Maximum Capability of Feedback Control for Network Systems

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Abstract: In this paper, we investigate the capabilities of feedback mechanism in dealing with uncertainties for network systems. The study of maximum capability of feedback control was pioneered in Xie and Guo (2000) for scalar systems with nonparametric nonlinear uncertainty. In a network setting, nodes with unknown and nonlinear dynamics are interconnected through a directed interaction graph. Nodes can design feedback controls based on all available information, where the objective is to stabilize the network state. Using information structure and decision pattern as criteria, we specify three categories of network feedback laws, namely the global-knowledge/global-decision, network-flow/local-decision, and local-flow/local-decision feedback. We establish a series of network capacity characterizations for these three fundamental types of network control laws.

Key Words: adaptive control, nonlinear systems, feedback mechanism, network systems

1 Introduction

With original ideas traced back to the 1980s [2], network systems have been widely studied in the past decade. The central aim lies in resilient and scalable solutions for systems with a large number of interconnected agents to achieve collective goals ranging from consensus and formation to optimization and filtering [3–7]. Multi-agent control has evolved to a discipline in its own right with its boundaries being generalized even to the domain of quantum mechanics [8] and intrinsic equation solvers [9]. New tools such as graph theory and computational complexity have led to insightful new perspectives to the structural effects within the classical control problems [10, 11]. Besides these tremendous successes, however, it is also important to understand the limitations of feedback mechanism over network dynamics facing uncertainty. More specifically, a clear characterization to the capacity of feedback mechanism over a network in dealing with uncertainty, for centralized and distributed controllers, respectively, will help us understand the boundaries of controlling complex networks from a theoretical perspective.

In the seminal work [12], Xie and Guo established fundamental results on the capability of feedback mechanism with nonparametric nonlinear uncertainty for the following discrete-time model

$$\mathbf{y}(t+1) = f(\mathbf{y}(t)) + \mathbf{u}(t) + \mathbf{w}(t),\ t = 0, 1, \ldots$$

where the $\mathbf{y}(t)$, $\mathbf{u}(t)$, and $\mathbf{w}(t)$ are real numbers representing output, control, and disturbance, respectively. It was shown in [12] that with completely unknown plant model $f(\cdot) : \mathbb{R} \to \mathbb{R}$ and bounded but unknown disturbance signal $\mathbf{w}(t)$, a necessary and sufficient condition for the existence of stabilizing feedback control of the above system is that a type of Lipschitz norm of $f(\cdot)$ must be strictly smaller than $3/2 + \sqrt{2}$. This number, now referred to as the *Xie-Guo constant* in the literature, points to fundamental limitations of *all* feedback laws.

In this paper, we consider a network setting of the nonparametric uncertainty model in [12], where nodes with unknown nonlinear self-dynamics are interconnected through a directed interaction graph. For the ease of presentation the dynamics of the nodes are assumed to be identical, corresponding to homogenous networks. The interaction graph defines neighbor relations among the nodes, where measurement and control take place. Nodes can design any feedback controller using the information they have, and the objective is to stabilize the entire network, i.e., every node state in the network.

Three basic categories of feedback laws over such networks are carefully specified. In global-knowledge/global-decision feedback, every node knows network structure (interaction graph) and network information flow, and nodes can coordinate to make control decisions; in network-flow/local-decision feedback, each node only knows the network information flow and carries out decision individually; in local-flow/local-decision feedback, nodes only know information flow of neighbors and then make their own control decisions. Note that various existing distributed controllers and algorithms can be naturally put into one of the three categories. A series of network feedback capacity results has been established:

(i) For global-knowledge/global-decision and network-flow/local-decision control, the generic network feedback capacity is fully captured by a critical value

$$\big(3/2 + \sqrt{2}\big)/\|A_{\mathrm{G}}\|_\infty$$

where A_{G} is the network adjacency matrix.

(ii) For local-flow/local-decision control, there exists a structure-determined value being an lower bound of the network feedback capacity.

(iii) Network flow can be replaced by max-consensus enhanced local flows, where nodes only observe information flows from their neighbors as well as network extreme (max and min) states via max-consensus, and then the same feedback capacity can be reached.

Additionally, for strongly connected graphs, we manage to establish a universal impossibility theorem on the existence of stabilizing feedback laws.

The remainder of the paper is organized as follows. Section 2 introduces the network model and defines the prob-

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lem of interest. Section 3 presents the main results, followed by Section 4 presenting the network stabilizing controllers. Finally Section 5 concludes the paper by a few remarks pointing out a few interesting future directions. Due to space limitations we refer the readers to [1] for the detailed proofs of all claimed results.

Notation: The set of real numbers is denoted by $\mathbb{R}$, and the set of integers is denoted by $\mathbb{Z}$. A sequence $a_0, a_1, \ldots$ is abbreviated as $\langle a_t \rangle_{t\geq 0}$. For any real number a, $(a)^+$ is defined as $(a)^+ = \max\{a, 0\}$. For convenience we use $\mathrm{dist}(X, Y)$ to denote the distance between two sets X and Y in $\mathbb{R}$ by $\mathrm{dist}(X, Y) = \inf_{x\in X, y\in Y} |x - y|$, and simply $\mathrm{dist}(a, Y) := \inf_{y\in Y} |a - y|$, $\mathrm{dist}(a, b) = |a - b|$ for real numbers a and b.

2 The Model

2.1 Network Dynamics with Uncertainty

Consider a network with n nodes indexed in the set $\mathrm{V} = \{1, ..., n\}$. The network interconnection structure is represented by a directed graph $\mathrm{G} = (\mathrm{V}, \mathrm{E})$, where E is the arc set. Each arc (i, j) in the set E is an ordered pair of two nodes $i, j \in \mathrm{V}$, and link (i, i) is allowed at each node i defining a self-arc. The neighbors of node i, that node i can be influenced by, is defined as nodes in the set $\mathrm{N}_i := \{j : (j, i) \in \mathrm{E}\}$. Let $a_{ij} \in \mathbb{R}$ be a real number representing the weight of the directed arc (j, i) for $i, j \in \mathrm{V}$. The arc weights a_{ij} comply with the network structure in the sense that $a_{ij} \neq 0$ if and only if $(j, i) \in \mathrm{E}$. Let A_{G} be the adjacency matrix of the graph G with $[A_{\mathrm{G}}]_{ij} = a_{ij}$.

