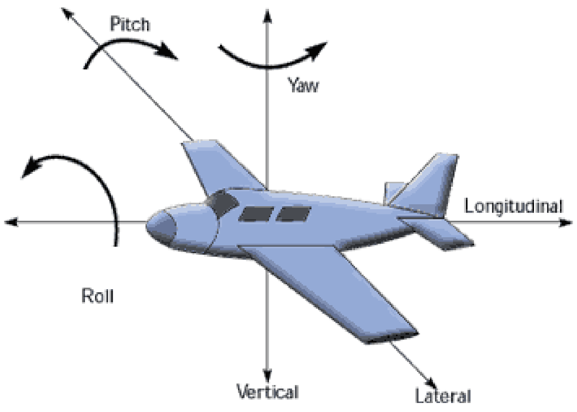
and usually only relative information

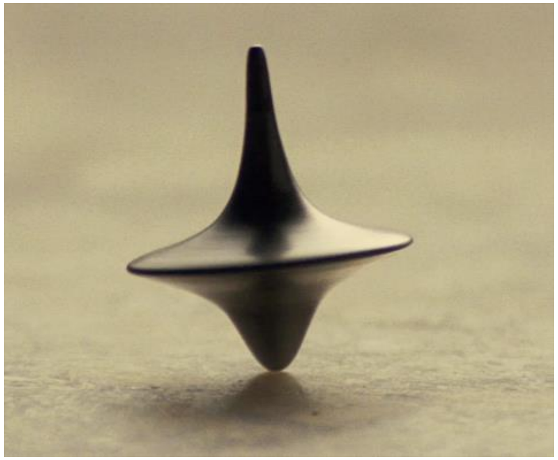
Is available

Some global emergence is the goal

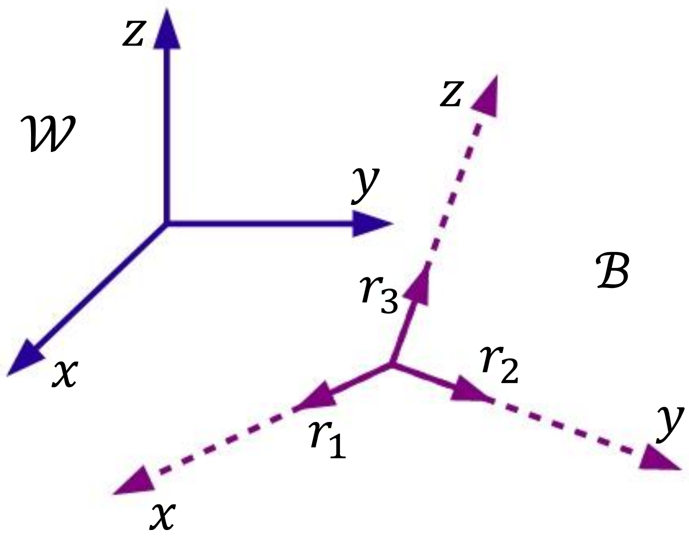


Rigid body systems





Rigid motion



State Space		
DOF	Translation	Rotation
3	$\mathbb{R}^3$	$SO(3)$
2	$\mathbb{R}^2$	S^2
1	$\mathbb{R}^1$	S^1

Attitude: $R = [r_1 \;\; r_2 \;\; r_3] \in SO(3)$

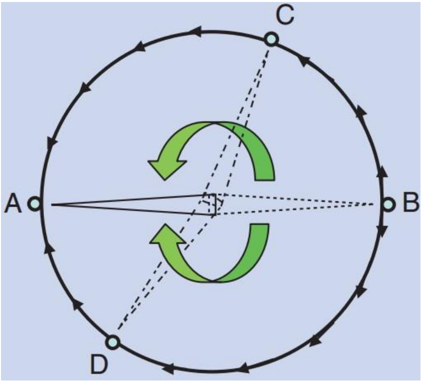
3D rotation group: $SO(3) = \{R \in \mathbb{R}^{3\times3}: R^T R = I, \det(R) = 1\}$

2-Sphere: $S^2 = \{x \in \mathbb{R}^3: x^T x = 1\}$

Attitude control vs position control

$SO(3), S^2, S^1$: compact (closed) manifold ➡ not contractible

Topological obstruction



continuous time-invariant attitude feedback

⬇

multiple closed-loop equilibria

It is impossible to stabilize the attitude globally using continuous time-invariant feedback.

The one-dimensional case: Synchronization of orientation

Main collaborator:

Zhixin Liu, Chinese Academy of Sciences

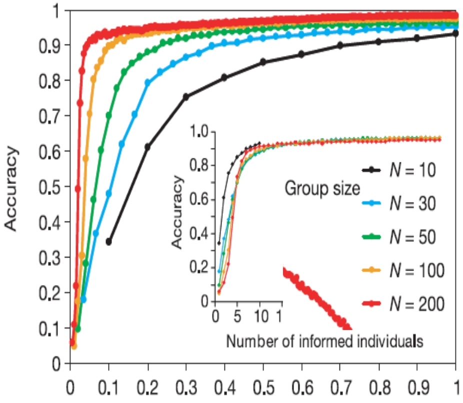
Z. Liu, L. Wang, J. Wang, D. Dong, X. Hu, Distributed sampled-data control of nonholonomic multi-robot systems with proximity networks, *Automatica*, vol. 77, 2017

Z. Liu, J. Han and X. Hu, The number of leaders needed for the expected consensus, *Automatica,* vol. 47, 2011

Leader-follower model

Role of "leaders":

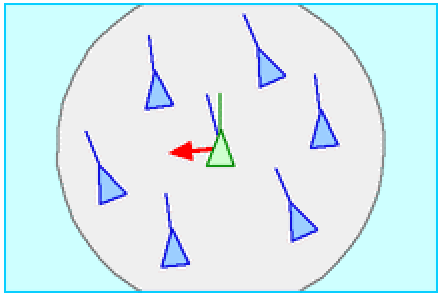
- have information about the desired state (destination)
- Improve accuracy of group motion
- For a given accuracy, the proportion of leaders needed is a nonlinear function of the total population



Proportion of informed individuals

Leader-follower model

Consider the problem of orientation synchronization for rigid body agents moving in 2D



The question is:

How many leaders are needed for the expected behavior?

To answer the above question,

We focus on the intervention of the flocking model, and provide lower bounds for the ratio of leaders to followers.

The continuous time case

The dynamics of both leaders and followers obey the following unicycle model

$$\begin{cases} \dot{x}_i(t) = v_i(t)\cos\theta_i(t) \\ \dot{y}_i(t) = v_i(t)\sin\theta_i(t) \\ \dot{\theta}_i(t) = \omega_i(t) \\ \dot{v}_i(t) = u_i(t) \end{cases} \qquad i = 1, \dots, n$$

where $\omega_i(t)$ and $u_i(t)$ are control inputs.

