Beyond consensus and polarisation: complex social phenomena in social networks

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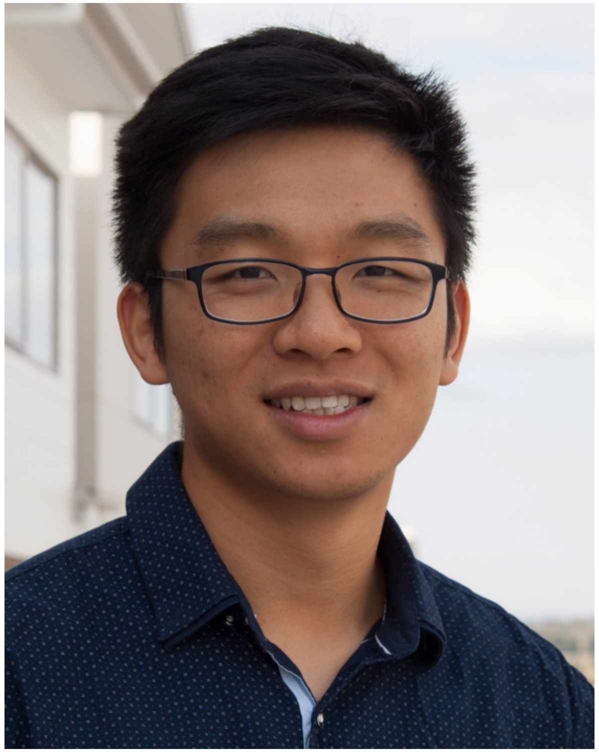
Thanks

- To the Chinese Control Conference organizers for inviting me
- To my major collaborator on this work, Research Associate Mengbin (Ben) Ye



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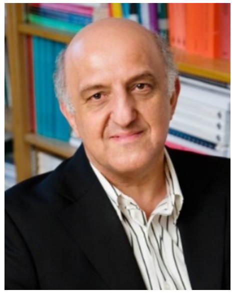

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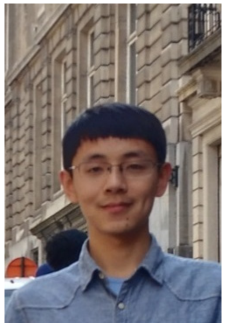

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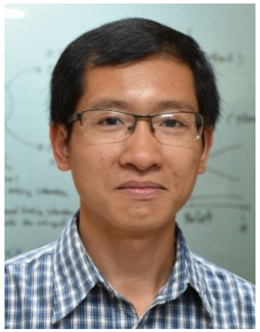

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Outline

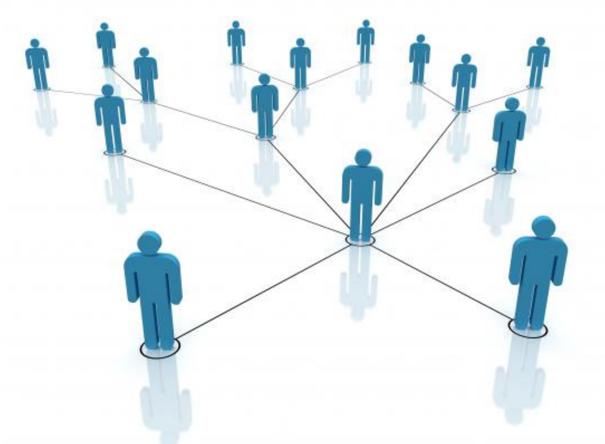
- Introduction & Motivation
  - Social Networks
  - Opinion Dynamics
  - Modelling with Systems ideas
- Part I: Differences in Private and Expressed Opinions
- Part II: Evolution of Individual Social Power
- Part III: Multiple related opinions

Social Networks

A set of social actors (individuals or organisations) who interact according to a set of social relationships/connections.

- Facebook, Tumblr, Instagram, Reddit, …… etc.

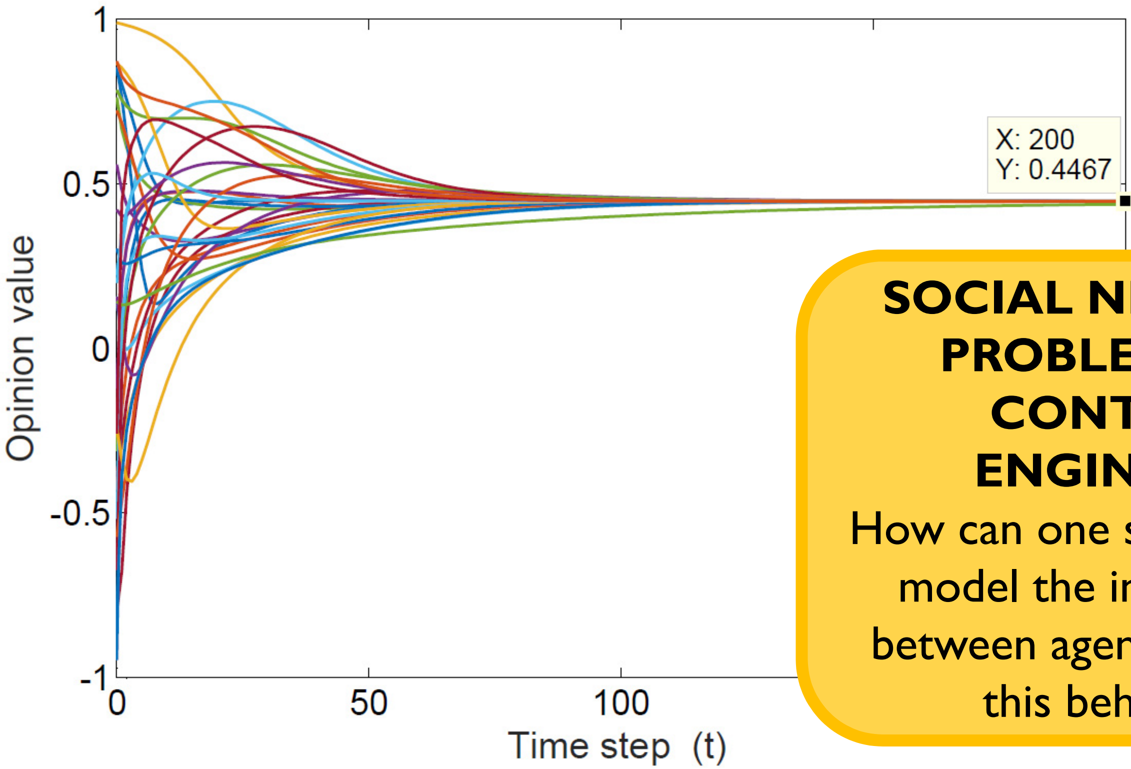
- Small networks – government cabinets, jury panels, board of directors.



Opinion Dynamics

Each individual has an opinion value (real number) on an issue/topic and individuals interact and discuss their opinions.

E.g. was the 2003 invasion of Iraq justified?



SOCIAL NETWORK PROBLEM FOR CONTROL ENGINEERS

How can one systematically model the interactions between agents to reflect this behavior.?

Opinion Dynamics Modelling

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- Different models have been proposed to describe different sociological, psychological behaviour that occur during interactions

- French-Harary-DeGroot discrete-time model [R1, R2]

60 years ago!

- Continuous time, bounded confidence, etc. etc,, e.g. [R3, R4, R5]

Many parallels now perceived with networked control and multiagent systems, e.g. robots following a leader, airborne formation control.

[R1] French Jr, J.R. A formal
[R2] DeGroot, M.H. Reaching a consensus. *Journal of the American Statistical Association*, *69*(345), pp.118-121, 1974.
[R3] Abelson, R.P. Mathematical models of the distribution of attitudes under controversy. *Contributions to mathematical psychology*, *14*, pp.1-160, 1964.
[R4] Ren, W. and Beard, R.W. Consensus seeking in multiagent systems under dynamically changing interaction topologies. *IEEE Transactions on automatic control*, *50*(5), pp.655-661, 2005.
[R5] Hegselmann, R. and Krause, U. Opinion dynamics and bounded confidence models, analysis, and simulation. *Journal of Artificial Societies and Social Simulation*, *5*(3), 2002.

High Level Issues

Objective: Develop mathematical models for describing opinion dynamics, which reflect sociology and social psychology concepts and phenomena

—*and conceivably predict some!*

Motivation: Better understand the key factors that determine how opinions evolve over time and make predictions on the dynamics of the opinions [Customer responses, employee satisfaction, etc]

Example of Specific Objective: We mainly interact with people whose opinions are similar to our own. How is this captured in a model? What phenomena occur?

