A Complete Solution to Mean Field Linear Quadratic Control

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Abstract: This paper studies uniform stabilization and social optimality for mean field linear quadratic control systems, where subsystems are coupled via dynamics and individual costs. For the finite horizon case, we first obtain a set of forward-backward stochastic differential equations (FBSDE) from the analysis of the social cost variation, and then design a feedback-type control by decoupling the FBSDE. The set of decentralized control laws is shown to have asymptotic social optimality. For the infinite horizon case, we design an asymptotically social optimal decentralized control using solutions of two Riccati equations. Two equivalent conditions are further given for uniform stabilization of the systems in different cases. Finally, we show that such decentralized control is a representation of the feedback control in previous works.

Key Words: Mean field game, open-loop control, social optimality, forward-backward stochastic differential equation

1 Introduction

Mean field games have drawn increasing attention in many fields including system control, applied mathematics and economics [6, 8, 13]. The mean field game involves a very large population of small interacting players with the feature that while the influence of each one is negligible, the impact of the overall population is significant. By combining mean field approximations and individual’s best response, the dimensionality difficulty is overcome. Mean field games and control have found wide applications, including smart grids [10, 27], finance, economics [9, 14, 31], and social sciences [5], etc.

By now, mean field games have been intensively studied in the LQ (linear-quadratic) framework [7, 19, 20, 25, 29, 32]. Huang et al. developed the Nash certainty equivalence (NCE) based on the fixed-point method and designed an $\epsilon$-Nash equilibrium for LQ mean field games with discount costs based on the NCE approach [19, 20]. The NCE approach was then applied to the cases with long run average costs [25] and with Markov jump parameters [32], respectively. Bensoussan et al. applied the adjoint equation approach and the fixed-point theorem to obtain a sufficient condition for the unique existence of the equilibrium strategy over a finite-time horizon [7]. For other aspects of mean field games, readers are referred to [11, 22, 24, 37] for nonlinear mean field games, [36] for oblivious equilibrium in dynamic games, [18, 33, 34] for mean field games with major players, [17, 29] for robust mean field games.

Besides noncooperative games, social optima in mean field models have also drawn much attention. The social optimum control refers to that all the players cooperate to optimize the common social cost—the sum of individual cost, which is usually regarded as a type of team decision problem [15, 30]. Huang et al. considered social optima in mean field LQ control, and provided an asymptotic team-optimal solution [21]. Wang and Zhang [35] investigated a mean field social optimal problem where a Markov jump parameter appears as a common source of randomness. For further literature, see [23] for social optima in mixed games, [3] for team-optimal control with finite population and partial information.

Most previous results on mean field games and control are given by using the fixed-point method [20, 21, 25, 33, 35]. However, the widely used tool in fixed-point analysis is contraction mapping, which is very conservative for the general systems. In this paper, we break away from the fixed-point method and solve the mean field control problem by tackling forward-backward stochastic differential equations (FBSDE). In recently years, some substantial progress for the optimal LQ control has been made by solving the FBSDE. See [38, 40] for details.

This paper investigates uniform stabilization and social optimality for mean field LQ control systems, where subsystems are coupled via dynamics and individual costs. For the finite horizon case, we first obtain a set of forward-backward stochastic differential equations (FBSDE) by examining the variation of the social cost, and then give a centralized feedback-type control laws by decoupling the FBSDE. With mean field approximations, we design a set of decentralized control laws, which is further shown to have asymptotic social optimality. For the infinite horizon case, we design a set of decentralized control laws using solutions of two Riccati equations, which is shown to be asymptotically social optimal. Two equivalent conditions are further given for uniform stabilization of the systems when the state weight $Q$ is semi-positive definite or only symmetric. Finally, we show such set of decentralized control laws is a representation of the feedback control in previous works.

The main contributions of the paper are summarized as follows.

- From the analysis of the social cost variation we obtain a set of open-loop control laws, and then design a feedback-type decentralized control by tackling FBSDE with mean field approximations.
- In the case $Q \geq 0$, the necessary and sufficient conditions are given for uniform stabilization of the systems with the help of the Riccati equations’ solutions and the system’s observability.
- In the case that $Q$ is only symmetric, the necessary and sufficient conditions are given for uniform stabilization of the systems using the Hamiltonian matrices.
- It is under nonconservative assumptions that we obtain an asymptotically social optimal decentralized control,
and such control is shown to be equivalent to the feedback control given by the fixed-point method in previous works [21, 35].