Compared with followers, the leaders have the information of the desired orientation and desired velocity

Leader-follower synchronization of orientation

Sampled-data control design for leaders for $t \in [t_k, t_{k+1})$

$$\begin{cases} \omega_i(t) = \dfrac{1}{\tau_n}\left\{\mu\big(\theta_n - \theta_i(t_k)\big) + \dfrac{1-\mu}{d_i(t_k)} \sum_{j \in N_i(t_k)} \big(\theta_j(t_k) - \theta_i(t_k)\big)\right\} \\ u_i(t) = \dfrac{1}{\tau_n}\left\{\mu\big(\mathrm{v_n} - v_i(t_k)\big) + \dfrac{1-\mu}{d_i(t_k)} \sum_{j \in N_i(t_k)} \big(v_j(t_k) - v_i(t_k)\big)\right\} \end{cases}$$

Sampled-data control law design for followers for $t \in [t_k, t_{k+1})$

$$\begin{cases} \omega_i(t) = \dfrac{1}{\tau_n d_i(t_k)} \sum_{j \in N_i(t_k)} \big(\theta_j(t_k) - \theta_i(t_k)\big) \\ u_i(t) = \dfrac{1}{\tau_n d_i(t_k)} \sum_{j \in N_i(t_k)} \big(v_j(t_k) - v_i(t_k)\big) \end{cases}$$

$N_i(t) = \left\{j: \left\|X_i(t) - X_j(t)\right\| \le r_n\right\}$, $X_i(t) = (x_i(t), y_i(t))^T$

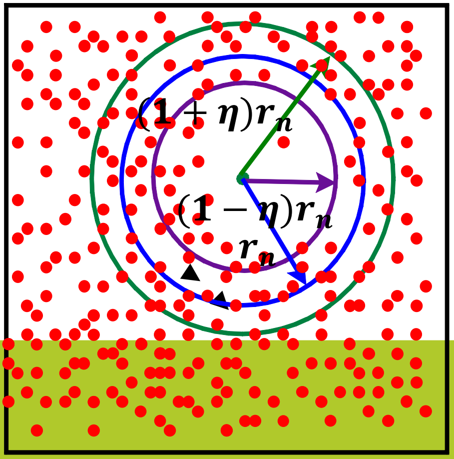
Leader-follower synchronization of orientation

Theorem: If the ratio of the number of leaders to the number of followers satisfies

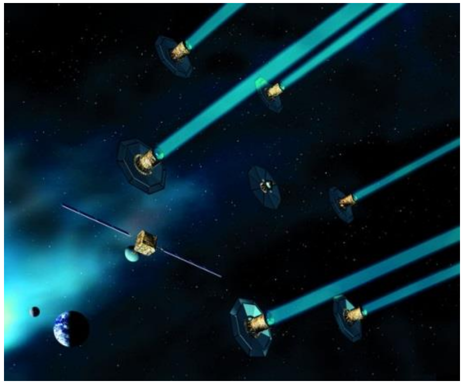
1) $\alpha_n \geq \frac{8 v_n \tau_n (1+\bar{\theta}_0)(1+o(1))}{\mu \eta r_n}$, provided that $v_n \tau_n \gg \frac{\log n}{n r_n}$;

2) $\alpha_n \geq \frac{\log n}{n r_n^2}$ provided that $v_n \tau_n \ll \frac{\log n}{n r_n}$ or $v_n \tau_n = \Theta\left(\frac{\log n}{n r_n}\right)$,

then all agents move with the desired speed v_n and orientation θ_n asymptotically.



Full attitude synchronization



Main collaborator:

Johan Thunberg, University of Luxembourg

J. Thunberg, W. Song, E. Montjano, Y. Hong, X. Hu, Distributed At-titude Synchronization Control of Multi-Agent Systems with Switching Topologies, *Automatica*, vol. 50, 2014

J. Thunberg, J. Goncalves, X. Hu, Consensus and formation control on SE (3) for switching topologies, *Automatica*, vol. 66, 2016

Rotational motion

	Full Attitude	Reduced Attitude
Notation	$R \in SO(3)$	$\Gamma = Rb \in S^2$
Kinematics	$\dot{R} = \widehat{\omega} R$	$\dot{\Gamma} = \widehat{\omega} \Gamma$

$$\widehat{\omega} = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix}$$

Local representations of rotation

We consider a broad class of local representations for the rotations. These are on the forms

$$y_i = \left(f(R_i)\right)^\vee = g(\theta_i)u_i \quad \text{and} \qquad f(\cdot) \text{ skew symmetric}$$

$$y_{ij} = \left(f(R_{ij})\right)^\vee = g(\theta_{ij})u_{ij} \quad \text{where} \quad R_{ij} = R_j^T R_i$$

$$\begin{bmatrix} 0 & -y_3 & y_2 \\ y_3 & 0 & -y_1 \\ -y_2 & y_1 & 0 \end{bmatrix}^\vee = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \text{ and } \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}^\wedge = \begin{bmatrix} 0 & -y_3 & y_2 \\ y_3 & 0 & -y_1 \\ -y_2 & y_1 & 0 \end{bmatrix}.$$

Some examples

- Axis-Angle Representation, where , $f(R_i) = \mathrm{Log}(R_i)$
- Rodrigues Parameters, where $f(R_i) = (R_i - I)(R + I)^{-1},$
- Modified Rodrigues Parameters, where
  $$f(R_i) = (R_i - I)^2(R + I)^{-2},$$
- sin-representation, where $f(R_i) = R_i - R_i^T,$
- Unit quaternions (or rather parts of it).

Axis-Angle Representation

Rodrigues' formula:

$$R = e^{\theta \widehat{n}} = I + \widehat{n} \sin\theta + (\widehat{n})^2 (1 - \cos\theta)$$

Then,

$$\theta = arccos(\tfrac{1}{2} tr(R) - 1), \ \widehat{n} = \frac{1}{2\sin\theta}(R - R^T)$$

From this we can define

$$Log\ R = \frac{\theta}{2\sin\theta}(R - R^T)$$

$y = \theta n$ Axis-Angle representation

Kinematics in Axis-angle

Let us denote $y_i = \theta_i n_i$, then

$$\dot{y}_i = L_{y_i} \omega_i$$

where $L_{y_i} = L_{\theta_i n_i} = I_3 + \frac{\theta_i}{2}\widehat{n}_i + \left(1 - \frac{\operatorname{sinc}(\theta_i)}{\operatorname{sinc}^2(\frac{\theta_i}{2})}\right)\widehat{n}_i^2.$