Time is slotted at $t = 0, 1, 2, \ldots$. Each node i holds a state $\mathbf{s}_i(t) = \big(\mathbf{x}_i(t), \mathbf{z}_i(t)\big)^\top \in \mathbb{R}^2$ at time t. The network dynamics are described by

$$\begin{aligned} \mathbf{z}_i(t+1) &= f(\mathbf{x}_i(t)) + \mathbf{e}_i(t) \\ \mathbf{x}_i(t+1) &= \sum_{j\in \mathrm{N}_i} a_{ij}\mathbf{z}_j(t+1) + \mathbf{u}_i(t) + \mathbf{w}_i(t), \end{aligned} \tag{1}$$

for $i \in \mathrm{V}$ and $t = 0, 1, 2, \ldots$, where f is a function mapping from $\mathbb{R}$ to $\mathbb{R}$, $\mathbf{u}_i(t) \in \mathbb{R}$ is the control input, and $\mathbf{e}_i(t), \mathbf{w}_i(t) \in \mathbb{R}$ are disturbances and noises. The system (1) describes the following node interactions: $\mathbf{x}_i(t)$ is the internal state of node i at time t, based on which an external state $\mathbf{z}_i(t + 1)$ is generated at that node; at time $t + 1$, the external states $\mathbf{z}_i(t + 1)$ are exchanged over the interaction graph G, defining the update of the internal states $\mathbf{x}_i(t + 1)$. In this way, $\big(\mathbf{x}_i(t), \mathbf{z}_i(t + 1)\big)$ is an input-output pair at node i for time t. We impose the following standing assumptions.

Assumption 1. (*Dynamics Uncertainty*) The function f is unknown, and the arc weight a_{ij} is known to the node i.

Assumption 2. (*Disturbance Boundedness*) The process disturbances $\mathbf{e}_i(t)$ and $\mathbf{w}_i(t)$ are unknown but bounded, i.e., there exist $e_*, w_* > 0$ such that

$$\big|\mathbf{e}_i(t)\big| \leq e_*, \quad \big|\mathbf{w}_i(t)\big| \leq w_*$$

for all t and for all $i \in \mathrm{V}$. Furthermore, the bounds e_* and w_* are unknown.

The Assumptions 1–2 are quite natural and general, which are adopted throughout the remainder of the paper without specific further mention. An illustration of this dynamical network model can be seen in Fig. 1. The dynamics of the internal node states $\mathbf{x}_i(t)$ can be written in a compact form as

$$\mathbf{x}_i(t+1) = \sum_{j\in \mathrm{N}_i} a_{ij} f(\mathbf{x}_j(t)) + \mathbf{u}_i(t) + \mathbf{d}_i(t), \tag{2}$$

where $\mathbf{d}_i(t) = \sum_{j\in \mathrm{N}_i} a_{ij}\mathbf{e}_j(t) + \mathbf{w}_i(t)$. It is clear that various work on network control, optimization, and computation would fall to this form of network dynamics [8, 9].

2.2 Feedback Laws over Networks

We now classify all possible network feedback control laws into categories determined by *information patterns* and *decision structures*. Such a classification is not straightforward at all bearing the following questions in mind:

(i) (*Knowledge*) How much would nodes know about the network itself, e.g., number of nodes n, network connectivity, or even the network topology G?

(ii) (*Flows*) How much would nodes know about the network information flows, e.g., availability of $\mathbf{x}_i(t)$, $\mathbf{z}_i(t)$, and $\mathbf{u}_i(t)$ for a neighbor, or a neighbors' neighbor of the node i?

(iii) (*Decisions*) To what level nodes could cooperate in determining the control actions, e.g., can a node i tell a neighbor j to stand by with $\mathbf{u}_j(t) = 0$ at time t to implement its own control input $\mathbf{u}_i(t)$?

Different answers to these questions will lead to drastically different scopes of network control rules. In this paper, we focus on a few fundamental forms of network feedback laws that from a theoretical perspective represent a variety of network control and computation results in the literature.

Denote $\mathbf{S}(t) = (\mathbf{s}_1(t)^\top \ldots \mathbf{s}_n(t)^\top)^\top$ and $\mathbf{U}(t) = (\mathbf{u}_1(t) \ \ldots \ \mathbf{u}_n(t))^\top$ for $t = 0, 1, \ldots$. Here without loss of generality we assume $\mathbf{z}_i(0) = 0$ for all i. The following definition specifies network and local flows.

Definition 1 *The network flow vector up to time t is defined as*

$$\Theta(t) := \Big(\mathbf{S}(0), \ldots, \mathbf{S}(t); \mathbf{U}(0), \ldots, \mathbf{U}(t-1)\Big)^\top.$$

The local network flow vector for node i up to time t is defined as

$$\begin{aligned} \Theta_i(t) := \Big(&\mathbf{s}_j(0)^\top, \ldots, \mathbf{s}_j(t)^\top; \\ &\mathbf{u}_j(0), \ldots, \mathbf{u}_j(t-1) : j \in \mathrm{N}_i \bigcup \{i\}\Big)^\top. \end{aligned}$$

Note that, here we have assumed that the $\mathbf{s}_i(t)$ and $\mathbf{u}_i(t)$ are known to a node i even if it does not hold a self arc $(i, i) \in \mathrm{E}$ (therefore $i \notin \mathrm{N}_i$). This is indeed quite natural and general which simplifies the presentation considerably.

2.2.1 Global-Knowledge/Global-Decision Feedback

Recall that A_{G} is the adjacency matrix of the graph G. Network controllers that have omniscient narration and omnipotent actuators at all nodes are certainly of primary interest.