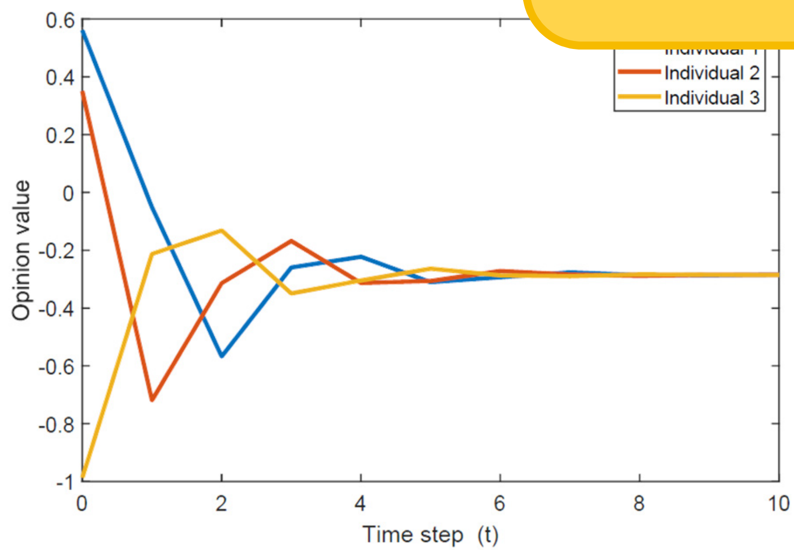
Consider a single top... is updated:

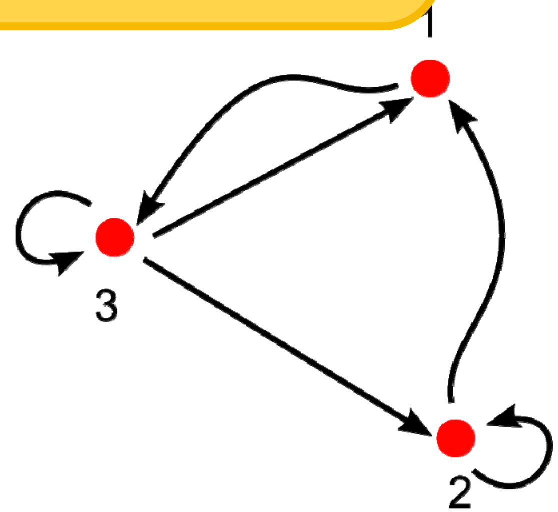
$$y_i(t+1) \;\ldots\; {}_j = 1 \quad w_{ij} \geq 0$$

Connectivity of the graph and the row-stochastic property produce consensus!

$$\boldsymbol{y}(t+1) = \boldsymbol{W}\boldsymbol{y} \;\ldots\; \begin{matrix} .7 & 0.3 \\ .2 & 0.8 \\ 0 & 0.5 \end{matrix}\Bigg] \begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{bmatrix}$$

Individual *i*'s new opinion is **linear convex combination** of his prior opinion and of neighbor's opinions





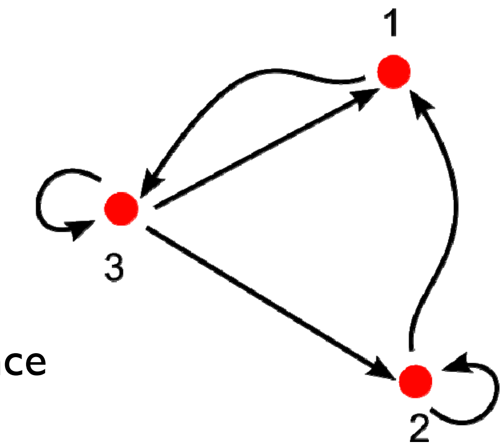
W **is row-stochastic**

Social Networks and Graphs

Graphs: a convenient and powerful way to represent a social network (in some literature, also called *influence network* because of w_{ij})

A graph is $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \boldsymbol{W})$ where

- $\mathcal{V} = \{1, \dots, n\}$ is the set of nodes representing individuals
- $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ is the set of ordered edges representing unidirectional interpersonal influence
- $\boldsymbol{W}$ is the *influence matrix* capturing the interpersonal influence
- Connectivity means what you think it should mean.



Topology conditions on $\mathcal{G}$, e.g. connectivity

Structural conditions on $\boldsymbol{W}$, e.g. row stochastic

Social Phenomenon e.g. consensus

This talk

This talk will explore three recent developments of social network modelling. They all build (substantially) on the DeGroot model.

Outline

- Introduction & Motivation
- Part I: Differences in Private and Expressed Opinions
  - Opinion Evolution Under Pressure to Conform
  - Relation to Social Psychology Concepts
  - Conclusions & Future Work
- Part II: Evolution of Individual Social Power
- Part III: Multiple related opinions

High Level Motivation

Background: In many situations, we have for one reason or another expressed a view which is different to our private view. Pressures from group dynamics *altered* our expression.

Example: I secretly support Donald Trump, but in the presence of everyone here, I express the opposite position.

Question: How does the pressure from group dynamics affect the process of opinion dynamics?

Background Literature

Literature from social psychology, sociology, political science, and economics *qualitatively* studies private vs. expressed opinions/actions and pressure to conform.

- Group pressure can modify and distort an individual's judgement even in the face of overwhelming facts [R1]

- Preference falsification occurs when an individual knowingly expresses an altered form of his/her true opinion [R2]

- Pluralistic ignorance is a phenomenon whereby an individual believes the public majority support position A, but in reality, the majority support position B [R3]

- Example: in factories, group pressure can force individuals with high productivity rates to lower their rates to match a desired group standard [R4]

70 years ago!

[R1] Asch, S.E. and Guetzkow, H., 1951. Effects of Group Pressure Upon the Modification and Distortion of Ju... *and Men*, pp.222-236, Pittsburgh: Carnegie Press.
[R2] Kuran, T., 1997. *Private Truths, Public Lies: The Social Consequences of Preference Falsification*. Harvard University Press.
[R3] O'Gorman H.J. 1975. Pluralistic Ignorance and White Estimates of White Support for Racial Segregation. *Public Opinion Quarterly*, 39(3):313–330.
[R4] Coch, L. and French Jr, J.R., 1948. Overcoming Resistance to Change. *Human Relations*, 1(4), pp.512-532.

The Opinion Dynamics Process

The social network undergoes the following iterative process

Individual i expresses his/her opinion, and

We will build a model which incorporates:

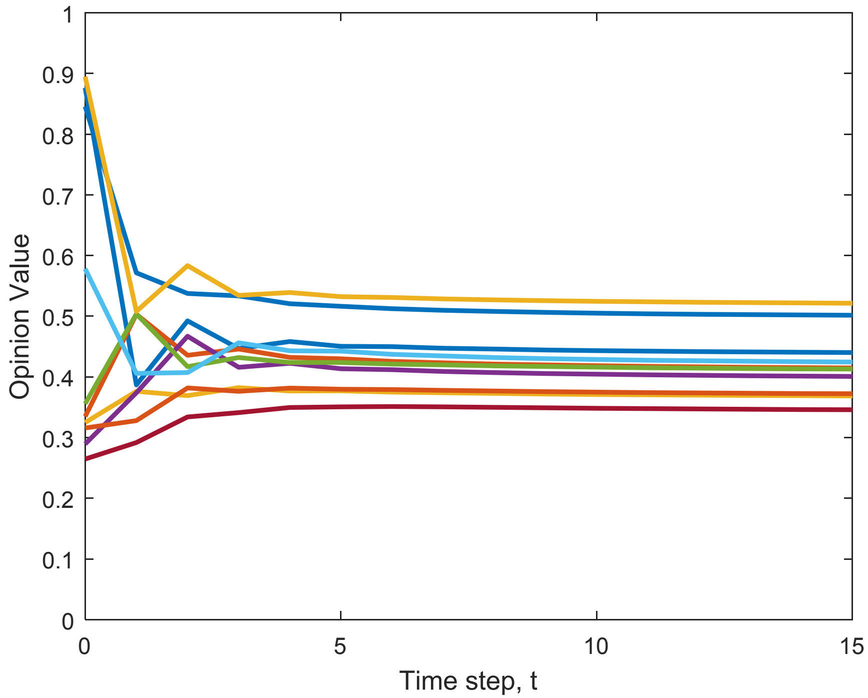
(a) Stubbornness

(b) Resilience to pressure from others

Individual i determines what his/her new expressed opinion should be, altered by group pressure effects

The discrete-time FJ model [R1,R2] is an established model of opinion dynamics

$$y_i(t+1) = \lambda_i w_{ii} y_i(t) + \lambda_i \sum_{j \neq i}^{n} w_{ij} y_j(t) + (1-\lambda_i) y_i(0)$$



Key difference with DeGroot model is **stubbornness** term, which creates disagreement.

[R1] Friedkin N.E. , Johnsen E.C., 1990. Social Influence and Opinions. *Journal of Mathematical Sociology* 15(3-4):193–206.
[R2] Friedkin N.E., 2006. *A Structural Theory of Social Influence*. Cambridge University Press

The Newly Proposed Model

Let individual i's private opinion be y_i, and expressed opinion be $\hat{y}_i$:

$$y_i(t+1) = \lambda_i w_{ii} y_i(t) + \lambda_i \sum_{j \neq i}^{n} w_{ij} \hat{y}_j(t) + (1 - \lambda_i) y_i(0)$$

Except for the use of the expressed opinion of neighbors of individual *i*, this is the same as the Friedkin-Johnsen model

Resilience in the new Model

Let individual i's private opinion be y_i, and expressed opinion be $\hat{y}_i$:

$$y_i(t+1) = \lambda_i w_{ii} y_i(t) + \lambda_i \sum_{j \neq i}^{n} w_{ij} \underline{\hat{y}_j(t)} + (1-\lambda_i) y_i(0)$$

$$\hat{y}_i(t) = \underline{\phi_i} y_i(t) + \underline{(1-\phi_i)} \underline{\hat{y}_{avg}}(t-1)$$

- $\hat{y}_{avg} = \hat{\boldsymbol{y}}^\top \mathbf{1}_n / n$ is the average expressed view: **the public opinion**
- $\phi_i \in [0,1]$ is individual i's resilience to the pressure of the public opinion

Model Analysis

The entire opinion dynamics model can be expressed in compact form as a linear time-invariant system, with $\boldsymbol{\Lambda} = \mathrm{diag}(\lambda_i)$.

$$\begin{bmatrix} \boldsymbol{y}(t+1) \\ \hat{\boldsymbol{y}}(t) \end{bmatrix} = \begin{bmatrix} \boldsymbol{P}_{11} & \boldsymbol{P}_{12} \\ \boldsymbol{P}_{21} & \boldsymbol{P}_{22} \end{bmatrix} \begin{bmatrix} \boldsymbol{y}(t) \\ \hat{\boldsymbol{y}}(t-1) \end{bmatrix} + \begin{bmatrix} (\boldsymbol{I}_n - \boldsymbol{\Lambda})\boldsymbol{y}(0) \\ \boldsymbol{0}_n \end{bmatrix}$$

Under the mild assumptions

- on the connectivity of the graph representing the social network (standard)
- that $\phi_i, \lambda_i \in (0,1)$ for all i

the opinion dynamics system is exponentially convergent to a steady state..