Synchronizing Control

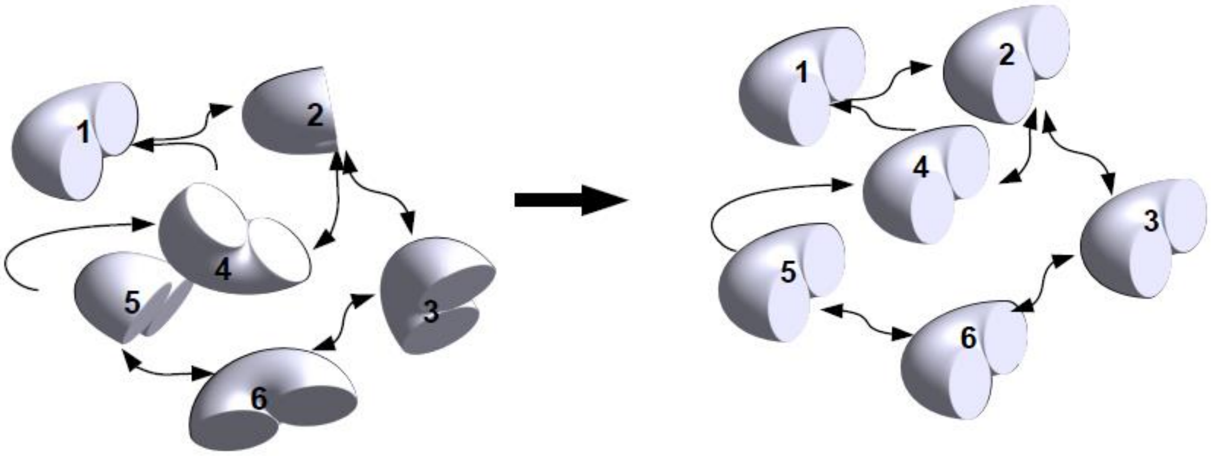
Now given N rigid-body agents and we want to synchronize their attitude:

$$R_1 = R_2 = \cdots = R_N$$

or equivalently in axis-angle representation or any other representation

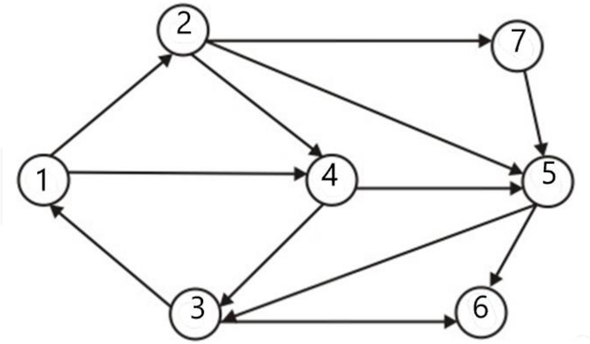
$$y_1 = y_2 = \cdots = y_N$$

as $t \to \infty$

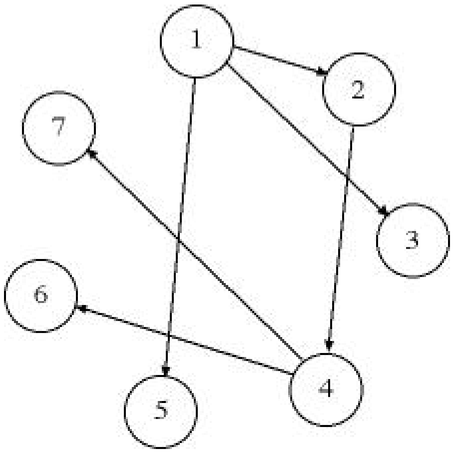


Graph connectivity

The type of connectivity between the agents (or rather connectivity of the graph) plays an important role for convergence. We need two types of connectivity.



Strongly connected



Quasi-strongly connected

Result 1

Feedback control law

$$\omega_i = \sum_{j \in \mathcal{N}_i(t)} a_{ij}(y_j - y_i)$$

Uniformly strongly connected ➡

Up to almost globally attractive to the consensus manifold

Result 2

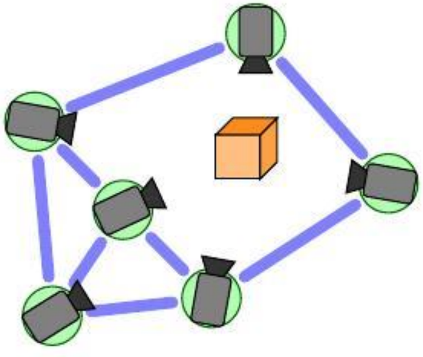
Feedback control law

$$\omega_i = \sum_{j \in \mathcal{N}_i(t)} a_{ij} y_{ij}$$

Uniformly quasi-strongly connected ⇒

Locally uniformly asymptotically stable to the consensus manifold

Intrinsic formation control for reduced attitude



Main collaborators:

Wenjun Song, Beijing KuWeather Science and Technology
Silun Zhang, KTH Royal Institute of Technology

W. Song, J. Markdahl, S. Zhang, X. Hu, Y. Hong, Intrinsic reduced attitude formation with ring inter-agent graph, *Automatica*, vol. 85, 2017

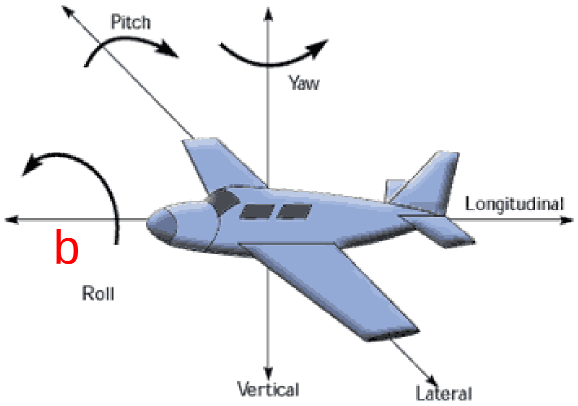
S. Zhang, W. Song, F. He, Y. Hong, X. Hu, Intrinsic tetrahedron formation of reduced attitude, *Automatica*, vol. 87, 2018

S. Zhang, F. He, Y. Hong, X. Hu, Intrinsic Formation Control of Regular Polyhedra for Reduced Attitudes, Proc. CDC, 2017

Rotational motion

	Full Attitude	Reduced Attitude
Notation	$R \in SO(3)$	$\Gamma = Rb \in S^2$
Kinematics	$\dot{R} = \hat{\omega} R$	$\dot{\Gamma} = \hat{\omega}\Gamma$

$$\hat{\omega} = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix}$$



Motivation of studying reduced attitude

- Easy to visualize
- Pointing applications

Cooperative reduced attitude control

Consider the system

$$\dot{\Gamma}_i = \hat{\omega}_i \Gamma_i, \qquad i = 1,2,\dots,n$$

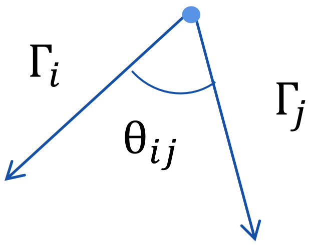
$\Gamma_i \in S^2$ is the reduced attitude of agent i

- control at kinematic level
- information exchange $\mathcal{G} = (\mathcal{V}, \mathcal{E})$
- available information: $\{\hat{\Gamma}_i \Gamma_j : j \in \mathcal{N}_i\}$

Objective: make $\Gamma_1, \Gamma_2, \dots, \Gamma_n$ reach

consensus or

a desired formation on the sphere



$|\hat{\Gamma}_i \Gamma_j| = |\sin \theta_{ij}|$

$$\hat{\Gamma}_i \Gamma_j = \Gamma_i \times \Gamma_j$$

Cooperative reduced attitude control

Consensus

$$\omega_i = \sum_{j \in \mathcal{N}_i} \hat{\Gamma}_i \Gamma_j, \qquad i = 1,2,\ldots,n$$

A sufficient condition to reach consensus

- $\mathcal{G}$ is strongly connected
- $\Gamma_1(0), \Gamma_2(0), \ldots, \Gamma_n(0)$ lie on the surface of an open hemisphere

Formation

If the desired reference formation is not available to the agents, is it possible to achieve it with only relative attitude information?