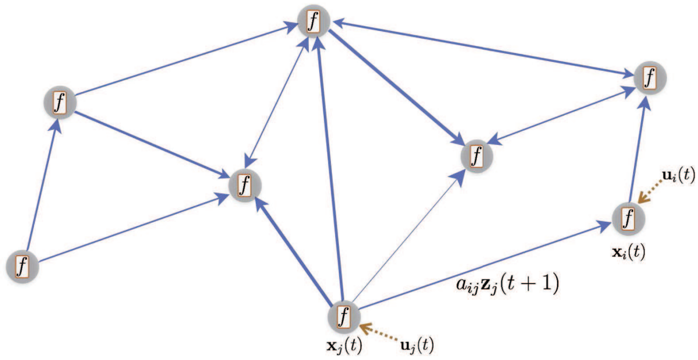

Fig. 1: A graphical diagram of the considered network model: (i) Interaction structure forms a directed graph where nodes are influenced by their in-neighbors and influence their out-neighbors; (ii) Node interaction rules are governed by completely unknown nonlinear dynamics and link width indicates the weight (strength) of interactions; (iii) Control inputs are applied to individual nodes subject to unknown disturbances.

Definition 2 *A network control rule in the form of*

$$\mathbf{U}(t) = \hbar_t\Big(\Theta(t); A_{\rm G}\Big), \quad t = 0, 1, \dots \tag{3}$$

where $\hbar_t$ is an arbitrary function mapping from $\mathbb{R}^{n(3t+1)}$ to $\mathbb{R}^n$ with $A_{\rm G}$ being a common knowledge, is termed a Global-Knowledge/Global-Decision Feedback Law for the network system (1).

To implement a global-knowledge/global-decision network control, one requires a network operator who knows the structure of the network (topology and arc weights), collects states and signals across the entire network, and then enforces control decisions on each individual node.

2.2.2 Network-Flow/Local-Decision Feedback

Knowing the network flow, nodes can still carry out individual control decisions even without knowledge of the entire network structure G. This will incur restrictions on feasible control rules, leading to the following definition.

Definition 3 *A network control rule in the form of*

$$\mathbf{u}_i(t) = \hbar_t^i\Big(\Theta(t); [A_{\rm G}]_{ij}, j \in {\rm N}_i\Big) \tag{4}$$

where independent with other nodes, $\hbar_t^i$ is an arbitrary function mapping from $\mathbb{R}^{n(3t+1)}$ to $\mathbb{R}$ for any $t = 0, 1, \dots,$ is termed a Network-Flow/Local-Decision Feedback Law for the network system (1).

The $\hbar_t^i$ being independent means that a node m can determine its control rule $\hbar_t^m$ without knowing or influencing the exact control decision values at any other node and for any given time. The following example helps clarify the ambiguity in the notion of independent decisions.

Example 1. Consider two nodes 1 and 2. The following control rule with q_t being a function with proper dimension for its argument

$$\begin{aligned} \mathbf{u}_1(t) &= q_t(\Theta(t)) \\ \mathbf{u}_2(t) &= 1 - q_t(\Theta(t)) \end{aligned} \tag{5}$$

implicitly holds the identity

$$\mathbf{u}_1(t) + \mathbf{u}_2(t) = 1$$

and therefore can only be implemented if the two nodes coordinate their respective inputs. In this sense (5) is a global-knowledge/global-decision feedback rather than a network-flow/local-decision feedback law. □

2.2.3 Local-Flow/Local-Decision Feedback

The notion of distributed control consists of three basis elements [7]: nodes only have a local knowledge of the network structure; nodes only receive and send information to a few neighbors; control and decision are computed by each node independently. Inspired by these criteria we impose the following definition.

Definition 4 *Any feedback control rule in the form of*

$$\mathbf{u}_i(t) = \hbar_t^i\Big(\Theta_i(t); [A_{\rm G}]_{ij}, j \in {\rm N}_i\Big) \tag{6}$$

with $\hbar_t^i : \mathbb{R}^{|{\rm N}_i \bigcup \{i\}|(3t+1)} \to \mathbb{R}$ being an arbitrary function independent with other nodes, is termed a Local-Flow/Local-Decision Feedback Law for the network system (1).

The three classes of network feedback laws are certainly not disjoint. In fact the set of global-knowledge/global-decision feedback contains the set of network-flow/local-decision feedback, which in turn contains the set of local-flow/local-decision feedback.

2.3 Network Stabilizability

We are interested in the existence of feedback control laws that stabilize the network dynamics (1) for the closed loop, as indicated in the following definition.

Definition 5 *A feedback law stabilizes the network dynamics (1) if there holds*

$$\sup_{t \geq 0} \Big(\big|\mathbf{x}_i(t)\big| + \big|\mathbf{z}_i(t)\big| + \big|\mathbf{u}_i(t)\big|\Big) < \infty, \ i \in {\rm V} \tag{7}$$

for the closed loop system.

2.4 Function Space

We need a metric quantifying the uncertainty in the node dynamical mode f. Let $\mathcal{F}$ denote the space that contains all $\mathbb{R} \to \mathbb{R}$ functions, where the $f \in \mathcal{F}$ are equipped with a quasi-norm defined by

$$\|f\|_{\mathrm{q}} := \lim_{\alpha\to\infty} \sup_{x,y\in\mathbb{R}} \frac{|f(x)-f(y)|}{|x-y|+\alpha}.$$

We refer to [12] for a thorough explanation of this quasi-norm and the resulting function space $\mathcal{F}$. Define

$$\mathcal{F}_L := \{f \in \mathcal{F} : \|f\|_q \le L\}$$

as a subspace in $\mathcal{F}$ consisting of functions bounded by $L > 0$ under quasi-norm $\|\cdot\|_{\mathrm{q}}$. Functions in $\mathcal{F}_L$ can certainly be discontinuous, but they are closely related to Lipschitz continuous functions. The following lemma holds, whose proof can be found in [12].