The organization of the paper is as follows. Section 2 formulates the socially optimal control problem. In Section 3, we design a decentralized control by tackling BSDE for the finite horizon case, which is shown to be asymptotically social optimal. In Section 4, we design asymptotically social optimal control laws for the infinite horizon case and further give some equivalent conditions for uniform stabilization of the systems. Such set of decentralized control laws is compared with the feedback control of previous works in Section 5. In Section 6, two numerical examples are provided to show the effectiveness of the proposed control. Section 7 concludes the paper.

The following notation will be used throughout this paper. \( \| \cdot \| \) denotes the Euclidean vector norm or matrix spectral norm. For a vector \( z \) and a matrix \( Q \), \( \| z \|^2_Q = z^T Q z \), and \( Q > 0 \) \((Q \geq 0)\) means that \( Q \) is positive definite (semi-positive definite). For two vectors \( x, y \), \( (x, y) = x^T y \). \( C([0, \infty), \mathbb{R}^n) \) is the space of all the \( n \)-dimensional continuous functions on \([0, \infty)\), and \( C_{\rho/2}([0, \infty), \mathbb{R}^n) \) is a subspace of \( C([0, \infty), \mathbb{R}^n) \) which is given by \( \{ f \mid f_0^\infty e^{-\rho t} \| f(t) \|^2 dt < \infty \} \). For convenience of presentation, we use \( C, C_1, C_2, \cdots \) to denote generic positive constants, which may vary from place to place.

2 Problem Formulation

Consider a large population systems with \( N \) agents. Agent \( i \) evolves by the following stochastic differential equation:

\[
\begin{align*}
\dot{x}_i(t) & = [A x_i(t) + B u_i(t) + G x_i(N)(t) + f(t)] dt + \sigma(t) dW_i(t), \quad 1 \leq i \leq N, \\
\end{align*}
\]

where \( x_i \in \mathbb{R}^n \) and \( u_i \in \mathbb{R}^r \) are the state and input of the \( i \)th agent, \( x_i(N)(t) = \frac{1}{N} \sum_{j=1}^N x_j(t), f, \sigma \in C_{\rho/2}([0, \infty), \mathbb{R}^n) \). \( \{ W_i(t), 1 \leq i \leq N \} \) are a sequence of independent \( 1 \)-dimensional Brownian motions. The cost function of agent \( i \) is given by

\[
\begin{align*}
J_i(u) & = \frac{1}{2} \mathbb{E} \int_0^\infty e^{-\rho t} \left\{ \| x_i(t) - \Gamma x_i(N)(t) - \eta \|^2_Q + \| u_i(t) \|^2_R \right\} dt, \\
\end{align*}
\]

where \( Q \) is symmetric and \( R > 0 \). \( u = \{ u_1, \ldots, u_N \} \) are the admissible control set is given by

\[
\mathcal{U}_{d,i} = \left\{ u_i \mid u_i(t) \in \sigma(x_i(s), 0 \leq s \leq t) \right\}.
\]

For comparison, define the centralized control set as

\[
\mathcal{U}_{c,i} = \left\{ u_i \mid u_i(t) \in \sigma \left( \bigcup_{i=1}^N \mathcal{F}_i \right) \right\},
\]

where \( \mathcal{F}_i = \sigma(x_i(0), W_i(s), 0 \leq s \leq t), i = 1, \ldots, N \).

In this paper, we mainly study the following problem:

\((0)\). Seek a set of decentralized control strategies to minimize social cost for the system (1)-(2), i.e., \( \inf_{u_i \in \mathcal{U}_i} J_{soc} \) where

\[
J_{soc} = \sum_{i=1}^N J_i(u).
\]

Assume

\( A1 \) \( x_i(0), i = 1, \ldots, N, \) are mutually independent and have the same mathematical expectation. \( \mathbb{E} x_i(0) \equiv \bar{x}_0 \), and there exists a constant \( C_0 \) (independent of \( N \)) such that \( \max_{1 \leq i \leq N} \mathbb{E} \| x_i(0) \|^2 < C_0 \).

3 Finite Horizon Problem

For the convenience of design, we first consider the finite horizon problem \((P1)\).

\[
\inf_{u_i \in \mathcal{U}_{d,i}} J^F_{soc}(u),
\]

where \( J^F_{soc}(u) = \inf_{u_i \in \mathcal{U}_{d,i}} \sum_{i=1}^N J^F_i(u) \) and

\[
J^F_i(u) = \frac{1}{2} \mathbb{E} \int_0^T e^{-\rho t} \left\{ \| x_i(t) - \Gamma x_i(N)(t) - \eta \|^2_Q + \| u_i(t) \|^2_R \right\} dt.
\]

We first give an equivalent condition for the convexity of Problem \((P1)\).