Intrinsic reduced attitude formation

The geometry of $(S^2)^n$ makes the closed-loop system

$$\dot{\Gamma}_i = -\hat{\Gamma}_i \sum_{j \in \mathcal{N}_i} \hat{\Gamma}_i \Gamma_j, \qquad i = 1,2,\dots,n$$

have multiple disjoint equilibrium sets

- The consensus manifold is an intrinsic equilibrium set
- Other equilibrium sets vary according to the inter-agent graph $\mathcal{G}$

Is it possible to achieve a desired formation by imposing some proper inter-agent graph to the system and then making that (intrinsic) formation asymptotically stable?

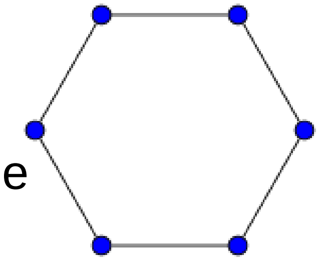
Intrinsic reduced attitude formation

Reverse the sign of the consensus protocol

$$\dot{\Gamma}_i = \hat{\Gamma}_i \sum_{j \in \mathcal{N}_i} \hat{\Gamma}_i \, \Gamma_j, \qquad i = 1,2,\ldots,n$$

- Let $\Gamma_1^*, \Gamma_2^*, \ldots, \Gamma_n^*$ be an equilibrium
- Suppose $\mathcal{G}$ is an undirected ring (cycle)

⇒ $\Gamma_1^*, \Gamma_2^*, \ldots, \Gamma_n^*$ must lie on a great circle



undirected ring graph

Different formations are achieved depending on if n is even or odd

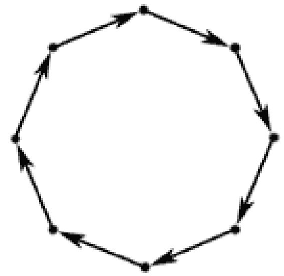
Antipodal formation

n is even

- Asymptotically stable equilibria:

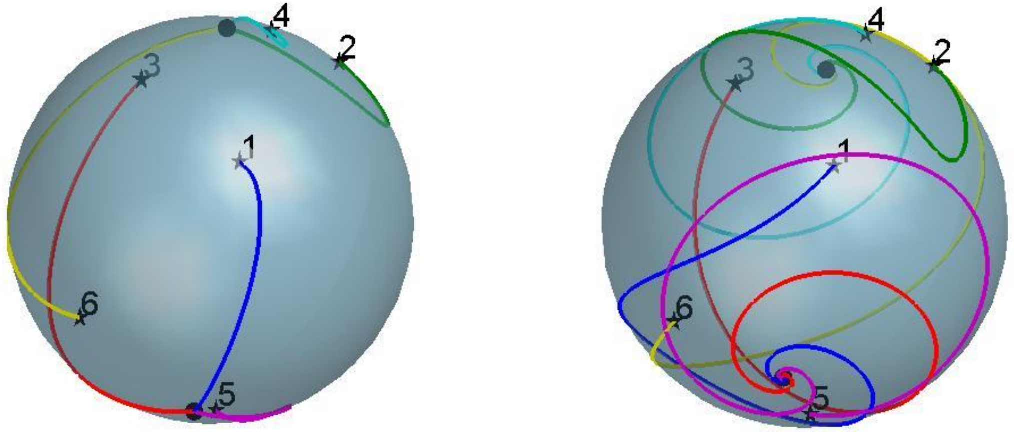
$$\Gamma_1^* = -\Gamma_2^* = \cdots = \Gamma_{n-1}^* = -\Gamma_n^*$$

the region of attraction is almost all $\left(S^2\right)^n$.



directed ring graph

- Same formation can also be achieved when $\mathcal{G}$ is a directed ring



Trajectories under undirected/directed ring inter-agent graph

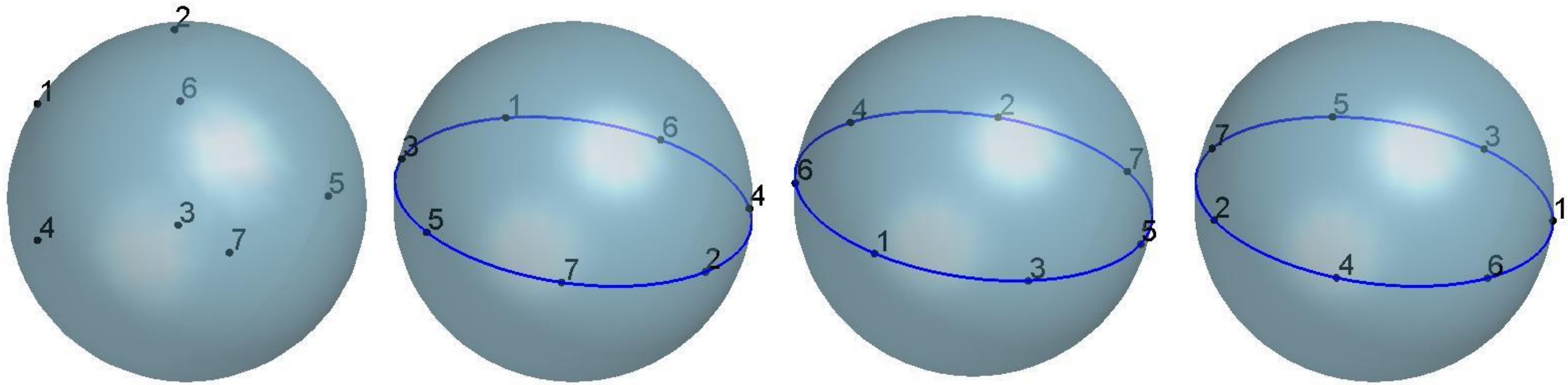
Cyclic formation

n is odd

- Asymptotically stable equilibria: $\Gamma_i^* = \exp\left(\left(\pi - \frac{\pi}{n}\right)\hat{u}\right)v$, $u, v \in S^2, u^T v = 1$

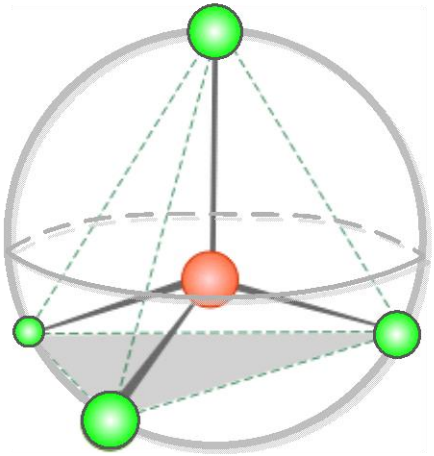
  the region of attraction is almost all $\left(S^2\right)^n$.