Lemma 1 *Let $\|f\|_{\mathrm{q}} \le L$. Then for any $\eta > 0$, there exists $c \ge 0$ such that*

$$|f(x)-f(y)| \le (L+\eta)|x-y| + c,\ \forall x,y \in \mathbb{R}. \tag{8}$$

As a result of Lemma 1, as long as $\|f\|_{\mathrm{q}}$ admits a finite number, the stabilizability condition (7) is equivalent to

$$\sup_{t\ge 0}(|\mathbf{x}_i(t)| + |\mathbf{u}_i(t)|) < \infty,\ i \in \mathrm{V},$$

which is in turn equivalent to

$$\sup_{t\ge 0}|\mathbf{x}_i(t)| < \infty,\ i \in \mathrm{V}.$$

Moreover, from Lemma 1, the set $\Gamma_L(f) := \big\{(\eta, c) : \text{Eq. (8) holds}\big\}$ is nonempty for any $f \in \mathcal{F}_L$. We further define a constant $W_f(r)$ associated with any $f \in \mathcal{F}_L$ and $r > L$

$$W_f(r) := \inf\big\{c : L+\eta < r, (\eta,c) \in \Gamma_L(f)\big\}. \tag{9}$$

3 Network Stabilizability Theorems

In this section, we present a series of possibility and/or impossibility results for the stabilizability of the network dynamics (1) for the three categories of feedback laws.

3.1 Global-Knowledge/Global-Decision Feedback

With global-knowledge/global-decision feedback, it turns out that, the infinity norm $\|A_{\mathrm{G}}\|_\infty$ of the the adjacency matrix A_{G}, i.e.,

$$\|A_{\mathrm{G}}\|_\infty = \max_{i\in\mathrm{V}} \sum_{j\in\mathrm{N}_i} \big|[A_{\mathrm{G}}]_{ij}\big|$$

plays a critical role.

Recall that $W_f(\cdot)$ is the function defined in (9). The following theorem characterizes a generic fundamental limit for the capacity of global-knowledge/global-decision feedback laws.

Theorem 1 (Generic Fundamental Limit) *Consider $\mathcal{F}_L$ in the function space $\mathcal{F}$. Then there exists a generic Global-Knowledge/Global-Decision feedback law that stabilizes the network dynamics (1) if and only if*

$$L < (3/2+\sqrt{2})/\|A_{\mathrm{G}}\|_\infty.$$

To be precise, the following statements hold.

(i) If $L < (3/2+\sqrt{2})/\|A_{\mathrm{G}}\|_\infty$, then there exists a global-knowledge/global-decision feedback control law that stabilizes the system (1) for all $f \in \mathcal{F}_L$ and for all interaction graphs G. *In fact, with $L < (3/2+\sqrt{2})/\|A_{\mathrm{G}}\|_\infty$ we can find a global-knowledge/global-decision feedback control law that ensures*

$$\limsup_{t\to\infty}|\mathbf{x}_i(t)| \le \Big(W_f\big(3/2+\sqrt{2})/\|A_{\mathrm{G}}\|_\infty\big) + 2e_*\Big) \cdot \|A_{\mathrm{G}}\|_\infty + w_*,\ \forall i \in \mathrm{V}.$$

(ii) If $L \ge (3/2+\sqrt{2})/\|A_{\mathrm{G}}\|_\infty$, then for any global-knowledge/global-decision feedback law (3) and any initial value $\mathbf{X}(0)$, there exist an interaction graph G *and a function $f \in \mathcal{F}_L$ under which there holds*

$$\limsup_{t\to\infty}\max_{i\in\mathrm{V}}|\mathbf{x}_i(t)| = \infty.$$

Note that the error bound of the internal state $\mathbf{x}_i(t)$ in statement (i) can be extended to the external state $\mathbf{z}_i(t)$ by

$$\begin{aligned}&\limsup_{t\to\infty}|\mathbf{z}_i(t)|\\ &\le (5/2+\sqrt{2})\Big(W_f\big(3/2+\sqrt{2})/\|A_{\mathrm{G}}\|_\infty\big)+2e_*\Big)\\ &\quad + w_*(3/2+\sqrt{2})/\|A_{\mathrm{G}}\|_\infty + |f(0)|.\end{aligned}$$

utilizing the fact that $L < (3/2+\sqrt{2})/\|A_{\mathrm{G}}\|_\infty$. Moreover, we should emphasize that the critical value $L < (3/2+\sqrt{2})/\|A_{\mathrm{G}}\|_\infty$ established in Theorem 1 is for general interaction graphs. In fact, as will be shown in its proof, the graph constructed for the necessity proof is a very special one containing exactly one self arc. For a given graph G, e.g., a complete graph or a directed cycle, it is certainly possible that the corresponding network dynamics are stabilizable even with $L \ge (3/2+\sqrt{2})/\|A_{\mathrm{G}}\|_\infty$. Finding such feedback capacity values for any given interaction graph seems to be rather challenging, as illustrated in the following example.

Example 2. Consider two nodes, indexed by 1 and 2, respectively, which both possess a self link with unit weight and have no link between them (see Fig. 2). From our standing network model their internal dynamics read as

$$\begin{aligned}\mathbf{x}_1(t+1) &= f(\mathbf{x}_1(t)) + \mathbf{d}_1(t) + \mathbf{u}_1(t)\\ \mathbf{x}_2(t+1) &= f(\mathbf{x}_2(t)) + \mathbf{d}_2(t) + \mathbf{u}_2(t).\end{aligned} \tag{10}$$

A first sight indicates that (10) appears to be merely two copies of the scaler model considered in [12]. Indeed, directly from results established in [12], we know that if $f \in \mathcal{F}_L$ with $L < (3/2+\sqrt{2})$, we can stabilize each $\mathbf{x}_i(t)$ with control input $\mathbf{u}_i(t)$ being a feedback from its own dynamics. However, note that with global information, one cannot rule out the case where

(i) Node 1 stabilizes itself;
(ii) Node 2 uses the information flow vector[1] at the node 1:

$$\Theta_1^*(t) := \Big(\mathbf{s}_1(0)^\top,\ldots,\mathbf{s}_1(t)^\top;\mathbf{u}_1(0),\ldots,\mathbf{u}_1(t-1)\Big)^\top$$

to design its controller.

[1] Node that $\mathbf{z}_1(t)$ can simply be chosen as $\mathbf{x}_1(t+1) - \mathbf{u}_1(t)$.

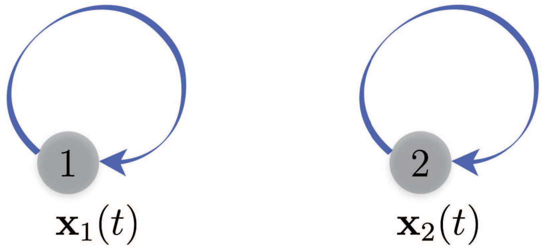

Fig. 2: A simple two-node network with two self links only.