**Proposition 1** Problem \((P1)\) is convex in \( u \) if and only if for any \( u_i \in \mathcal{U}_i, i = 1, \ldots, N, \)

\[
\sum_{i=1}^N \int_0^T \left\{ \| y_i - \Gamma y_i(N) \|^2_Q + \| u_i \|^2_R \right\} dt \geq 0,
\]

where \( y_i \) satisfies

\[
dy_i = [Ay_i + G y_i(N) + Bu_i] dt,
\]

\( y_i(0) = 0, i = 1, 2, \ldots, N. \)

By examining the variation of \( J^F_{soc} \), we obtain the existence conditions of the centralized optimal control.

**Theorem 1** Suppose that \((P1)\) is convex and \( R > 0 \). Then \((P1)\) has a (unique) optimal control if and only if the following equation system admits a (unique) set of solutions \((x_i, p_i, i = 1, \cdots, N)\):

\[
\begin{align*}
\dot{x}_i & = (Ax_i - BR^{-1} B p_i + G x_i(N) + f) dt + \sigma dW_i, \\
p_i & = -((A - \rho I) x_i + GT p_i + G^T p_i(N)) dt \\
& - (Q x_i - \Sigma x_i(N) - \bar{\eta}) dt + \sum_{j=1}^N \beta_j dW_j, \\
x_i(0) & = x_{i0}, \quad p_i(T) = 0, \quad i = 1, \cdots, N,
\end{align*}
\]

where \( \Sigma \Delta \equiv \Gamma^T Q + \Gamma^T Q \Gamma, \bar{\eta} \Delta \equiv \eta - \Gamma^T Q \eta, p_i(N) = \sum_{i=1}^N p_i, \) and furthermore the optimal control is given by \( \hat{u}_i = -R^{-1} B^T p_i \).

**Proof.** See Appendix A. \( \square \)
It follows from (5) that
\[
\begin{align*}
\dot{x}^{(N)} &= ((A + G)x^{(N)} - BR^{-1}B^T p^{(N)} + f)dt \\
&\quad + \frac{1}{N} \sum_{i=1}^{N} \sigma_i dW_i, \\
dp^{(N)} &= - \left[ (A + G - \rho I)^T p^{(N)} \\
&\quad - (I - \Gamma)^T Q(I - \Gamma) x^{(N)} + \tilde{\eta} \right] dt \\
&\quad + \frac{1}{N} \sum_{i=1}^{N} \sum_{j=1}^{N} \beta_{ij}^l dW_j, \\
x^{(N)}(0) &= \frac{1}{N} \sum_{i=1}^{N} x_{i0}, \quad p^{(N)}(T) = 0.
\end{align*}
\]

Let \( p_i = P x_i + K x^{(N)} + s \). Then by (5) and Itô’s formula,
\[
dp = \dot{P} x + P (Ax - BR^{-1}B^T(P x_i + K x^{(N)} + s) + G x^{(N)} + f) dt + \sigma_i dW_i + \dot{s} + \dot{K} x^{(N)} + K \left[ [(A + G)x^{(N)} - BR^{-1}B^T(P + K)x^{(N)} + s] + f \right] dt + \frac{1}{N} \sum_{i=1}^{N} \sigma_i dW_i.
\]

This implies that \( \beta_{ij}^l = \frac{1}{N} K \sigma + P \sigma, \beta_{ij}^r = \frac{1}{N} K \sigma, \ j \neq i, \rho P = \dot{P} + A^T P + PA - PBR^{-1}B^T P + Q, \ P(T) = 0, \),
\[
\rho K = \dot{K} + (A + G)^T K + K(A + G) - PBR^{-1}B^T K - KBR^{-1}B^T P + G^T P + PG - KB \bar{R}^{-1}B^T K - \Sigma_1, \ K(T) = 0, \rho s = s + [A + G - BR^{-1}B^T(P + K)]^T s + (P + K) f - \tilde{\eta}, \ s(T) = 0.
\]

Then \( \dot{u}_i = - R^{-1}B^T(P x_i + K x^{(N)} + s) \).

**Theorem 2** Let A1 hold and \( Q \geq 0 \). Then Problem (P1) has an optimal control
\[
\dot{u}_i = - R^{-1}B^T(P x_i + K x^{(N)} + s),
\]
where \( P, K \) and \( s \) are determined by (7)-(9).