- Rotating cyclic formation can be achieved when $\mathcal{G}$ is a directed ring



t=0 (sec) t=40 (sec) t=45 (sec) t=50 (sec)

Regular tetrahedron formation



Expected formation: Regular tetrahedron

$$\Omega_T = \{\Gamma \in (S^2)^4 : \Gamma_i^T \Gamma_j = -\frac{1}{3}, \forall i \neq j\}$$

Let $\mathcal{G}$ be complete.

Equilibria by applying the former antipodal/cyclic control:

$$\left\{\Gamma \in (S^2)^4 : \sum_{i=1}^{4} \Gamma_i = 0 \right\}$$

- It is a connected submanifold of $(S^2)^4$, so Ω_T is not asymptotically stable.

Regular tetrahedron formation

Redesign the control as:

$$\omega_i = -\sum_{j \in \mathcal{N}_i} f(\theta_{ij})\hat{\Gamma}_i \Gamma_j, \qquad i \in \{1,2,3,4\} \qquad (*)$$

where $f: [0, \pi] \to \mathbb{R}$ and $\theta_{ij} = \arccos(\Gamma_i^T \Gamma_j)$.

Theorem:

Under the control (*) , the regular tetrahedron formation manifold Ω_T is almost globally asymptotically stable if $f(\cdot)$ satisfying

$$f(\cdot)>0 , \ \dot{f}(\cdot) <0 \ \text{ on } [0, \pi] .$$

Regular tetrahedron formation

Idea of proof:

(a) With the proposed control, the equilibria set of the closed-loop system is

$$\Omega = \Omega_T \cup \Omega_L,$$

where $\Omega_T = \{\Gamma \in (S^2)^4 : \Gamma_i^T \Gamma_j = -\frac{1}{3},\ \forall i \neq j\}$ ➡ **Regular tetrahedron**

$$\Omega_L = \{\Gamma \in (S^2)^4 : \hat{\Gamma}_i \Gamma_j = 0,\ \forall i \neq j\}$$ ➡ **Mutually parallel**

(b) When $t \to \infty$, the trajectory converges to an equilibrium. Meanwhile any $\mathrm{x} \in \Omega_L$ is anti-stable.

(c) Ω_T is locally asymptotically stable.

Regular tetrahedron formation

To prove Ω_T is locally asymptotically stable

- We first do some coordinate change and then we can show that, $\forall x_0 \in \Omega_T$, the spectrum of linearization $A_\eta^{x_0}$ satisfies

$$\lambda_i \in C^- \qquad i = 1,2,\cdots,5$$
$$\lambda_i \in C^0 \qquad i = 6,7,8$$

→ **Exist a 3-D center manifold**

- With a further coordinate change, we can show that the center manifold of the original system is exactly Ω_T.

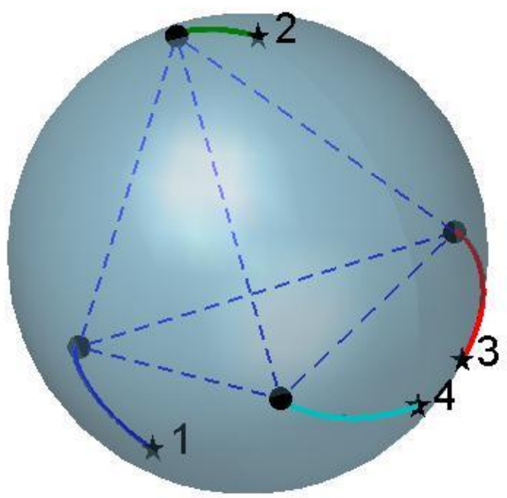
Thus: Ω_T is locally asymptotically stable.
Regular tetrahedron is almost globally a. s. !

Regular tetrahedron formation-simulation

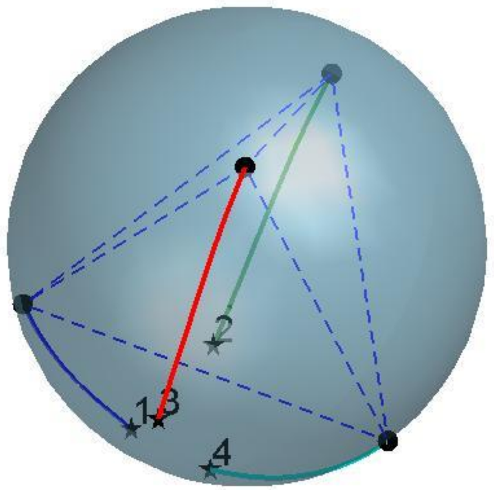
Only relative information is available:

$$\omega_i = -\sum_{j \in \mathcal{N}_i} f(\theta_{i,j}) \hat{\Gamma}_i \Gamma_j, \qquad i \in \{1,2,3,4\}$$

$f(\theta)$ satisfies $f(\theta)>0$ and $\dot{f}(\theta)<0$ for $\forall\theta \in [0,\pi]$.



when $f(\theta) = \mathrm{e}^{-\theta}$



when $f(\theta) = \cos(\theta) + 1$

Figure: Trajectories under complete inter-agent graph

Rotating tetrahedron formation

Inter-agent graph is set to be a weighed directed one:

3 coplanar edges are changed into directed edges.

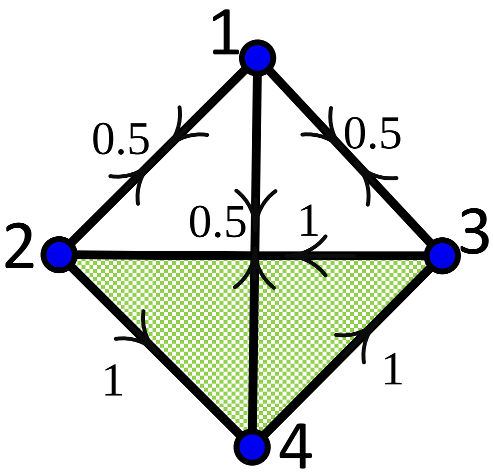
Double the weight of these 3 edges.