In fact, $\Theta_1^*(t)$ can be rather informative even for node 2 because it can be utilized putting an effective estimate to the unknown function $f(\cdot)$, which is essential for $\mathbf{u}_2(t)$. Furthermore, one cannot rule out an even more interesting scenario where nodes 1 and 2 design their controllers *cooperatively* since now they share a common information set. Therefore, it is not clear whether the critical feedback capacity value $3/2+\sqrt{2}$, which applies to the two nodes respectively when they are separate [12], will continue to apply when they form a network with shared information. An intuitive way of understanding this is that while the two nodes in system (10) share no *dynamical* interaction, a global view of the network information flow will create hidden *intellectual* interaction through their control inputs. □

Furthermore, we introduce

$$\|A_{\mathrm{G}}\|_\sharp = \min_{i,j\in \mathrm{V}} \Big\{ \big|[A_{\mathrm{G}}]_{ij}\big| : [A_{\mathrm{G}}]_{ij} \neq 0 \Big\}$$

where of course $\|A_{\mathrm{G}}\|_\sharp = 0$ if $A_{\mathrm{G}} = 0$. It is easy to verify that $\|\cdot\|_\sharp$ is not even a proper matrix semi-norm. The following result however provides a further impossibility characterization of global-knowledge/global-decision feedback laws for networks with strong connectivity based on the metric $\|A_{\mathrm{G}}\|_\sharp$.

Theorem 2 (Impossibility Theorem with Connectivity) *Suppose the underlying graph* G *is strongly connected. Assume that either* $[A_{\mathrm{G}}]_{ij} \geq 0$ *for all* $i,j \in \mathrm{V}$ *or* $[A_{\mathrm{G}}]_{ij} \leq 0$ *for all* $i,j \in \mathrm{V}$. *If* $L \geq 4/\|A_{\mathrm{G}}\|_\sharp$, *then for any Global-Knowledge/Global-Decision Feedback Law (3) and any initial value* $\mathbf{X}(0)$, *there exists a function* $f \in \mathcal{F}_L$ *under which there always holds*

$$\limsup_{t\to\infty} |\mathbf{x}_i(t)| = \infty.$$

3.2 Network-Flow/Local-Decision Feedback

It is obvious from its definition that any network-flow/local-decision feedback law is by itself a global-knowledge/global-decision control as well. In other words, any possibility result for network stabilization achieved by network-flow/local-decision feedback laws can also be viewed as a possibility result for global-knowledge/global-decision controls. Remarkably enough, the contrary also holds true for generic graphs, as indicated in the following result.

Theorem 3 (Generic Fundamental Limit) *Consider* $\mathcal{F}_L$ *in the function space* $\mathcal{F}$. *Then there exists a generic Network-Flow/Local-Decision Feedback Law that stabilizes the network dynamics (1) if and only if* $L < (3/2+\sqrt{2})/\|A_{\mathrm{G}}\|_\infty$.

In fact, the error bound in Theorem 1.(i) continues to hold for network-flow/local-decision feedback laws. Putting Theorem 1 and Theorem 3 together we learn that, for generic interaction graphs, information flow plays a more critical role for feedback capacity compared to decision structures.

3.3 Local-Flow/Local-Decision Feedback

Recall that $a_{ij} = [A_{\mathrm{G}}]_{ij}$ is the weight of arc $(j,i) \in \mathrm{E}$. Let $\langle p_t^i\rangle_{t=1}^\infty$ and $\langle q_t^i\rangle_{t=1}^\infty$ be non-negative sequences for $i \in \mathrm{V}$ that satisfy the following recursive relations:

$$\begin{aligned} p_{t+1}^i &\leq \Big(M \sum_{j\in \mathrm{N}_i} |a_{ij}| \max_{1\leq s\leq t} \{p_s^j, q_s^j\} + \omega - \sum_{s=1}^t p_s^i \Big)^+, \\ q_{t+1}^i &\leq \Big(M \sum_{j\in \mathrm{N}_i} |a_{ij}| \max_{1\leq s\leq t} \{p_s^j, q_s^j\} + \omega - \sum_{s=1}^t q_s^i \Big)^+. \end{aligned} \tag{11}$$

Induced by recursion (11), we present the following metric for the matrix A_{G}

$$\begin{aligned} \|A_{\mathrm{G}}\|_\dagger := \sup \Big\{ & M : \text{For any } \omega > 0 \text{ Eq.(11)} \\ & \text{implies } \sum_{t=1}^\infty (p_t^i + q_t^i) < \infty \text{ for all } i \in \mathrm{V} \Big\}. \end{aligned} \tag{12}$$

Note that the positivity of $\|A_{\mathrm{G}}\|_\dagger$ can be shown for nontrivial graphs G by establishing $\|A_{\mathrm{G}}\|_\dagger \geq 1/\|A_{\mathrm{G}}\|_\infty$.

The following theorem establishes a sufficiency condition for feedback stabilizability of the network dynamics, effectively providing a lower bound of the feedback capacity for local-flow/local-decision feedback laws.

Theorem 4 (Generic Possibility Theorem) *Consider* $\mathcal{F}_L$ *in the function space* $\mathcal{F}$. *There exists a generic Local-Flow/Local-Decision Feedback Law that stabilizes the network dynamics (1) if*

$$L/\|A_{\mathrm{G}}\|_\dagger < 1.$$

More precisely, if $L < \|A_{\mathrm{G}}\|_\dagger$, *then there exists a Local-Information/Local-Decision feedback law that stabilizes the network dynamics (1) for all* $f \in \mathcal{F}_L$ *and all graphs* G.

3.4 Max-Consensus Enhanced Feedback Capacity

It is evident from the above discussions that knowledge of information flows heavily influences the capacity of feedback laws. Network flow enables universal feedback laws that apply to generic graphs as shown in Theorem 1 and Theorem 3, while local flows can be rather insufficient in stabilizing a network with uncertainty.