The convergence result itself is not unexpected or difficult to conclude.
Much deeper insight is obtained by study of the final opinion distribution.

Evaluating the Final Opinions

$$\boldsymbol{y}(\infty) = \boldsymbol{y}^* = \boldsymbol{R}\boldsymbol{y}(0)$$
$$\hat{\boldsymbol{y}}(\infty) = \hat{\boldsymbol{y}}^* = \boldsymbol{S}\boldsymbol{y}^*$$

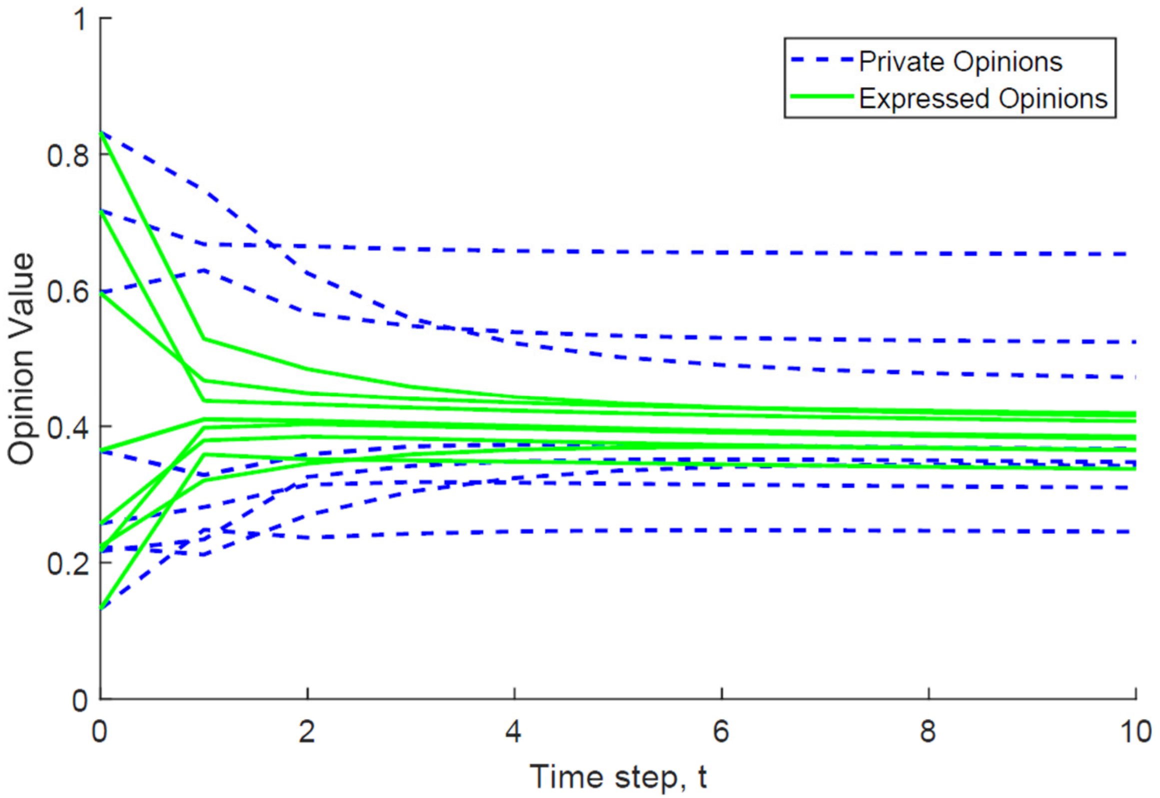
Key result:

- $\boldsymbol{R}, \boldsymbol{S} \in \mathbb{R}^{n\times n}$ are invertible, positive, row-stochastic matrices

Social Interpretations:

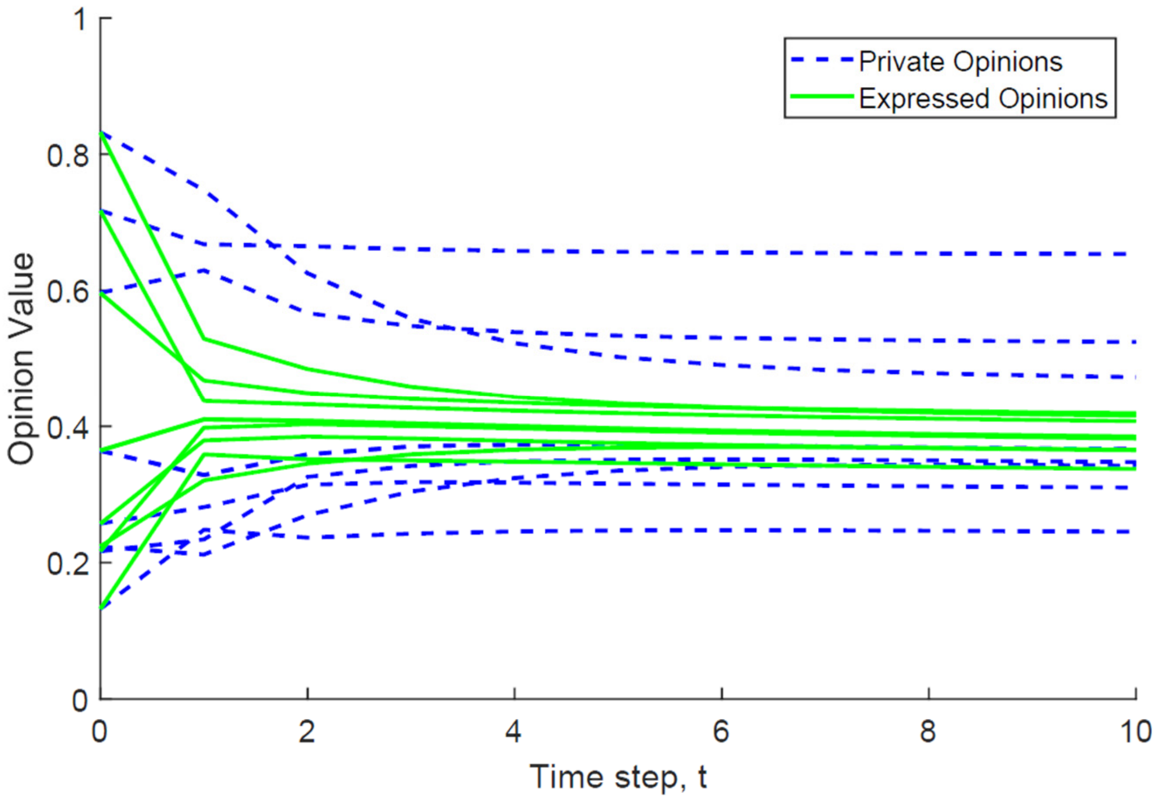
- Private opinions $\boldsymbol{y}^*$ are a convex combination of entries of initial private opinion vector $\boldsymbol{y}(0)$. Entries of expressed opinion vector $\hat{\boldsymbol{y}}^*$ are a convex combination of entries of $\boldsymbol{y}^*$.
- **No dependence on $\hat{\boldsymbol{y}}(\boldsymbol{0})$.**
- Final expressed and private opinions are at a persistent disagreement.