**Proof.** Denote \( \Pi = P + K \). Then from (8) and (9), \( \Pi \) satisfies
\[
\dot{\Pi} + (A + G)^T \Pi + \Pi(A + G) - \Pi BR^{-1}B^T \Pi + (I - \Gamma)^T Q(I - \Gamma) = 0, \quad \Pi(T) = 0.
\]

Note that \( Q \geq 0 \) and \( R > 0 \). By [2, 39], (7) and (10) admit unique solutions \( P \geq 0 \) and \( \Pi \geq 0 \), respectively, which implies that (8) and (9) have unique solutions \( K \) and \( s \), respectively. Then by [26, 40], FBSDE (5) admits a solution \((x_i, p_i, i = 1, \ldots, N)\). By Theorem 1, Problem (P1) has an optimal control given by \( \dot{u}_i = - R^{-1}B^T(P x_i + K x^{(N)} + s) \), where \( P, K \) and \( s \) are determined by (7)-(9).

As an approximation to \( x^{(N)} \) in (6), we have
\[
\ddot{x} = (A + G)x - BR^{-1}B^T \Pi x + f, \quad \dot{x}(0) = \bar{x}_0.
\]

Then we may design a set of decentralized control laws as follows:
\[
\dot{u}_i(t) = - R^{-1}B^T(P \dot{x}_i(t) + K \dot{x}(t) + s(t)), \quad 0 \leq t \leq T, \quad i = 1, \ldots, N,
\]
where \( P, K \) and \( s \) are determined by (7)-(9), and \( \dot{x} \) and \( \dot{x}_i \) satisfies (11) and
\[
d\dot{x}_i = [(A - BR^{-1}B^T P) \dot{x}_i - BR^{-1}B^T (K \dot{x} + s) + G \dot{x}^{(N)} + f] dt + \sigma_i dW_i.
\]

We now give the result of asymptotic optimality. Due to page limitations, the proof is omitted. Denote
\[
\mathcal{U}_c = \{ (u_1, \ldots, u_N) \mid u_i(t) \in \sigma(\bigcup_{i=1}^{N} F_i) \}.
\]

**Theorem 3** Let A1 hold and \( Q \geq 0 \). For (P1), the set of decentralized control laws \( \{ \dot{u}_1, \ldots, \dot{u}_N \} \) given by (12) has asymptotic social optimality, i.e.,
\[
\left| \frac{1}{N} J_{soc}(\dot{u}) - \frac{1}{N} \inf_{u \in \mathcal{U}_c} J_{soc}(u) \right| = O\left( \frac{1}{\sqrt{N}} \right).
\]

**4 Infinite Horizon Problem**

Based the analysis in Section 3, we may design the following decentralized control for (PO):
\[
\dot{u}_i(t) = - R^{-1}B^T(P \dot{x}_i(t) + (\Pi - P) \dot{x}(t) + s(t)), \quad t \geq 0, \quad i = 1, \ldots, N,
\]
where \( P, \Pi \) are determined by
\[
\rho P = A^T P + PA - PBR^{-1}B^T P + Q, \quad P(T) = 0, \rho \Pi = (A + G)^T \Pi + \Pi(A + G) - \Pi BR^{-1}B^T \Pi + (I - \Gamma)^T Q(I - \Gamma),
\]
and \( s, \dot{x} \in C_{p/2}([0, \infty), \mathbb{R}^n) \) are determined by
\[
\rho s = s + [A + G - BR^{-1}B^T \Pi]^T s + \Pi f - \tilde{\eta}, \quad \frac{d\dot{x}}{dt} = (A + G) \dot{x} - BR^{-1}B^T (\Pi \dot{x} + s) + f, \quad \dot{x}(0) = \bar{x}_0.
\]

Here the existence conditions of \( P, \Pi, s \) and \( \dot{x} \) need to be investigated further.

We provide the following assumptions:
**A2** \((A - \frac{\rho}{2} I, B) \) is stabilizable, and \((A + G - \frac{\rho}{2} I, B) \) is stabilizable.

**A3** \( Q \geq 0 \), \((A - \frac{\rho}{2} I, \sqrt{Q}) \) is observable, and \((A + G - \frac{\rho}{2} I, \sqrt{Q}(I - \Gamma)) \) is observable.
Lemma 1 Under A1)-A3), (15) and (16) admit unique solutions $P > 0$, $\Pi > 0$, respectively, and (17)-(18) admits a set of unique solutions $s, \bar{x} \in C_{\rho/2}([0, \infty), \mathbb{R}^n)$.

Proof. From A2)-A3) and [2], (15) and (16) admit unique solutions $P > 0$, $\Pi > 0$ such that $\hat{A} - \frac{\rho}{2} I$ and $A + G - BC^{-1} B^T \Pi - \frac{\rho}{2} I$ are Hurwitz. It is straightforward that $s, \bar{x} \in C_{\rho/2}([0, \infty), \mathbb{R}^n)$.

We further introduce the following assumption.