Apply the control law:

$$\omega_i = -\sum_{j \in \mathcal{N}_i} f(\theta_{i,j}) \cdot w(i,j) \cdot \hat{\Gamma}_i \, \Gamma_j, \qquad i \in \{1,2,3,4\}$$

where $w(i,j)$ is the weight of edge $(i,j) \in E$, $f(\theta)>0$,

$\dot{f}(\theta) < 0$ for $\forall \theta \in [0, \pi]$.



graph

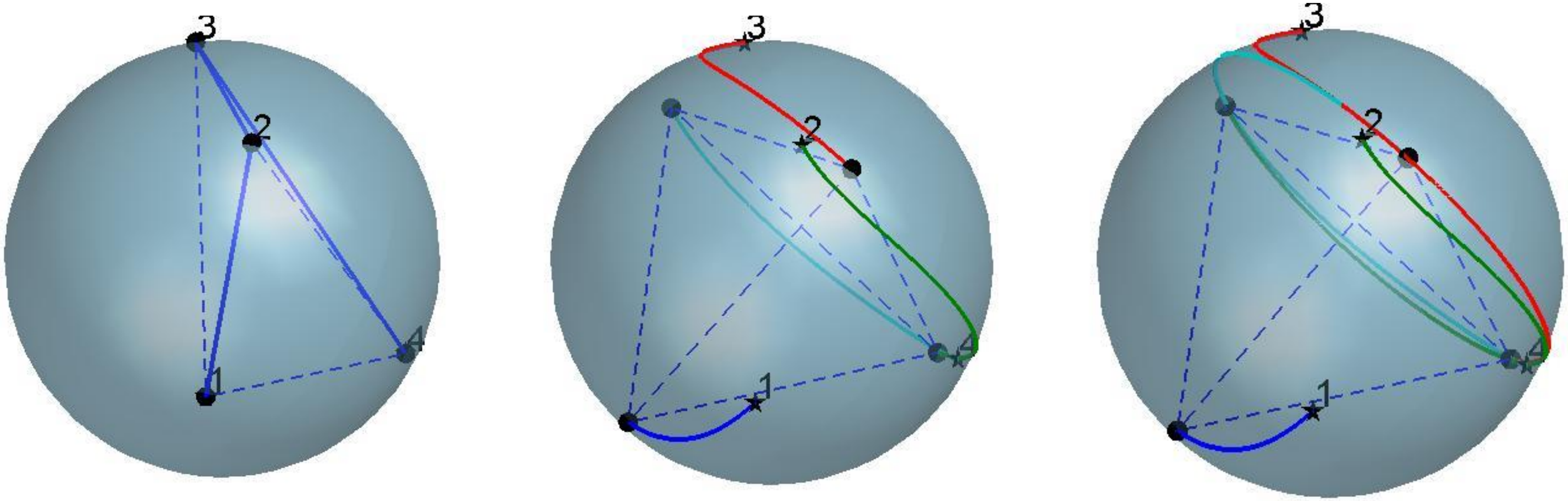
- A rotating tetrahedron formation can be obtained: center manifold is the same, but the dynamics on the center manifold has changed!

Rotating tetrahedron formation-simulation

$f(\theta)$ still satisfies

$$f(\theta)>0 \quad \text{and} \quad \dot{f}(\theta)<0 \quad \text{for } \forall\theta \in [0,\pi] .$$

We take $f(\theta) = \mathrm{e}^{-\theta}$.

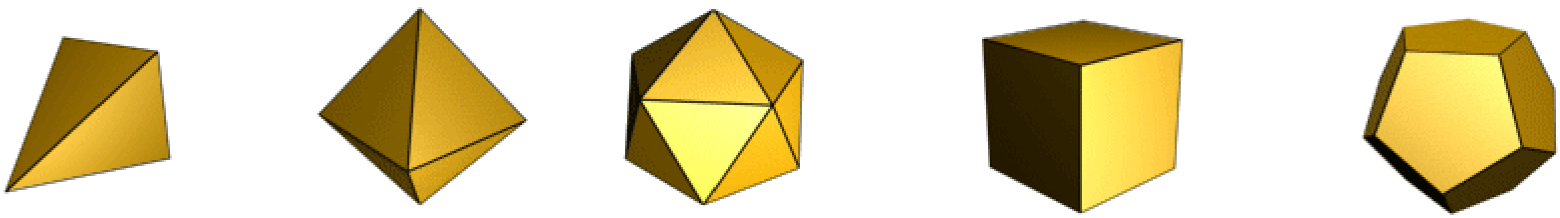


t=0 (sec) t=20 (sec) t=40 (sec)

Figure: Trajectories of rotating regular tetrahedron formation

Platonic solids formation

Five regular polyhedra



{3,3}	{3,4}	{3,5}	{4,3}	{5,3}
Tetrahedron	*Octahedron*	*Icosahedron*	*Cube*	*Dodecahedron*

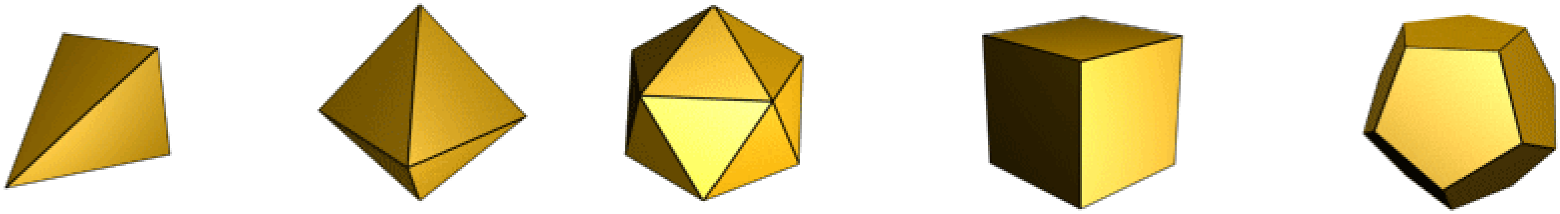
The Platonic solids are convex polyhedra with equivalent faces composed of congruent convex regular polygons

Schläfli symbol gives a combinatorial description of the polyhedron

$$V = \frac{4p}{4-(p-2)(q-2)}, \quad E = \frac{2pq}{4-(p-2)(q-2)}, \quad F = \frac{4q}{4-(p-2)(q-2)}.$$

Platonic solids formation

Five regular polyhedra



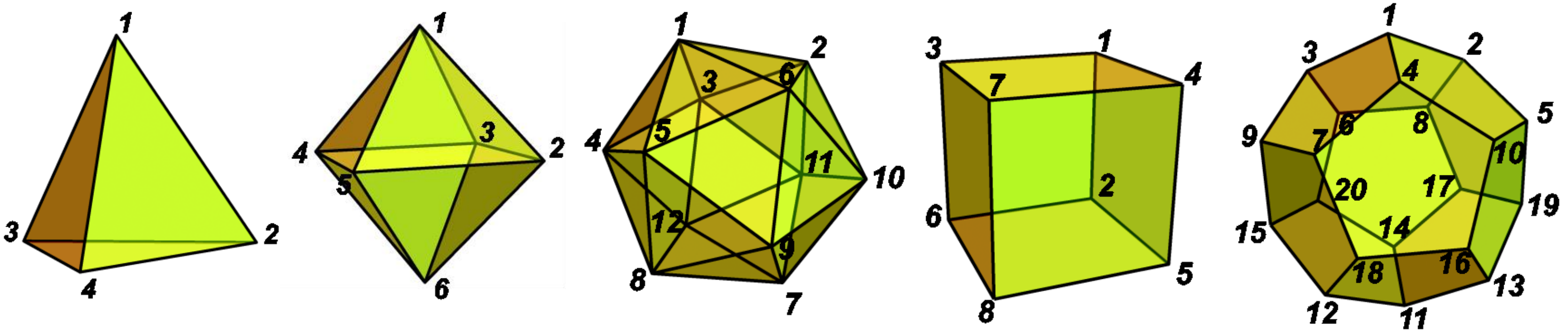
{3,3} **Tetrahedron** | *{3,4}* **Octahedron** | *{3,5}* **Icosahedron** | *{4,3}* **Cube** | *{5,3}* **Dodecahedron**

The Platonic solids are everywhere: crystals, gems, microscopic organisms

Geodesic grids in climatology, geometry of space frames, platonic hydrocarbons, satellites, dice

Platonic solids formation

Five regular polyhedra



***{3,3}* Tetrahedron** · ***{3,4}* Octahedron** · ***{3,5}* Icosahedron** · ***{4,3}* Cube** · ***{5,3}* Dodecahedron**

Three solids with triangular faces can be formed directly by the previous control protocol, when the inter-agent graph is complete.