However, various distributed algorithms have been developed in the literature serving the aim of achieving collective goals using local node interactions only, which often leads to propagation of certain global information to local levels. One particular type of such algorithms is the so-called *max-consensus*, where a network of nodes holding real values can agree on the network maximal value in finite time steps by distributed interactions [4, 15]. In this subsection, we show simple max-consensus algorithms can fundamentally change the nature of network feedback capacity.

[**Max-Consensus Enhancement**] At time t, each node i has the knowledge of the vector $(\mathbf{x}_i(t-1), \mathbf{z}_i(t))^\top$. From time t to $(t+1)^-$, nodes run a max-consensus algorithm on the first entry by

$$\mathbf{s}_i[k+1] = \big(\mathbf{x}_{\arg\max_{j\in \mathrm{N}_i} \mathbf{x}_j[k]}, \mathbf{z}_{\arg\max_{j\in \mathrm{N}_i} \mathbf{x}_j[k]}\big)^\top$$

where with slight abuse of notation we neglect the time index $t-1$ in $\mathbf{x}_i$, and t in $\mathbf{z}_i$, and use $[k]$ to represent time steps in the max-consensus algorithm. It is clear [15] that in a finite number of steps in k (therefore it is safe to assume before time $t+1$), all nodes will hold

$$\overline{\mathbf{s}}(t) = \big(\overline{\mathbf{x}}(t-1), \overline{\mathbf{z}}(t)\big)^\top$$

with $\overline{\mathbf{x}}(t-1) = \max_i \mathbf{x}_i(t-1)$ and $\overline{\mathbf{z}}(t) = \mathbf{z}_{\arg\max_{j\in \mathrm{V}} \mathbf{x}_j(t-1)}(t)$.

Similarly, $\underline{\mathbf{s}}(t) = \big(\underline{\mathbf{x}}(t-1), \underline{\mathbf{z}}(t)\big)^\top$ with $\underline{\mathbf{x}}(t) = \min_i \mathbf{x}_i(t)$ and $\underline{\mathbf{z}}(t) = \mathbf{z}_{\arg\min_{j\in \mathrm{V}} \mathbf{x}_j(t-1)}(t)$ can also be possessed by all nodes i before time $t+1$ with another parallel min-consensus algorithm. We are now ready to introduce the following definition.

Definition 6 *The max-consensus enhanced local flow vector for node i up to time t is defined as*

$$\Theta_i^{\mathrm{e}}(t) := \Big(\Theta_i(t)^\top,\ \overline{\mathbf{s}}(1)^\top, \dots, \overline{\mathbf{s}}(t)^\top,\ \underline{\mathbf{s}}(1)^\top, \dots, \underline{\mathbf{s}}(t)^\top\Big)^\top.$$

Moreover, any feedback control rule in the form of

$$\mathbf{u}_i(t) = \hbar_t^i\Big(\Theta_i^{\mathrm{e}}(t); [A_{\mathrm{G}}]_{ij}, j \in \mathrm{N}_i\Big) \tag{13}$$

with $\hbar_t^i$ being an arbitrary function independent with other nodes, is termed a Max-Enhanced-Local-Flow/Local-Decision Feedback Law for the network system (1).

It turns out that, max-consensus-enhanced-local-flow/local-decision feedback laws have the same capacity in stabilizing the generic network dynamics (1) as the global-knowledge/global-decision feedback.

Theorem 5 (Generic Fundamental Limit) *Consider $\mathcal{F}_L$ in the function space $\mathcal{F}$. Then there exists a generic Max-Consensus-Enhanced-Local-Flow/Local-Decision Feedback Law that stabilizes the network dynamics (1) if and only if $L < (3/2+\sqrt{2})/\|A_{\mathrm{G}}\|_\infty$.*

Although Theorem 5 exhibits the same fundamental limit as Theorem 1, the error bound of $\limsup_{t\to\infty} |\mathbf{x}_i(t)|$ becomes inevitably more conservative. This suggests potential difference at performance levels for the two different types of controllers.

4 The Feedback Laws

In this section, we present the control rules that are used in the possibility claims of the above network stabilization theorems.

4.1 Local Feedback with Network Flow

We now present a local feedback controller in the form of Definition 3 with entire network flow information. Denote

$$\overline{\mathbf{y}}(t) := \max\{\mathbf{x}_i(s) : s = 0, \dots, t; i = 1, \dots, n\}, \tag{14}$$
$$\underline{\mathbf{y}}(t) := \min\{\mathbf{x}_i(s) : s = 0, \dots, t; i = 1, \dots, n\}. \tag{15}$$

as the maximal and minimal states at all nodes and among all time steps up to t, respectively. The controller contains two parts, an estimator and a distributed feedback rule.

[**Estimator**] For each $i \in \mathrm{V}$, $t \geq 1$, there exists $[v_i]_t \in \mathrm{V}$ and $0 \leq [s_i]_t \leq t-1$ that satisfies

$$\mathbf{x}_{[v_i]_t}([s_i]_t) \in \arg\min_{\mathbf{x}_j(\tau)} \Big\{ |\mathbf{x}_i(t) - \mathbf{x}_j(\tau)| : j \in \mathrm{V}, \tau \in [0, t-1] \Big\}. \tag{16}$$

Then at time t, an estimator for $f(\mathbf{x}_i(t))$ made by nodes that are i's neighbors is given by

$$\widehat{f}(\mathbf{x}_i(t)) := \mathbf{z}_{[v_i]_t}([s_i]_t + 1). \tag{17}$$

[**Feedback**] Fix any positive ϵ. Let $\mathbf{u}_i(0) = 0$ for all $i \in \mathrm{V}$. For all $t \geq 1$ and all $i \in \mathrm{V}$, we define

$$\mathbf{u}_i(t) = \begin{cases} -\sum\limits_{j\in \mathrm{N}_i} a_{ij} \widehat{f}(\mathbf{x}_j(t)), \\ \qquad \text{if } |\mathbf{x}_k(t) - \mathbf{x}_{[v_k]_t}([s_k]_t)| \leq \epsilon \text{ for all } k \in \mathrm{V}; \\ -\Big(\sum\limits_{j\in \mathrm{N}_i} a_{ij} \widehat{f}(\mathbf{x}_j(t))\Big) + \frac{1}{2}(\underline{\mathbf{y}}(t) + \overline{\mathbf{y}}(t)), \\ \qquad \text{otherwise.} \end{cases} \tag{18}$$

It is clear that Eq. (17)–(18) lead to a well defined Network-Flow/Local-Decision feedback control law that is consistent with Definition 3. In the following, we will prove that it suffices to use the control law (17)–(18) to establish the stabilizability statements in Theorem 1 and Theorem 3.