A4) $\hat{A} + G - \frac{\rho}{2} I$ is Hurwitz.

Lemma 2 Let A1)-A4) hold. Then for $(P0)$

$$\mathbb{E} \int_0^\infty e^{-\rho t}\|\bar{x}^{(N)}(t) - \bar{x}(t)\|^2 dt = O\left(\frac{1}{N}\right).$$

Proof. We have

$$\bar{x}^{(N)}(t) - \bar{x}(t) = e^{(\hat{A} + G) t}[\bar{x}^{(N)}(0) - \bar{x}(0)] + \frac{1}{N} \sum_{i=1}^N \int_0^t e^{(A + G)(t-s)}\sigma dW_i(s).$$

Thus,

$$\mathbb{E} \int_0^\infty e^{-\rho t}\|\bar{x}^{(N)}(t) - \bar{x}(t)\|^2 dt \leq 2\mathbb{E} \int_0^\infty \left\| e^{(A+G-t)\rho} \right\|^2 \left\| \bar{x}^{(N)}(0) - \bar{x}(0) \right\|^2 dt + 2\mathbb{E} \int_0^\infty e^{-\rho t} \frac{1}{N} \left\| \int_0^t e^{(A+G)(t-s)}\sigma dW_i(s) \right\|^2 dt$$

$$\leq 2\int_0^\infty \left\| e^{(A+G-t)\rho} \right\|^2 \mathbb{E} \left\| \bar{x}^{(N)}(0) - \bar{x}(0) \right\|^2 dt + \frac{2}{N} \int_0^\infty e^{-\rho t} \int_0^t \text{tr} \left[ \sigma^T e^{(A+G+\hat{A}T+GT)(t-s)} \right] ds dt$$

$$\leq \frac{2}{N} \int_0^\infty \left\| e^{(A+G-t)\rho} \right\|^2 \mathbb{E} \left\| \bar{x}^{(N)}(0) \right\|^2 dt + \frac{2}{N} \int_0^\infty e^{-\rho t} C \left\| 1 - e^{\delta t} \right\| dt \leq O\left(\frac{1}{N}\right),$$

where $\delta < \rho/2$.

The systems are shown to be uniformly stabilized.

Theorem 4 Let A1)-A4) hold. Then for any $N$,

$$\sum_{i=1}^N \mathbb{E} \int_0^\infty e^{-\rho t} \left( \|\hat{x}_i(t)\|^2 + \|\hat{u}_i(t)\|^2 \right) dt < \infty.$$ 

Proof. See Appendix B.

Remark 1 $M_1$ and $M_2$ are Hamilton matrices. The Hamilton matrix plays a significant role in studying general algebraic Riccati equations. See more details of the property of Hamilton matrices in [1, 28].

To show Theorem 6, we need two lemmas.

Lemma 3 Let A1) hold. Assume that (15) and (16) admit stabilizing solutions, respectively, and $\hat{A} + G$ is Hurwitz. Then

$$\sum_{i=1}^N \mathbb{E} \int_0^\infty e^{-\rho t} \left( \|\hat{x}_i(t)\|^2 + \|\hat{u}_i(t)\|^2 \right) dt < \infty.$$ 

Proof. From the definition of stabilizing solutions, $A - BR^{-1}B^T P$ and $A + G - BR^{-1}B^T \Pi$ are Hurwitz. By the argument in the proof of Theorem 4, the lemma follows.

Lemma 4 [28] Equations (15) and (16) admit stabilizing solutions if and only if A2) holds and both $M_1$ and $M_2$ have no eigenvalues on the imaginary axis.

The Proof of Theorem 6. By a similar argument in the proof of Theorem 4 combined with Lemmas 3 and 4, the Theorem follows.

Example 1 Consider a scalar system with $A = a$, $B = b$, $G = g$, $Q = q$, $\Gamma = \gamma$, $R = r > 0$. Then

$$M_1 = \begin{bmatrix} a - \rho/2 & b^2/r \\ b^2/r & -a + \rho/2 \end{bmatrix},$$

$$M_2 = \begin{bmatrix} a + g - \rho/2 & b^2/r \\ b^2/r & q(1-\gamma)^2 - (a + g - \rho/2) \end{bmatrix}.$$ 

By direct computations, neither $M_1$ nor $M_2$ has eigenvalues in imaginary axis if and only if

$$a - \frac{\rho}{2} > \frac{b^2}{r} q > 0,$$

$$a + g - \frac{\rho}{2} > \frac{b^2}{r} (1-\gamma)^2 q > 0.$$ 

For a Riccati equation (15), $P$ is called a stabilizing solution if $P$ satisfies (15) and all the eigenvalues of $A - BR^{-1}B^T P$ are in left half-plane.
Note that if $q > 0$, then (23) holds, and if $(1 - \gamma)^2 q > 0$, then (24) holds.