$y_i^* \neq \hat{y}_i^*$, i.e. stubbornness (λ_i) and pressure to conform (ϕ_i) create different private and expressed opinions in the same individual



Distribution of Private and Expressed Opinions

1. Larger disagreement among private opinions than observed from expressed opinions
2. Expressed opinions are enclosed in the private opinions



Estimating Private Disagreement

Resilience (opposite of pressure to conform)

$$\hat{y}_i(t) = \phi_i y_i(t) + (1 - \phi_i)\hat{y}_{avg}(t-1)$$

Expressed opinion spread

Private opinion spread

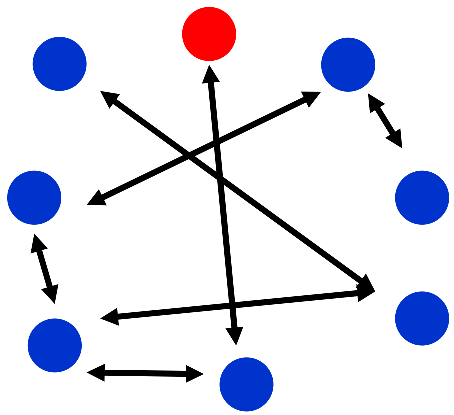
Result:

$$\frac{\max_i \hat{y}_i^* - \min_j \hat{y}_j^*}{1 - \frac{\phi_{\min}}{\phi_{\max}}(1 - \phi_{\max})} < \max_i y_i^* - \min_j y_j^*$$

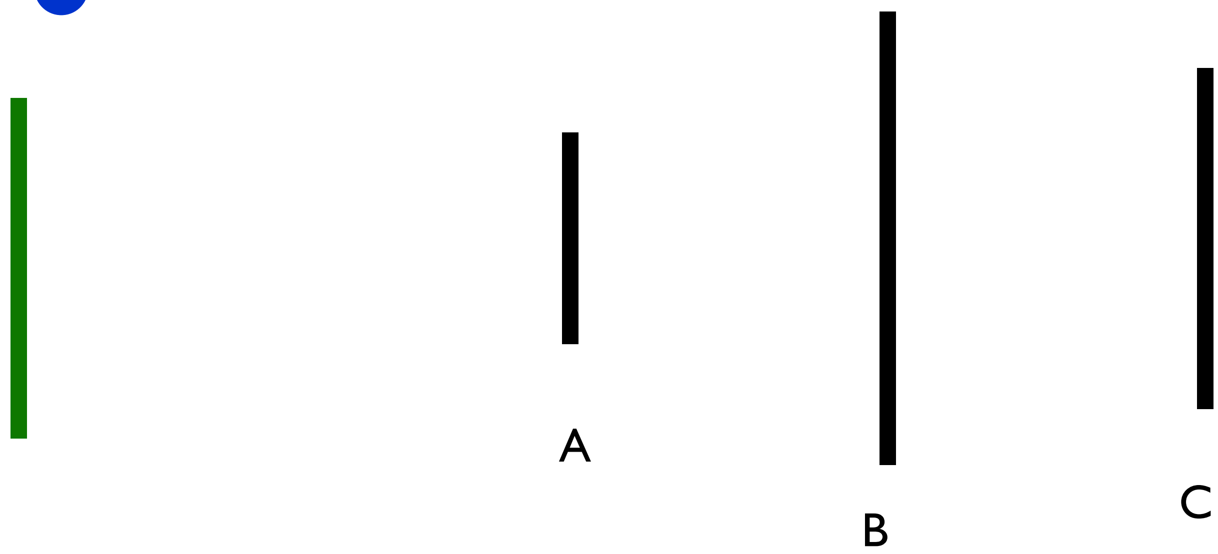
This gives a **lower bound** on the level of private disagreement, knowing the expressed disagreement and resilience parameters.

Asch Experiment Revisit (1951)

Perhaps one of the most famed sociological experiments on group pressures.



Which of lines A,B,C has the same length as the line on the left?



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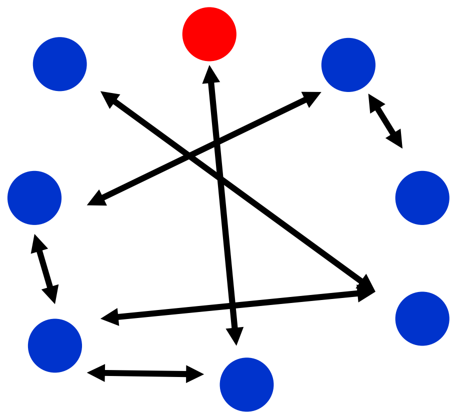
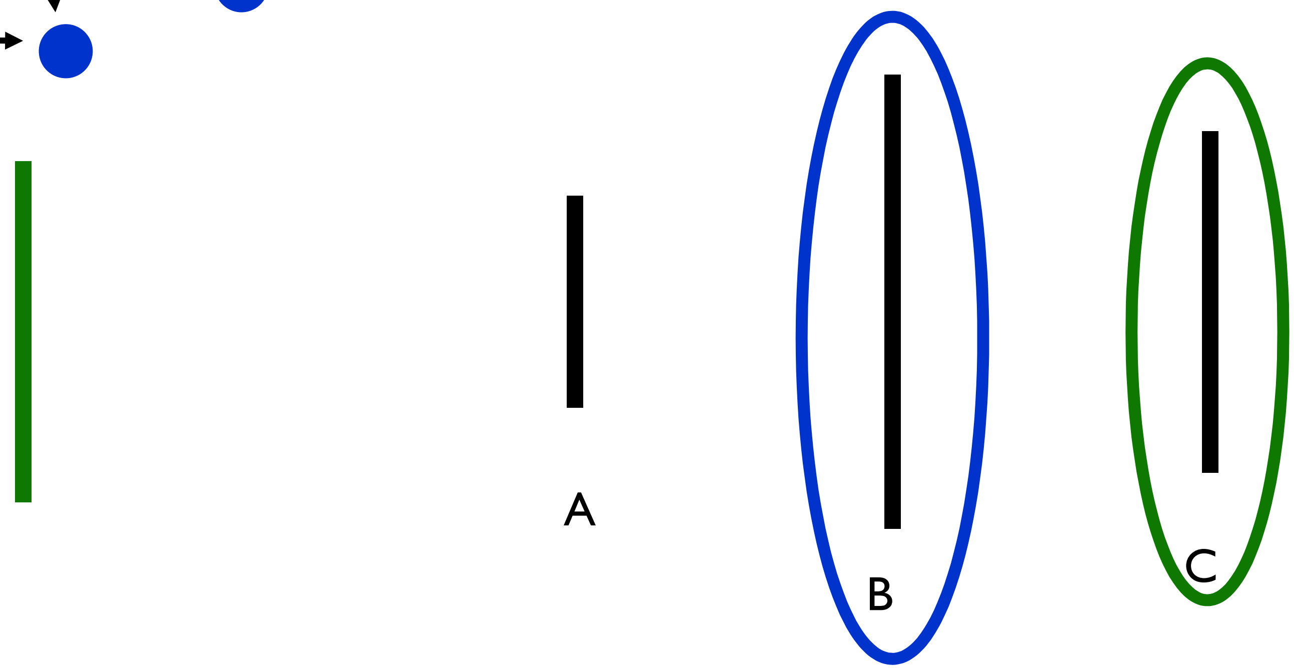


7 blue confederates choose B. How does the red person react?



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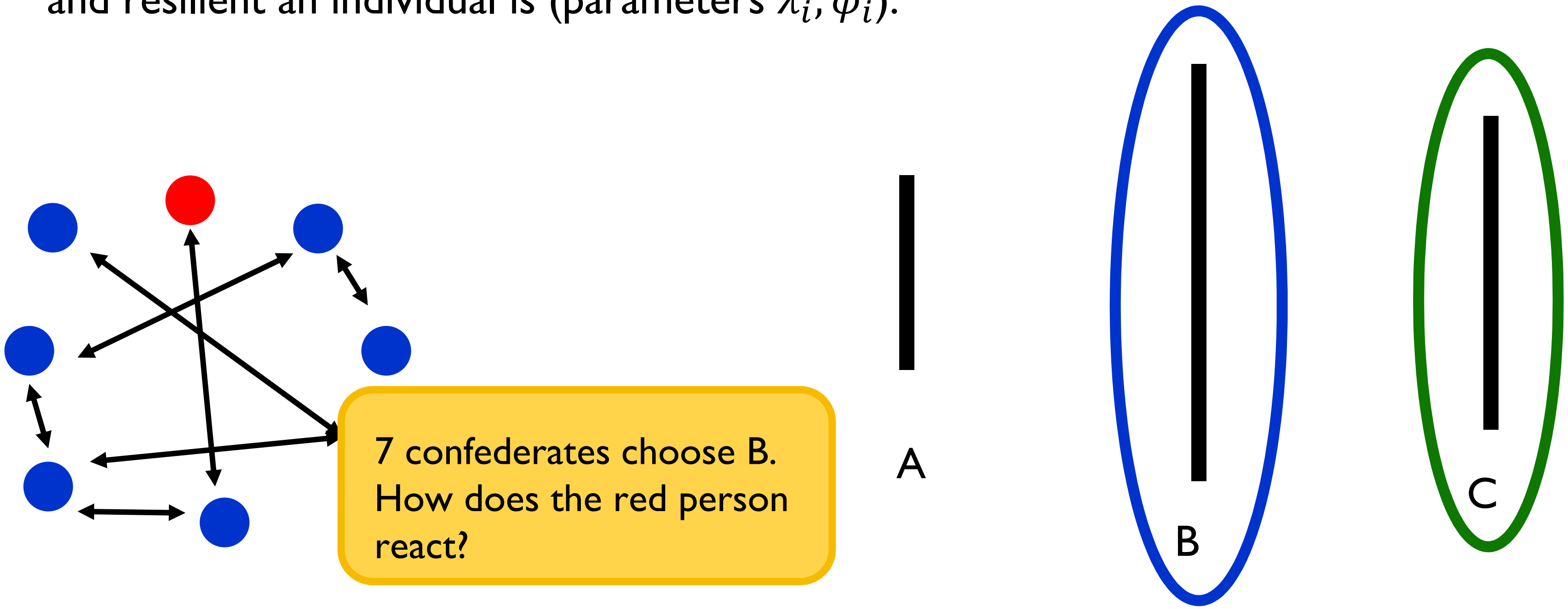
Asch Experiment Revisit (1951)

Observed Results of Red Person in different experiments:

1. He remained insistent that C was the correct answer
2. He expressed B as the correct answer but in a post-interview reaffirmed C as true.
3. He expressed B as the correct answer and in a post-interview still chose B.