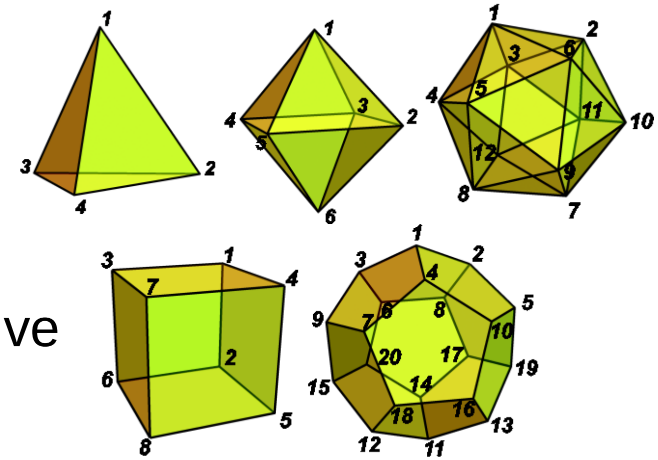
$$\omega_i = -\sum_{j\in\mathcal{N}_i} f(\theta_{ij})\hat{\Gamma}_i\Gamma_j, \qquad i \in \mathcal{V}$$

Under a particular graph, the other two can also be formed.

Formation description of five regular polyhedra

Polyhedral groups define the rotational symmetries of regular polyhedra.

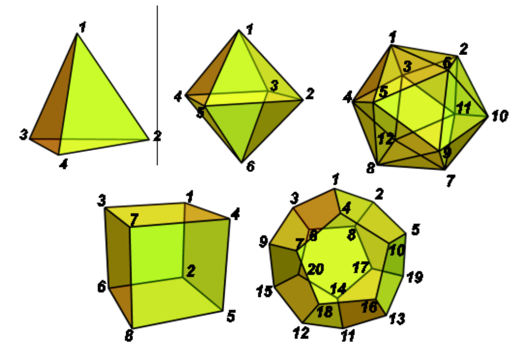
Due to rotational symmetries, vertex sets of five platonic solids satisfy



$$\Omega_{\{p,q\}} = \left\{ \boldsymbol{\Gamma} \in \mathbb{R}^{3\times N_0} : \left(I_{N_0} \otimes R_i - P_i \otimes I_3 \right) \boldsymbol{\Gamma} = 0, i = 1,2,\cdots,\mathcal{O}_{\{p,q\}} \right\},$$

where $\boldsymbol{\Gamma} = \left[\Gamma_1^T, \Gamma_2^T, \cdots, \Gamma_{N_0}^T \right]^T, N_0 = \frac{4p}{4-(p-2)(q-2)}$ is the number of vertices, P_i and R_i are permutation and rotation matrices corresponding to rotational symmetry i of solid $\{p,q\}$.

Graph design



Since five Platonic solids possess the most symmetries in all polyhedra, intuitively, some symmetries should be also inherited by the designed graph.

- Definition (graph automorphism): For a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, we say a permutation specified by mapping $\sigma : V \rightarrow V$ is a graph automorphism, when $\left(\sigma(i), \sigma(j)\right) \in \mathcal{E}$ if and only if $(i, j) \in \mathcal{E}$.

Assumption (graph symmetry):

The inter-agent graph $\mathcal{G}_{\{p,q\}}$ is connected and each permutation corresponding to rotational symmetry of solid $\{p, q\}$ is an automorphism of this graph.

Graph design

$\Omega_{\{p,q\}}$
$= \{\boldsymbol{\Gamma} \in \mathbb{R}^{3\times N_0}: \ \exists m \neq n \in V, s.t. \hat{\Gamma}_m \Gamma_n$

Velocity control

$$\omega_i = -\sum_{j\in\mathcal{N}_i} f(\theta_{ij})\hat{\Gamma}_i\Gamma_j, \qquad i \in \mathcal{V}$$

where the gain function $f\left(\theta_{ij}\right) = e^{2cos\left(\theta_{ij}\right)}$.

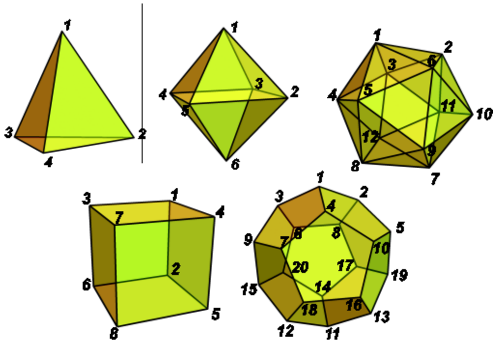
- With the above control, we can verify that the closed-loop system $\dot{\Gamma} = \mathrm{F}(\Gamma)$ is **Symmetric** under the transformation $I_{N_0} \otimes R_i$ and $P_i \otimes I_3$, i.e.

$$\boldsymbol{F}\left(I_{N_0} \otimes R_i \boldsymbol{\Gamma}\right) = I_{N_0} \otimes R_i \boldsymbol{F}(\boldsymbol{\Gamma}),$$

$$\boldsymbol{F}(P_i \otimes I_3 \boldsymbol{\Gamma}) = P_i \otimes I_3 \boldsymbol{F}(\boldsymbol{\Gamma}).$$

Theorem: Under the graph symmetry assumption, the regular polyhedra formation $\Omega_{\{p,q\}}$ is positively invariant in closed-loop system.

Graph design

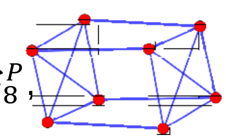
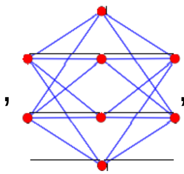
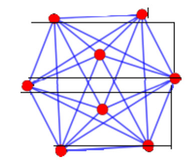
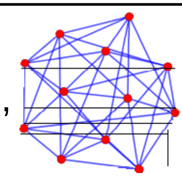
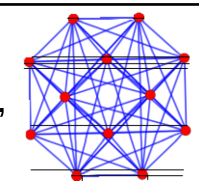
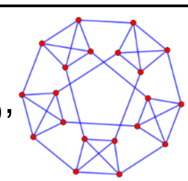
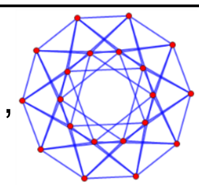


- It is obvious that the complete graph and Platonic graph *with* N_0 vertices satisfy graph symmetry Assumption.
- We can also compute all other possible graphs fulfilling such graph symmetries by the following remark.