For this model, the Riccati equation (15) is written as
\[
b^2 p^2 - (2a - \rho)p - q = 0.
\]
(25)

Let $\Delta = 4[(a - \rho/2)^2 + b^2q/r]$. If (23) holds then $\Delta > 0$, which implies (25) admits two solutions. If $q > 0$ then (25) has a unique positive solution such that $a - b^2p/r - \rho/2 = -\sqrt{\Delta}/2 < 0$.

Assume that (23) and (24) hold. By Theorem 6, the system is uniformly stable if and only if $(a - \rho/2, b)$ is stabilizable (i.e., $b \neq 0$ or $a - \rho/2 < 0$), and $a - b^2p/r - \rho/2 + g < 0$. Note that $a - b^2p/r - \rho/2 < 0$. When $g \leq 0$, we have $a - b^2p/r - \rho/2 + g < 0$.

We are in a position to state the asymptotic optimality of the decentralized control. Due to page limitations, the proof is omitted.

**Theorem 7** Let A1)-A4) hold. The set of decentralized control laws $\{\hat{u}_1, \cdots, \hat{u}_N\}$ given by (14) has asymptotic social optimality, i.e.,
\[
\left| \frac{1}{N} J_{soc}(\hat{u}) - \frac{1}{N} \inf_{u \in \mathcal{U}} J_{soc}(u) \right| = O\left(\frac{1}{\sqrt{N}}\right).
\]

**5 Comparison of Different Solutions**

In this section, we compare the decentralized control laws in (14) with the feedback decentralized control in previous works [21].

We first introduce a definition from [4].

**Definition 1** For a control problem with an admissible control set $\mathcal{U}$, a control law $u \in \mathcal{U}$ is said to be a representation of another control $u^* \in \mathcal{U}$ if
(i) they both generate the same unique state trajectory, and
(ii) they both have the same open-loop value on this trajectory.

For Problem (P0), let $f = 0, G = 0$. In [21, Theorem 4.3], the decentralized control law is given by
\[
\hat{u}_i = -BR^{-1}(P_i x_i + s), \quad i = 1, \cdots, N,
\]
(26)
where $P$ is the semi-positive solution of (15), and $s = \bar{K}\hat{x} + \phi$. Here $\bar{K}, \hat{x}$ and $\phi \in C_{\rho/2}([0, \infty), \mathbb{R}^n)$ determined by
\[
\begin{align*}
\rho \bar{K} &= \bar{K}(A - BR^{-1}B^TP) + (A - BR^{-1}B^TP)^T \bar{K} - KBR^{-1}B^T \bar{K}^T - \Sigma_1, \\
\frac{d\bar{x}^T}{dt} &= (A - BR^{-1}B^TP)\hat{x}^T - BR^{-1}B^T(\bar{K}\hat{x}^T + \phi), \\
\frac{d\phi}{dt} &= -[A - BR^{-1}B^T(P + \bar{K}) - \rho I] \phi + \bar{\eta}.
\end{align*}
\]

By comparing this with (16)-(18), one can obtain that $\bar{K} = \Pi - P, \hat{x} = \hat{x}^T$ and $\phi = s$. From the analysis above, we have the equivalence of the two decentralized control laws.

**Proposition 2** The set of decentralized control laws $\{\bar{u}_1, \cdots, \bar{u}_N\}$ in (14) is a representation of $\{\hat{u}_1, \cdots, \hat{u}_N\}$ given in (26).

**6 Numerical Examples**

In this section, two numerical examples are provided to illustrate the effectiveness of the proposed decentralized control laws.

We first consider a scalar system with 50 agents in Problem (P0). Take $B = 1, G = -0.2, f(t) = 1, \sigma(t) = 0.1, \rho = 0.6, \Gamma = -0.2, \eta = 5, Q = 1, R = 1$ in (1) - (2). The initial states of 50 agents are taken independently from a normal distribution $N(5, 0.5)$. Then, under the control law (14), the state trajectories of agents for the case with $A = 0.2$ and $A = 1$ are shown in Fig. 1 and Fig. 2, respectively. After the transient phase, the states of agents behave similarly and achieve agreement roughly.

Fig. 1: State trajectories of 50 agents with $A = 0.2$.

Fig. 2: State trajectories of 50 agents with $A = 1$.

For the case $A = 1$, the trajectories of $\bar{x}$ and $\hat{x}^{(N)}$ in Problems (P0) is shown in Fig. 3. It can be seen that $\bar{x}$ and $\hat{x}^{(N)}$ coincide well, which illustrate the consistency of mean field approximations.