Result: All three behaviours can be observed in our model depending on how stubborn and resilient an individual is (parameters λ_i, ϕ_i).



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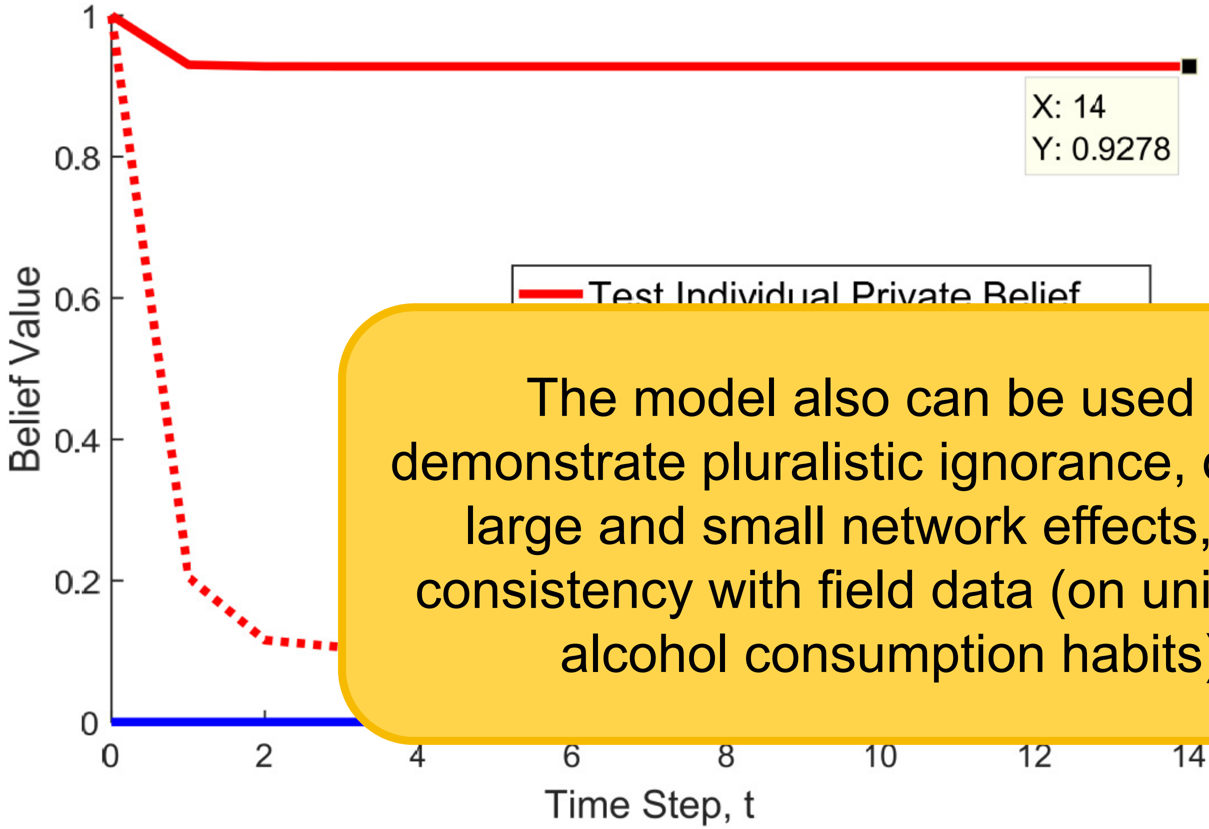
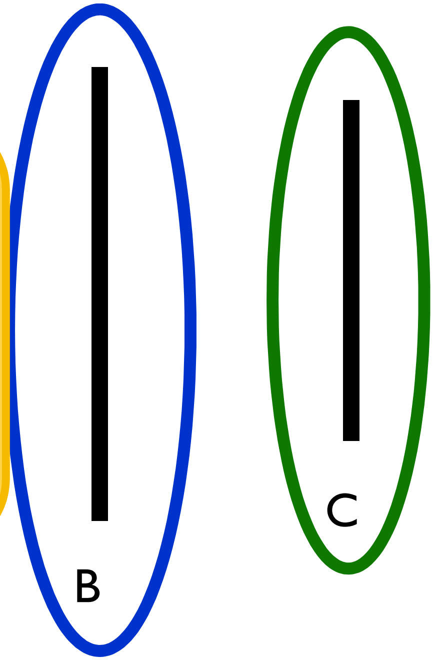
Asch Experiment Revisit (1951)

Example simulation showing a *yielding individual with distortion of action* [R1]:



How certain are you that C is the correct answer?

The model also can be used to demonstrate pluralistic ignorance, differing large and small network effects, and consistency with field data (on university alcohol consumption habits)



[R1] Asch, S.E. and Guetzkow, H., 1951. Effects of Group Pressure Upon the Modification and Distortion of Judgments. *Groups, Leadership, and Men*, pp.222-236, Pittsburgh: Carnegie Press.

Some conclusions

- A novel model was proposed to describe differences in expressed and private opinions due to pressure to conform
- Analytical results obtained giving relations between expressed and private beliefs
- Model was verified using the Asch Conformity Experiments
- Pluralistic ignorance was observed in some simulations
- Much more could be done, e.g. detailed study of zealots and pluralistic ignorance, event-based communication, reflecting rate of change somehow--but all should be done with an eye on the social science literature.

Outline

- Introduction & Motivation
- Part I: Differences in Private and Expressed Opinions
- Part II: Evolution of Individual Social Power
  - The DeGroot-Friedkin Model
  - Dynamic Topology: Exponential Forgetting of Perceived Power
  - Conclusions & Future Work
- Part III: Multiple related opinions

High Level Problem

Background: Recently proposed DeGroot-Friedkin model [R1] studies the evolution of each individual's social power (regarded as equivalent to self-confidence) as a network discusses opinions on a sequence of topics.

Example: As I see myself having less and less impact on the discussion, I become less and less confident of my own opinion.

Specific Problem: How does individual social power evolve when the interpersonal relationships are changing, i.e. dynamic network topology?

[R1] Jia, P., MirTabatabaei, A., Friedkin, N.E. and Bullo, F., 2015. Opinion dynamics and the evolution of social power in influence networks. *SIAM Review*, 57(3), pp.367-397.

Overview of DF Model

Consider discussion on a sequence of topics, indexed as $s = 0, 1, 2, \ldots, \infty$

E.g. are Toyota cars reliable?, is a dog man's best friend?, is chess a sport? … etc

Social network discusses opinions on topic s, and reaches a consensus

Each person evaluates their influence in the discussion (reflected appraisal)

Each individual uses appraisal to update his/her self-confidence (social power) for issue $s + 1$

Relative Interaction Matrix

Deals with relative effect of other individuals on a given individual.

- Will separately consider (next slide) the balance between attention given to other individuals and own opinion.

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Uses a row-stochastic, irreducible relative interaction matrix $\boldsymbol{C}$, i.e. $\sum_{j=1}^{n} c_{ij} = 1$ (with zero diagonal elements)

$c_{ij} > 0$ if individual i has directional interpersonal relationship with individual j

I.e. c_{ij} is the *relative* friendship/trust individual i allocates to individual j

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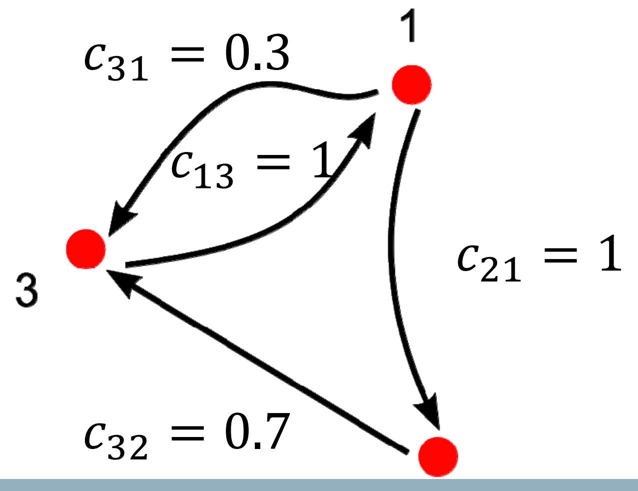
Note: Zeros on main diagonal since only other individuals are being considered

$$\boldsymbol{C} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0.3 & 0.7 & 0 \end{bmatrix}$$



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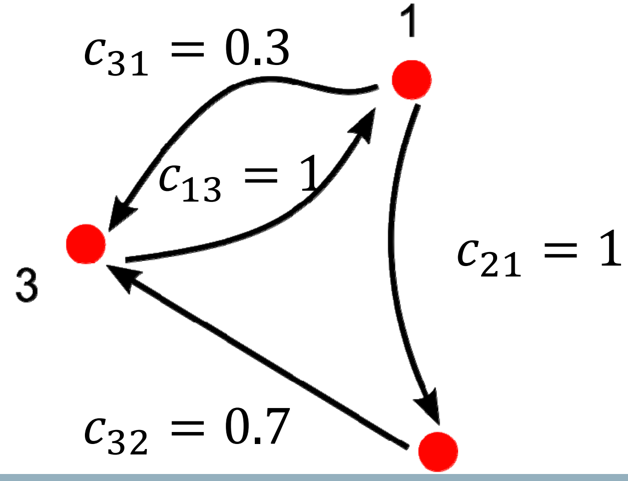
We will use:

$$\boldsymbol{\gamma}^\top \boldsymbol{C} = \boldsymbol{\gamma}^\top, \qquad \boldsymbol{\gamma}^\top \mathbf{1}_{\boldsymbol{n}} = 1$$

$\boldsymbol{\gamma}^\top$ is the normalized dominant left eigenvector of C, and its entries are positive and add to 1.

$$\boldsymbol{C} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0.3 & 0.7 & 0 \end{bmatrix}$$



The DeGroot-Friedkin Model

Individual i's opinion for issue s is $y_i(s,t) \in \mathbb{R}$ and is updated as:

$$y_i(s,t+1) = \underline{w_{ii}(s)} y_i(s,t) + \sum_{j \neq i}^{n} w_{ij}(s) y_j(s,t) , \qquad \sum_{j=1}^{n} w_{ij} = 1$$

$$w_{ij}(s) = (1 - w_{ii}(s)) c_{ij}$$

Key Observation: The DeGroot-Friedkin model operates on two time-scales.