4.2 Global Feedback with Global Information

The feedback controller given in (17)–(18) already manages to support the stabilization statement in Theorem 1 as well since by definition a local-decision controller is a special form of global-decision controllers. It is however of independent interest seeing how stabilizing network controllers with essentially centralized structure might work. A clear answer to this question for general graphs seems rather difficult. Nevertheless, we have been able to construct two insightful examples, with the interaction graphs being a directed path and a directed cycle (see Fig. 3), respectively, which partially illustrates some spirit of the problem.

4.2.1 Path Graph

Consider the path graph with exactly one self link at the root node[2] shown in Fig. 3 with $a_{11} = 1$. Let us consider the following network controller.
[**Control at root node**]: For each $t \geq 1$, there exists $0 \leq s_t \leq t-1$ that satisfies

$$\mathbf{x}_1(s_t) \in \arg\min_{\mathbf{x}_1(\tau)} \Big\{ |\mathbf{x}_1(t) - \mathbf{x}_1(\tau)| : \tau \in [0, t-1] \Big\}.$$

[2]This self link is added for the sake of providing a nontrivial example yet as simple as possible.

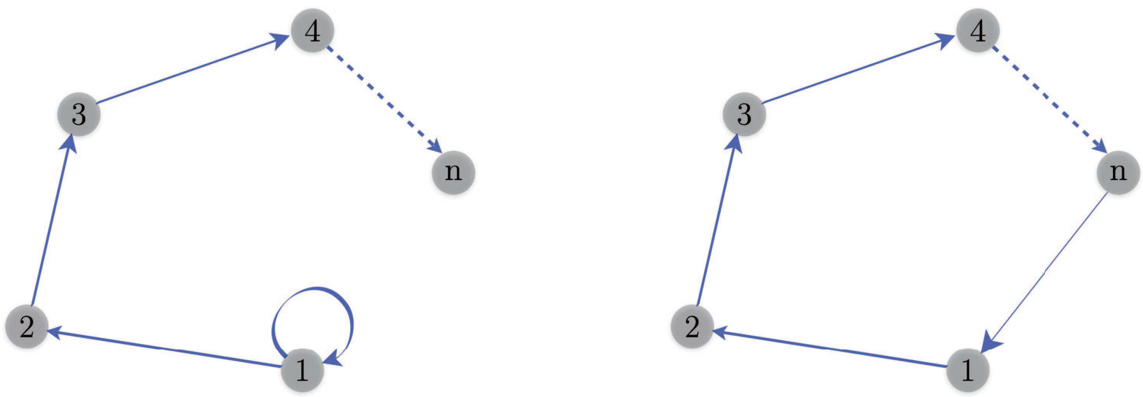

Fig. 3: A directed path graph with one self link at the root node (left), and a directed cycle graph (right). For these two graphs we can construct essential global-decision controllers that will stabilize the network states.

At time t, an estimator for $f(\mathbf{x}_1(t))$ is given by

$$\widehat{f}(\mathbf{x}_1(t)) := \mathbf{z}_1(s_t+1).$$

We define

$$\mathbf{u}_1(t) = -\widehat{f}(\mathbf{x}_1(t)) + \frac{1}{2}(\underline{\mathbf{x}}_1(t)+\overline{\mathbf{x}}_1(t)),$$

[**Control at other nodes**]: $\mathbf{u}_i(t) = 0$ for all $i = 2,\dots,n$ and all t.

The above network controller will stabilize the node states for any $f \in \mathcal{F}_L$ with $L < 3/2+\sqrt{2}$ citing the result of [12] directly. To implement such a controller, nodes need to know the network structure: node 1 must know it is a root. All nodes must know G is a directed path. Nodes $2,\dots,n$ must also know that the controller at node 1 will stabilize $\mathbf{x}_1(t)$. Therefore, the controller falls into the category of global-knowledge/global-decision network control, but not into other categories in our definition.

4.2.2 Cycle Graph

Consider the directed cycle graph shown in Fig. 3 and assume all arc weights are equal to one. Define $\kappa(b)$ for any (positive, negative, or zero) integer $b \in \mathbb{Z}$ by $\kappa(b)$ being the unique integer satisfying $1 \le \kappa(b) \le n$ and $\kappa(b) = b \mod n$.

[**Controller**] For each node $i \in \mathrm{V}$, there exists $0 \le [s_i]_t \le t-1$ that satisfies

$$\mathbf{x}_{\kappa([s_i]_t-t+i-1)}\big([s_i]_t\big) \in \arg\min_{\mathbf{x}_{\kappa(\tau-t+i-1)}(\tau)} \Big\{|\mathbf{x}_{\kappa(i-1)}(t)-\mathbf{x}_{\kappa(\tau-t+i-1)}(\tau)| : 0 \le \tau \le t-1\Big\}. \quad (19)$$

An estimator for $f(\mathbf{x}_{\kappa(i-1)}(t))$ is given by

$$\widehat{f}(\mathbf{x}_{\kappa(i-1)}(t)) := \mathbf{z}_{\kappa([s_i]_t-t+i-1)}\big([s_i]_t+1\big).$$

Let $\mathbf{u}_i(0)=0$. For $t \ge 1$, let

$$\mathbf{u}_i(t) = -\widehat{f}(\mathbf{x}_{\kappa(i-1)}(t)) + \Big(\max_{0\le\tau\le t} \mathbf{x}_{\kappa(\tau-t+i-1)}(\tau) + \min_{0\le\tau\le t} \mathbf{x}_{\kappa(\tau-t+i-1)}(\tau)\Big)/2. \quad (20)$$

Clearly (20) relies essentially on global decisions because the node number and the cycle structure are necessary knowledge and more importantly, the inherent symmetry in (20) requires coordination among the nodes. Suppose $f \in \mathcal{F}_L$ with $L < 3/2+\sqrt{2}$. Now we show the controller (20) indeed stabilizes the network dynamics.