Fig. 3: State trajectories of 50 agents with $A = 1$.

Next, we consider the 2-dimensional case of (P0). Take
parameters as follows: \( A = \begin{bmatrix} 0.1 & 0 \\ -1 & 0.2 \end{bmatrix}, B = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \), 
\( G = \begin{bmatrix} -0.5 & 0 \\ 0 & -0.3 \end{bmatrix}, B = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \), 
\( Q = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \), \( \Gamma = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} \), \( R = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} \), \( \eta = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \) and \( \sigma = 0.5 \). Denote \( \tilde{x}_i(t) = \begin{bmatrix} x_1^i(t) \\ x_2^i(t) \end{bmatrix} \), where both of \( \tilde{x}_i^1(0) \) and \( \tilde{x}_i^2(0) \) are taken independently from a normal distribution \( N(5, 0.5) \). Under the control (14), the trajectories of \( \tilde{x}_i^1 \) and \( \tilde{x}_i^2 \) are shown in Fig. 4 and Fig. 5, respectively.

![Fig. 4: Trajectories of \( \tilde{x}_i^1 \), \( i = 1, \ldots, 50. \)](image)

![Fig. 5: Trajectories of \( \tilde{x}_i^2 \), \( i = 1, \ldots, 50. \)](image)

7 Concluding Remarks

In this paper, we considered uniform stabilization and social control for mean field LQ control systems, where subsystems are coupled via dynamics and individual costs. For the finite and infinite horizon cases, we design open-loop decentralized control laws by using solutions of Riccati equations, respectively, which are further shown to be asymptotically social optimal. Two equivalent conditions are further given for uniform stabilization of the systems. Finally, we show such decentralized control is equivalent to the feedback control in previous works.

A Proof of Theorem 1

Necessity. Suppose that \( \{\tilde{u}_1, \ldots, \tilde{u}_N\} \) is a centralized optimal control to Problem (P1). \( \tilde{x}_i \) is the state of agent \( i \) under the control \( \tilde{u}_i \). Let \( \delta u_i = u_i - \tilde{u}_i, i = 1, 2, \ldots, N \), where \( \forall u_i, \tilde{u}_i \in \mathcal{U}_{c,i}, \mathbb{E} \int_0^T ||u_i||^2 dt < \infty \) and \( \mathbb{E} \int_0^T ||\tilde{u}_i||^2 dt < \infty \). Denote \( \delta x_i = x_i - \tilde{x}_i, i = 1, 2, \ldots, N \), and \( \delta x^{(N)} = \frac{1}{N} \sum_{j=1}^N \delta x_j \). Let \( \delta J_{soc}(\tilde{u}_i) \) be the first variation of \( J_{soc}^{(N)} \) with respect to \( (\delta u_1, \ldots, \delta u_N) \). By (1),

\[
d(\delta x_i) = [A(\delta x_i) + B(\delta u_i) + G(\delta x^{(N)})]dt,
\]

\(\delta x_i(0) = 0, \quad i = 1, 2, \ldots, N.\) (A.1)

Then we have

\[
0 = \delta J_{soc}(\tilde{u}_i) = \sum_{i=1}^N \mathbb{E} \int_0^T e^{-pT} \left[ \langle \delta x_i, - (\Gamma \hat{x}^{(N)} + \eta) \rangle \right] dt.
\]

Assume

\[
dp_i = \alpha_i dt + \beta_i^1 dW_i + \sum_{j \neq i} \beta_i^2 dW_j,
\]

\( p_i(T) = 0, \quad i = 1, \ldots, N, \)

where \( \alpha_i \) and \( \beta_i^2 \) are to be determined. Then by Itô’s formula,

\[
0 = \mathbb{E} \left[ e^{-pT} (p_i(T), \delta x_i(T)) - (p_i(0), \delta x_i(0)) \right]
\]

\[
= \mathbb{E} \int_0^T \left( \langle \alpha_i, \delta x_i \rangle + \langle p_i, (A - \rho I)\delta x_i + B\delta u_i \rangle \right) dt.
\]