Timescale t which appears only during the opinion dynamics process

Timescale s appears in both the opinion dynamics process and in the updating of self-confidence (social power)

Assume timescales are decoupled: we will only work in s timescale

Denote $w_{ii}(s) = x_i(s) \in [0,1]$ --measure of self-confidence.

The DeGroot-Friedkin Model

Self-confidence of individual i for issue $s+1$ is set to equal his contribution to the final consensus value for issue s (this is determinable). This results in a nonlinear equation:

$$\boldsymbol{x}(s+1) = \boldsymbol{F}(\boldsymbol{x}(s))$$

with the map $\boldsymbol{F} : \Delta_{\mathrm{n}} \rightarrow \Delta_{\mathrm{n}}$, where $\Delta_n = \{\boldsymbol{x} : x_i \geq 0, \sum x_i = 1\}$

$$\boldsymbol{F}(\boldsymbol{x}(s)) = \alpha(\boldsymbol{x}(s)) \begin{bmatrix} \frac{\gamma_1}{1-x_1(s)} \\ \vdots \\ \frac{\gamma_n}{1-x_n(s)} \end{bmatrix}, \quad \alpha(\boldsymbol{x}(s)) = \frac{1}{\sum_{i=1}^{n} \frac{\gamma_i}{1-x_i(s)}}$$

A nonlinear equation says how each individual's self-confidence updates immediately after discussing an issue. Roughly, the greater the influence on an issue, the greater the self-confidence for the next issue.

The DeGroot-Friedkin Model

Key result for almost all network structures [R1]:

If $\boldsymbol{C}$ is constant then $\lim_{s\to\infty} \boldsymbol{x}(s) = \boldsymbol{x}^*$ *asymptotically*, where $\boldsymbol{x}^*$ is the unique fixed point of $\boldsymbol{F}$, i.e. $\boldsymbol{x}^* = \boldsymbol{F}(\boldsymbol{x}^*)$

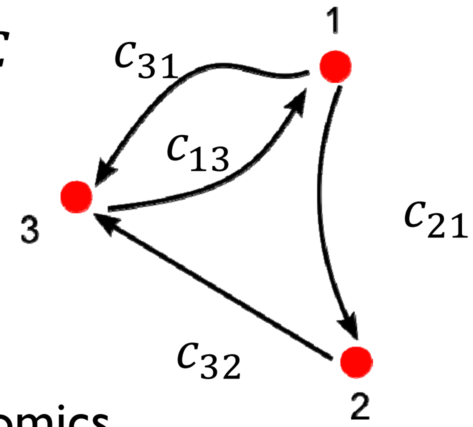
Our recent work [R2]: convergence is *exponentially fast.*

This indirectly paves the way to handle time-varying problems.

[R1] Jia, P., MirTabatabaei, A., Friedkin, N.E. and Bullo, F., 2015. Opinion dynamics and the evolution of social power in influence networks. *SIAM Review*, 57(3), pp.367-397.
[R2] Ye, M., Liu, J., Anderson, B.D.O., Yu, C. and Basar, T., Evolution of social power in social networks with dynamic topology, IEEE Transactions on Automatic Control, IEEE Explore Digital Object Identifier 10.1109/TAC.2018.2805261

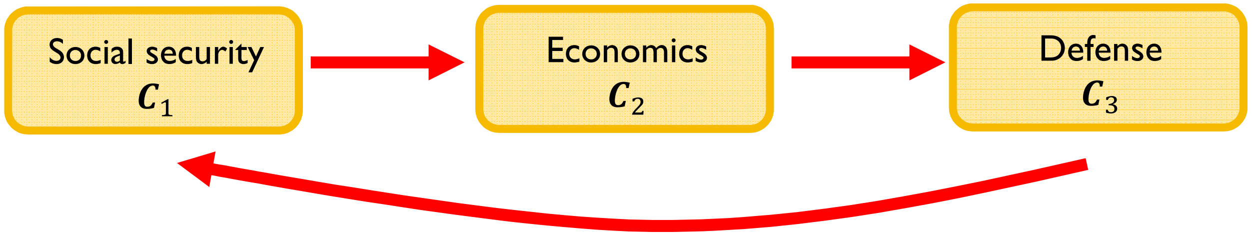
Dynamic Relative Interactions

The original DeGroot-Friedkin model assumed a constant $\boldsymbol{C}$



Topology can change over the sequence of issues to give $\boldsymbol{C}(s)$

1) Issue-driven topology change: If we firstly discuss economics, and then discuss environment, we expect $\boldsymbol{C}_{eco} \neq \boldsymbol{C}_{env}$
   - Special Case: Periodically changing $\boldsymbol{C}(s)$ (e.g. government cabinet weekly meetings)



2) Individual-driven topology change: Over time, individuals may form new relationships or eliminate old ones for any number of reasons, $\boldsymbol{C}(1) \neq \boldsymbol{C}(20)$

Dynamic Topology Main Result

Assume that $\boldsymbol{C}(s)$ changes independently of the state $\boldsymbol{x}(s)$:

$$\boldsymbol{x}(s+1) = \boldsymbol{F}_{\sigma(s)}(\boldsymbol{x}(s))$$

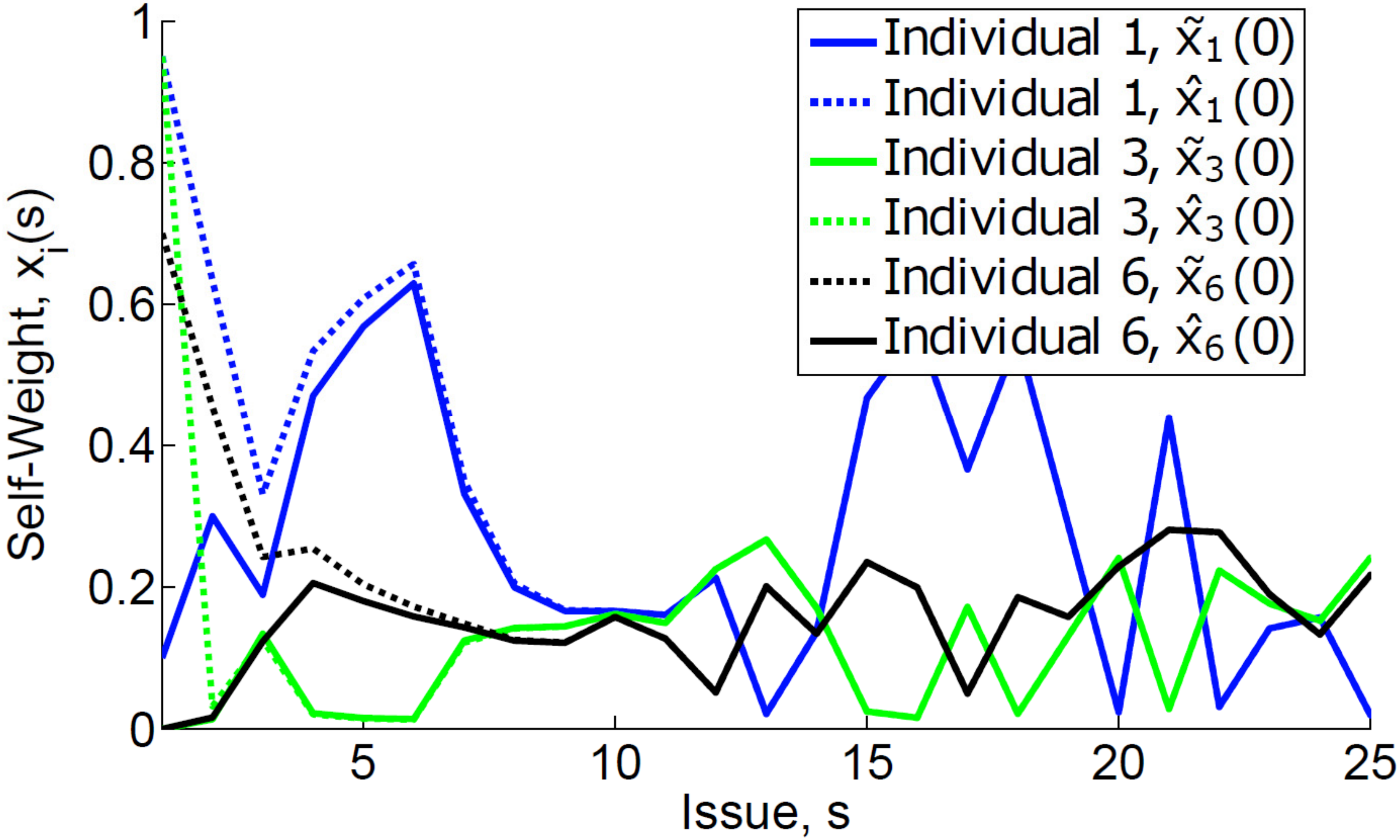
where $\sigma(s)$ is a switching signal. Then

1. Initial conditions are forgotten exponentially fast.
2. Convergence of *x(s)* sequence occurs to a unique trajectory independent of initial conditions.
3. Trajectory is constant if $\boldsymbol{C}(s)$ is constant, periodic if $\boldsymbol{C}(s)$ is periodic.