According to (1) and the cyclic network structure, for any $i \in \mathbb{Z}$, there holds

$$\mathbf{x}_{\kappa(i+t+1)}(t+1) = f(\mathbf{x}_{\kappa(i+t)}(t)) + \mathbf{u}_{\kappa(i+t+1)}(t) + \mathbf{d}_{\kappa(i+t+1)}(t). \quad (21)$$

We further write $[x_i]_t = \mathbf{x}_{\kappa(i+t)}(t)$, $[d_i]_t = \mathbf{d}_{\kappa(i+t+1)}(t)$, and also $\overline{[x_i]}_t = \max_{0\le s\le t}[x_i]_s$, $\underline{[x_i]}_t = \min_{0\le s\le t}[x_i]_s$. With these new variables (21) becomes

$$[x_i]_{t+1} = f([x_i]_t) + \Big(-\widehat{f}([x_i]_t) + \frac{1}{2}(\overline{[x_i]}_t + \underline{[x_i]}_t)\Big) + [d_i]_t, \quad (22)$$

which coincides with the closed loop dynamics for scalar system presented in [12]. Therefore, quoting the results in [12] we immediately know if $L < 3/2+\sqrt{2}$ then

$$\limsup_{t\to\infty} \big|[x_i]_t\big| < \infty,\ i \in \mathrm{V},$$

or equivalently, $\limsup_{t\to\infty} |\mathbf{x}_i(t)| < \infty$ and the network dynamics have been stabilized.

4.3 Local Feedback with Local Flow

We now present a local-flow/local-decision feedback law that will enable us to prove Theorem 4.

[**Estimator**] Fix $i \in \mathrm{V}$. For $j \in \mathrm{N}_i$ and $t \ge 1$, there exist $[\mathrm{v}_{ij}]_t \in \mathrm{V}$ and $0 \le [s_{ij}]_t \le t-1$ that satisfy

$$\mathbf{x}_{[\mathrm{v}_{ij}]_t}([s_{ij}]_t) \in \arg\min_{\mathbf{x}_k(s)} \Big\{|\mathbf{x}_j(t)-\mathbf{x}_k(s)| : k \in \mathrm{N}_i \bigcup \{i\}, s \in [0,t-1]\Big\}. \quad (23)$$

We define an estimator at node i for $f(\mathbf{x}_j(t))$, $j \in \mathrm{N}_i$ at time t by

$$\widehat{f}_i(\mathbf{x}_j(t)) = \mathbf{z}_{[\mathrm{v}_{ij}]_t}([s_{ij}]_t+1). \quad (24)$$

[Feedback] Let $\mathbf{u}_i(0) = 0$ for all $i \in \mathrm{V}$. Then for all $t \geq 1$ and all $i \in \mathrm{V}$, we let

$$\mathbf{u}_i(t) = -\sum_{j \in \mathrm{N}_i} a_{ij} \widehat{f}_i(\mathbf{x}_j(t)) + \mathbf{x}_i(0). \qquad (25)$$

It is also clear that Eq. (24)–(25) form a distributed controller with local information under Definition 4.

4.4 Local Feedback with Max-Consensus-Enhanced Local Flow

Let $i \in \mathrm{V}$ and $t \geq 1$. We denote

$$\mathcal{X}_i(t) = \Big\{\overline{\mathbf{x}}(s) : 0 \leq s \leq t-1\Big\} \bigcup \Big\{\underline{\mathbf{x}}(s) : 0 \leq s \leq t-1\Big\} \bigcup \Big\{\mathbf{x}_j(s) : j \in \mathrm{N}_i \bigcup \{i\}, 0 \leq s \leq t-1\Big\}$$

as the set of states whose estimated data under function f can be accessible to node i at time t. We define a function $\mathcal{K}_i^t(\cdot)$ over $\mathcal{X}_i(t)$ by

$$\mathcal{K}_i^t(x) = \begin{cases} \mathbf{z}_j(s+1), \\ \quad \text{if } x = \mathbf{x}_j(s),\ j \in \mathrm{N}_i \bigcup \{i\},\ 0 \leq s \leq t-1; \\ \overline{\mathbf{z}}(s+1),\ \text{if } x = \overline{\mathbf{x}}(s),\ 0 \leq s \leq t-1; \\ \underline{\mathbf{z}}(s+1),\ \text{if } x = \underline{\mathbf{x}}(s),\ 0 \leq s \leq t-1. \end{cases} \qquad (26)$$

[Estimator] Let node i estimate $f(\mathbf{x}_j(t))$ for $j \in \mathrm{N}_i \bigcup \{i\}$ at time t by

$$\widehat{f}_i(\mathbf{x}_j(t)) = \mathcal{K}_i^t\Big(\arg\min_{x \in \mathcal{X}_i(t)} \big\{|\mathbf{x}_j(t) - x|\big\}\Big). \qquad (27)$$

[Feedback] Let $\mathbf{u}_i(0) = 0$ for all $i \in \mathrm{V}$. Then for all $t \geq 1$ and all $i \in \mathrm{V}$, we let

$$\mathbf{u}_i(t) = -\sum_{j \in \mathrm{N}_i} a_{ij} \widehat{f}_i(\mathbf{x}_j(t)) + \frac{1}{2}(\overline{\mathbf{y}}(t) + \underline{\mathbf{y}}(t)). \qquad (28)$$

Eq. (27)-(28) form a Max-Consensus-Enhanced-Local-Flow/Local-Decision controller satisfying Definition 6.

5 Conclusions

This paper proposes a framework for studying the fundamental limitations of feedback mechanism in dealing with uncertainties over network systems. Using information structure and decision pattern as criteria, three classes of feedback laws over such networks were defined, under which critical or sufficient feedback capacities were established, respectively. These preliminary results reveal a promising path towards clear descriptions of feedback capabilities over complex network systems, many important problems yet remain open. It is very interesting to ask the same feedback capacity questions when only a subset of nodes can be monitors of the information flow and another subset of nodes can be controlled as anchors [10, 11]. Parametric network model as generalizations to the work of [13] and [14] would be intriguing because such a model will certainly yield a strong connection between distributed estimation and distributed control.

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