***Remark**: Let A be the adjacency matrix of $\mathcal{G}$, then a permutation σ with permutation matrix P_σ is an automorphism of G, if and only if $AP_\sigma = P_\sigma A$.*

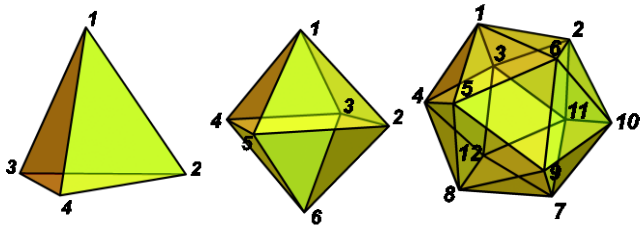
Graph design

- All possible graphs fulfilling graph symmetry Assumption:

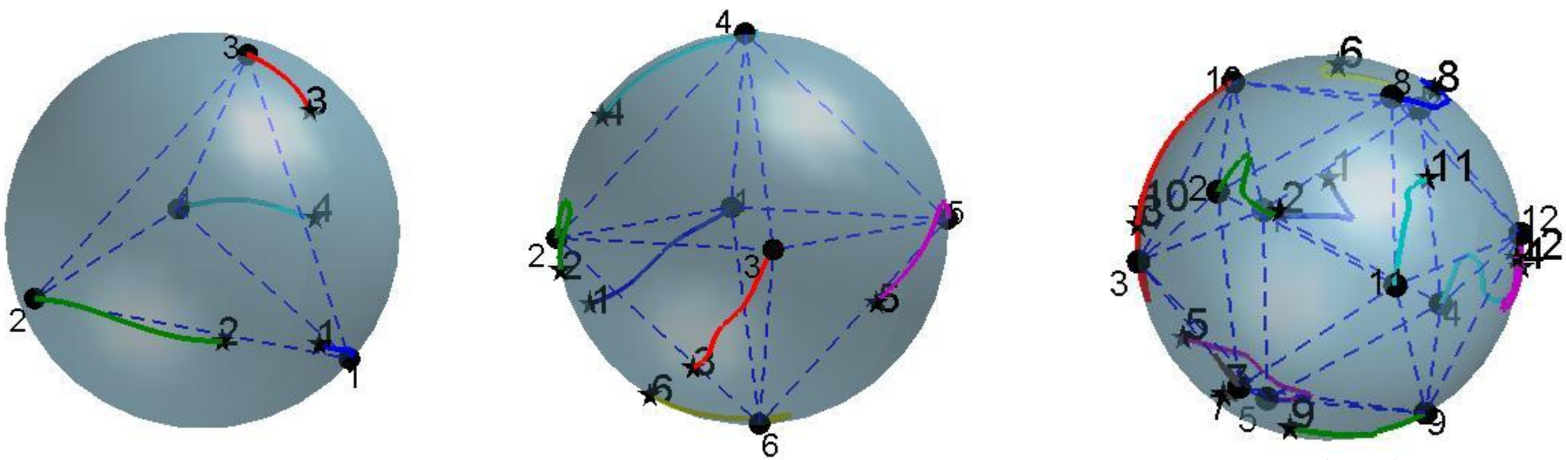
Formation $\{p, q\}$	Number of Vertices N_0	Number of Possible Graphs	Possible Graphs
$\{3, 3\}$	4	1	$\mathcal{G}_4^C$
$\{3, 4\}$	6	2	$\mathcal{G}_6^C$, $\mathcal{G}_6^P$
$\{4, 3\}$	8	5	$\mathcal{G}_8^C$, $\mathcal{G}_8^P$,  ,  , 
$\{3, 5\}$	12	4	$\mathcal{G}_{12}^C$, $\mathcal{G}_{12}^P$,  , 
$\{5, 3\}$	20	33	$\mathcal{G}_{20}^C$, $\mathcal{G}_{20}^P$,  ,  , ……

Where $\mathcal{G}_i^C$ is the complete graph with i vertices, and $\mathcal{G}_i^P$ is the Platonic graph with i vertices.

Stability of {3,3}, {3,4}, {3,5}



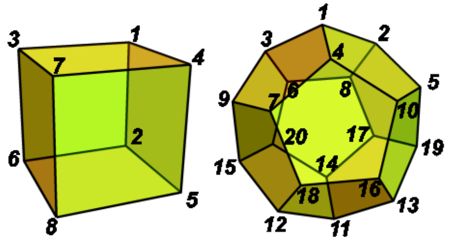
Theorem: Suppose the inter-agent graph $\mathcal{G}$ is complete, the invariant set $\Omega_{\{3,3\}}$, $\Omega_{\{3,4\}}$, $\Omega_{\{3,5\}}$ are asymptotically stable in the respective closed-loop system.



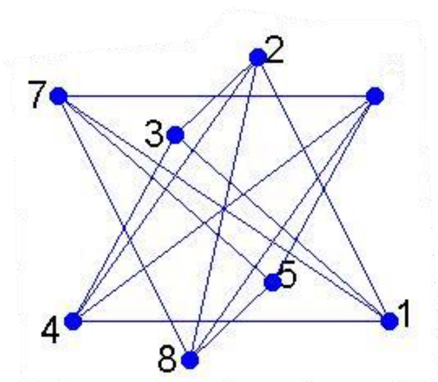
{3,3} {3,4} {3,5}

Figure: Trajectories of regular polyhedra formation.

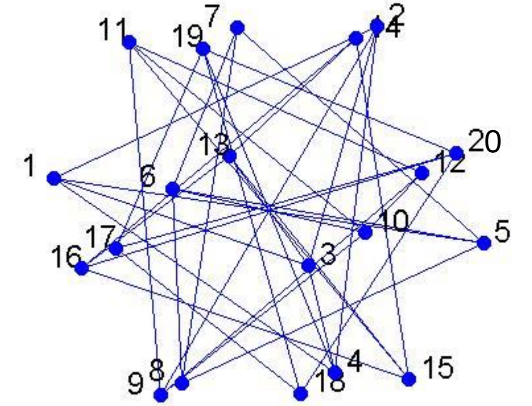
Stability of {4,3}, {5,3}



- **Polyhedron compound** is the composition of several identical polyhedra sharing a same center.
- We use the fact that
  - cube is the compound of 2 tetrahedra.
  - dodecahedron is the compound of 5 tetrahedra.



{4,3}



{5,3}

Figure: inter-agent graph for formation {4,3} and {5,3}

Theorem: Under the specific inter-agent graphs $\mathcal{G}$, the invariant set $\Omega_{\{4,3\}}$ and $\Omega_{\{5,3\}}$ are asymptotically stable in the respective closed-loop system.


Thank You!