Social Interpretation: *Sequential opinion discussion removes perceived (initial) social power/self-confidence exponentially fast. The social network is "self-regulating." True social power/self-confidence is dependent only on the sequence of topology structures.*

Simulations

Recall: $x_i(s) \in [0,1]$



Conclusions

- DeGroot-Friedkin model of evolution of individual social power
- Dynamic topology (changing relative interaction with neighbors) was incorporated
- Exponential convergence to unique limiting trajectory when there is dynamic topology (unique equilibrium when there is a fixed topology, unique periodic trajectory when there is periodic topology)

Outline

- Introduction & Motivation
- Part I: Differences in Private and Expressed Opinions
- Part II: Evolution of Individual Social Power
- Part III: Multiple related opinions
  - Puzzle of Strong Diversity
  - Multi-Topic Modelling
  - Two Topics
  - Three or more Topics

High Level Summary

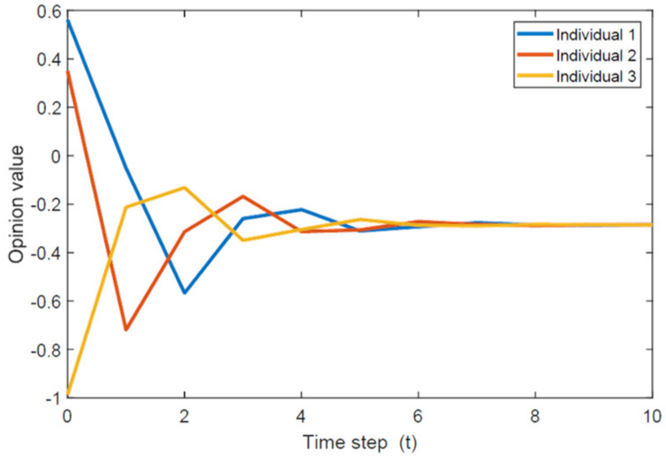
Objective: To study how disagreement in opinions can arise as a result of people viewing a set of topics with competing or different logical interdependencies

Motivation: Our opinion on Topic A is often dependent on our opinion on Topic B, because we build a set of logical interdependencies between multiple related topics.

Example: Everyone might agree that mentally challenging tasks can be as tough as physically challenging task but they might not agree that Go or Chess should be Olympic sports.

The Puzzle of Strong Diversity

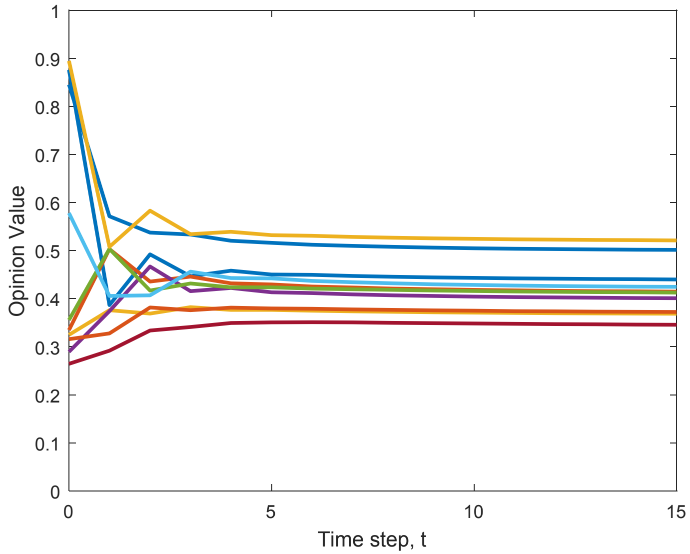
The DeGroot model: social influence in a strongly connected network leads to a consensus.



Some other models, e.g. Hegselmann-Krause or Altafini, leads to *weak diversity* where opinions are concentrated in clusters

What brings about *strong diversity* of opinions in strongly connected networks?

- Stubborn attachment to initial opinion [1]
- Desire to be unique [2]
- Heterogeneity in individuals' views on logically linked topics.



[1] N. E. Friedkin and E. C. Johnsen, "Social Influence and Opinions," *Journal of Mathematical Sociology*, vol. 15, no. 3-4, pp. 193–206, 1990

[2] M. Mas, A. Flache, and J. A. Kitts, "Cultural Integration and Differentiation in Groups and Organizations," in *Perspectives on Culture and Agent-based Simulations*. Springer, 2014, pp. 71–90

Multi-Topic DeGroot Model

The recent model [1] proposes that individual i's vector of opinions $\boldsymbol{x}_i(t) = \left[x_i^1(t), \ldots, x_i^m(t)\right]^\top$ evolves according to:

$$\boldsymbol{x}_i(t+1) = \sum_{j=1}^{n} w_{ij} \boldsymbol{C}_i \boldsymbol{x}_j(t) \,, \quad \sum_{j=1}^{n} w_{ij} = 1$$

$\boldsymbol{C}_i$ is the logic matrix capturing the logical interdependencies between the m topics for individual *i*. We will study it separately.

Consider a set of m related to

i's certainty on the truthfulnes

If $1 \geq x_i^p > 0$, then individual i considers p is true, while $-1 \leq x_i^p < 0$ means individual i considers p is false. The magnitude of x_i^p indicates the strength of i's certainty.

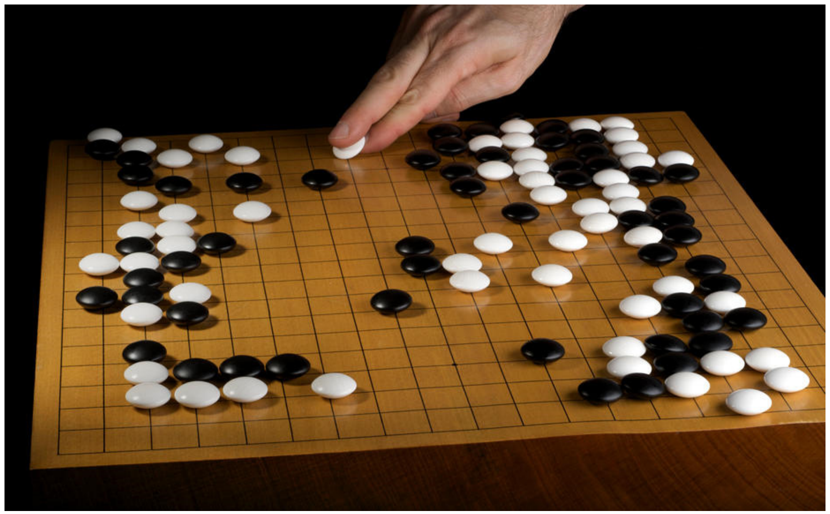
[1] S. E. Parsegov, A. V. Proskurnikov, R. Tempo, and N. E. Friedkin, "Novel Multidimensional Models of Opinion Dynamics in Social Networks," *IEEE Transactions on Automatic Control*, vol. 62, no. 5, pp. 2270–2285, 2017.

Multi-Topic DeGroot Model

If $x_i^p > 0$, then individual i considers p is true, while $x_i^p < 0$ means individual i considers p is false. The magnitude of x_i^p indicates the strength of i's certainty.

Statement 1: Mentally challenging activities can be as tough as physically challenging activities

Statement 2: Go and Chess should be Olympic sports.



Multi-Topic DeGroot Model

Statement 1: Mentally challenging activities can be as tough as physically challenging activities

Statement 2: Go and chess should be Olympic sports.

Let

$$\boldsymbol{C}_i = \begin{bmatrix} 1 & 0 \\ 0.5 & 0.5 \end{bmatrix}$$

Entry c_{jk} captures the logical dependence of topic j on topic k

Individual $i's$ opinion re Statement 1 is independent (obviously) of his opinion re statement 2. Hence $c_{12}=0$.

On the other hand, his opinion re statement 2 is clearly dependent on his opinion re statement 1. Hence c_{21} is nonzero.

If individual i has no neighbours, according to the multi-dimensional model:

$$\boldsymbol{x}_i(t+1) = \boldsymbol{C}_i \boldsymbol{x}_i(t)$$

Example: If $\boldsymbol{x}_i(0) = [1, -0.2]^\top$ then $\lim_{t\to\infty} \boldsymbol{x}_i(t) = [1, 1]^\top$.

Constraints on the Logic Matrix

From $\boldsymbol{x}_i(t+1) = \boldsymbol{C}_i \boldsymbol{x}_i(t)$ we can establish a number of constraints on $\boldsymbol{C}_i$ which reflect the fact that the logic matrix captures a logical process

Technical Assumptions:

- $\boldsymbol{C}_i$ has a semi-simple eigenvalue 1
- All other eigenvalues of $\boldsymbol{C}_i$ have modulus less than 1
- Diagonal entries are nonnegative

This ensures that $\boldsymbol{x}_i(t+1) = \boldsymbol{C}_i \boldsymbol{x}_i(t)$ always converges → individual i's belief structure is consistent

$\boldsymbol{C}_i$ represents individual i's logical processing of multiple related opinions to secure a consistent belief system

Constraints on the Logic Matrix

$$\boldsymbol{x}_i(t+1) = \sum_{j=1}^{n} w_{ij}\boldsymbol{C}_i\boldsymbol{x}_j(t)\,, \quad \sum_{j=1}^{n} w_{ij} = 1$$

Lower triangular logic matrices are very common. Specifically, we assume every individual i has a logic matrix with structure:

$$\boldsymbol{C}_i = \begin{bmatrix} 1 & & & \text{\Large 0} \\ c_{21,i} & c_{22,i} & & \\ \vdots & \vdots & \ddots & \\ c_{n1,i} & c_{n2,i} & \cdots & c_{nn,i} \end{bmatrix}$$

This is representative of a logical interdependence structure obtained by building upon an axiom or axioms

Convergence

With the required constraints on $\boldsymbol{C}_i$ and if the interaction network is suitably connected, and individual i's opinion evolves as:

$$\boldsymbol{x}_i(t+1) = \sum_{j=1}^{n} w_{ij}\boldsymbol{C}_i\boldsymbol{x}_j(t)\,, \quad \sum_{j=1}^{n} w_{ij} = 1$$

Then for every i, $\lim_{t\to\infty} \boldsymbol{x}_i(t) = \boldsymbol{x}_i^*$ exponentially fast, where $\boldsymbol{x}_i^* \in \mathbb{R}^m$ is the vector of final opinions for individual i.

Convergence always holds. But what about consensus in relation to each topic?

What are the final opinion values?

Strong Diversity in Two Topics

Suppose there are two topics, 1 and 2, and that 2 depends on 1 but not vice versa, i.e.

$$\boldsymbol{C}_i = \begin{bmatrix} 1 & 0 \\ c_{21,i} & c_{22,i} \end{bmatrix}$$

Then:

Consensus always occurs for topic 1

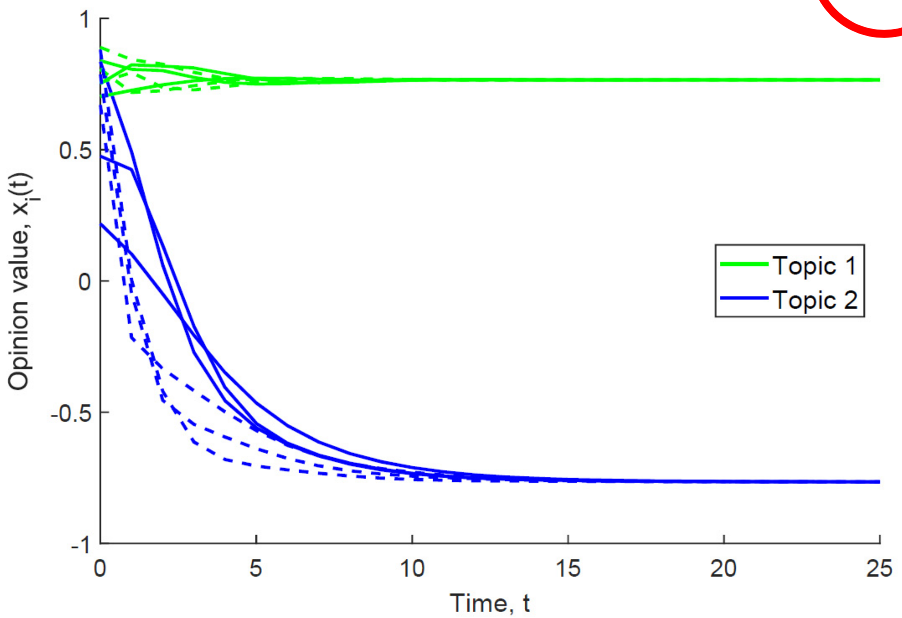
Consensus occurs for topic 2 if and only if there do not exist two individuals p, q such that $c_{21,q}$ and $c_{21,p}$ have opposite sign.

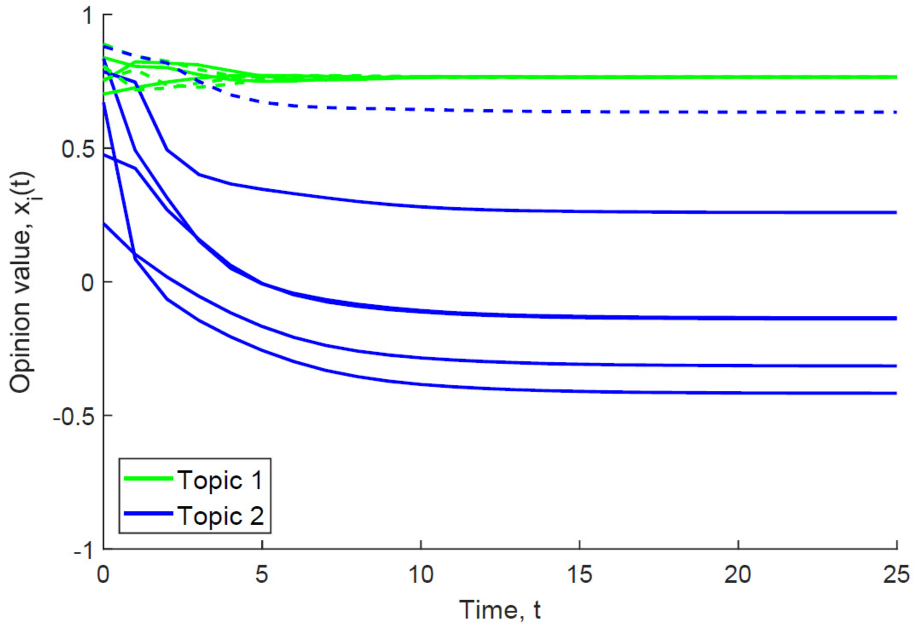
Disagreement in Topic 2 occurs iff there are individuals with **competing** logical interdependence structures in the network.

Strong Diversity in Two Topics

Disagreement in Topic 2 occurs if there are individuals with **competing** logical interdependence structures in the network

$$C_i = \begin{bmatrix} 1 & 0 \\ c_{21,i} & c_{22,i} \end{bmatrix}$$





No competition among C_i entries

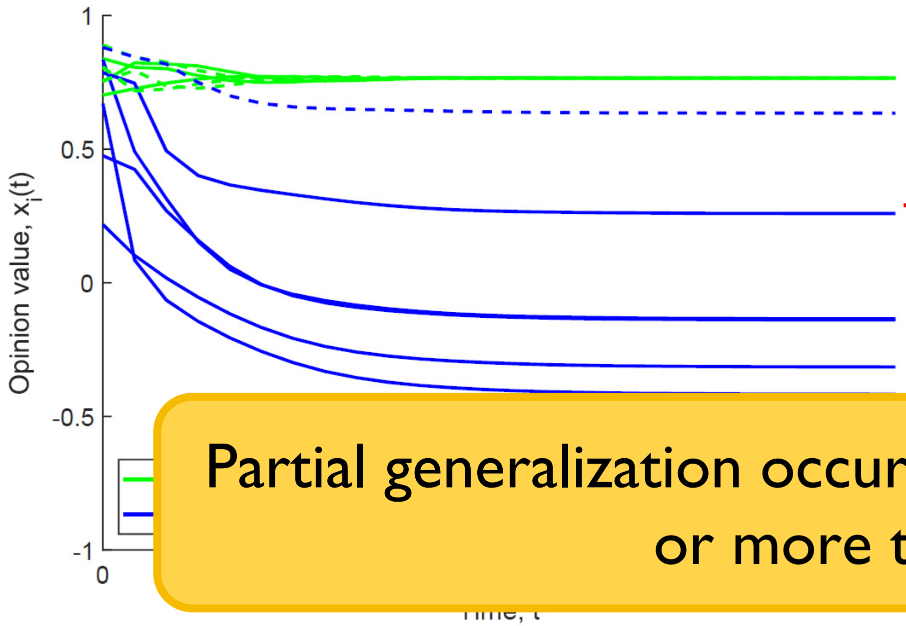
Competition among C_i entries

Strong Diversity in Two Topics

Disagreement in Topic 2 occurs if there are individuals with competing logical interdependence structures in the network

Statement 1: Mentally challenging activities can be as tough as physically challenging activities

Statement 2: Go and chess should be Olympic sports



$$C_i = \begin{bmatrix} 1 & 0 \\ 0.5 & 0.5 \end{bmatrix}$$

Partial generalization occurs when there are three or more topics.

Conclusions

- Introduced a recent model for discussion of logically interdependent topics

- Established conditions on the network and logic matrix for ensuring opinions converge to a steady state exponentially fast

- When there are two interdependent topics, strong diversity of opinions arises if and only if there are competing logical interdependence structures

Possible major future directions

- Using expressed and private opinion model to study the "spiral of silence", where people stop expressing opinions if they predict that others are moving away from their current opinion

- Capturing individual behaviour in social power dynamics: how does humility or arrogance during reflected self-appraisal change the evolution of social power?

- Comprehensive study of heterogeneity in logic structures: *do certain belief system structures more naturally lead to strong diversity?*