It follows by (A.2)-(A.4) that

\[
0 = \mathbb{E} \sum_{i=1}^N \int_0^T e^{-pT} \left[ \langle Q(\hat{x}_i, - (\Gamma \hat{x}^{(N)} + \eta)), \delta x_i \rangle - \Gamma \delta x^{(N)} \right] dt
\]

\[= \sum_{i=1}^N \mathbb{E} \int_0^T e^{-pT} \left[ \langle \alpha_i + (A - \rho I) p_i + G^T p^{(N)}, \delta x_i \rangle \right] dt
\]

\[= \sum_{i=1}^N \mathbb{E} \int_0^T e^{-pT} \left[ \langle R \delta u_i + B^T p_i, \delta u_i \rangle \right] dt
\]

\[+ \sum_{i=1}^N \mathbb{E} \int_0^T e^{-pT} \left[ Q(\hat{x}_i, (I - \Gamma) \hat{x}^{(N)} - \eta)) - \Gamma^T Q((I - \Gamma) \hat{x}^{(N)} - \eta) + \alpha_i + (A - \rho I) p_i + G^T p^{(N)}, \delta x_i \right] dt,
\]

which leads to

\[
\alpha_i = - \left[ (A - \rho I) p_i - \Gamma^T Q((I - \Gamma) \hat{x}^{(N)} - \eta) + Q(\hat{x}_i, (I - \Gamma) \hat{x}^{(N)} + \eta) + G^T p^{(N)} \right],
\]

\(\tilde{u}_i = - R^{-1} B^T p_i.\)
Thus, we have the following optimality system:

\[
\begin{aligned}
d\bar{x}_i &= (A\bar{x}_i - BR^{-1}B^T \bar{p}_i + G\bar{x}^{(N)} + f)dt + \sigma dW_i, \\
d\bar{p}_i &= -((A - \rho I)^T \bar{p}_i + G^T \bar{p}^{(N)} + Q\bar{x}_i - \Sigma_i \bar{x}^{(N)}) dt + \eta_i dt + \sum_{j=1}^{N} \beta_{ij} dW_j, \\
\bar{x}_i(0) &= x_{i0}, \quad \bar{p}_i(T) = 0, \quad i = 1, \ldots, N,
\end{aligned}
\]

(A.5)
such that \( \hat{u}_i = -R^{-1}B^T \bar{p}_i \). This implies that the equation systems (5) admits a solution \((\bar{x}_i, \bar{p}_i)\).

**Sufficiency.** Suppose (5) admits (a unique) solution \((\bar{x}_i, \bar{p}_i)\). Let \( \hat{u}_i = -R^{-1}B^T \bar{p}_i \). By (A.2)-(A.4), \( \delta J_{\text{loc}}(\hat{u}) = 0 \). Since Problem (P1) is convex in \( u \), then for any \( u = (u_1, \cdots, u_N) \), \( u_i \in U_i \),

\[
J_{\text{loc}}(u) \geq J_{\text{loc}}(\hat{u}).
\]

\( \square \)

**B Proofs of Theorems 4 and 5**

**Proof of Theorem 4.** By (A1)-(A4), Lemmas 1 and 2, we obtain that \( \bar{x} \in C_{\rho/2}([0, \infty), \mathbb{R}^n) \) and

\[
\mathbb{E} \int_0^\infty e^{-\rho t} \left( \|\bar{x}^{(N)}(t) - \bar{x}(t)\|^2 \right) dt = O\left(\frac{1}{N}\right),
\]

which further gives that \( x^{(N)} \in C_{\rho/2}([0, \infty), \mathbb{R}^n) \). Denote \( g \triangleq -BR^{-1}B^T((\Pi - P)\bar{x} + s) + G\bar{x}^{(N)} + f \). Then \( g \in C_{\rho/2}([0, \infty), \mathbb{R}^n) \) and

\[
\dot{x}_i(t) = A^t \bar{x}_i + \int_0^t e^{(A - \frac{\rho}{2} I)(t-s)} g(s) ds + \int_0^t e^{(A - \frac{\rho}{2} I)(t-s)} \sigma dW_i(s).
\]

(B.1)

Note that \( A - \frac{\rho}{2} I \) is Hurwitz. By Schwarz’s inequality,

\[
\int_0^\infty e^{-\rho t} \|\dot{x}_i(t)\|^2 dt \leq C + 3E \int_0^\infty e^{-\rho s} ||g(s)||^2 \int_s^\infty t ||(A - \frac{\rho}{2} I)(t-s)\|^2 dt ds + 3CE \int_0^\infty e^{-\rho s} ||g(s)||^2 \int_s^\infty ||(A - \frac{\rho}{2} I)(t-s)\|^2 dt ds \leq C + 3CE \int_0^\infty e^{-\rho s} ||g(s)||^2 ds + 3CE \int_0^\infty e^{-\rho s} ||\sigma(s)||^2 dt \leq C_1.
\]

(B.2)

\( \square \)

**Proof of Theorem 4.** i) \( \Rightarrow \) ii). By (13),

\[
\frac{d\mathbb{E}[\bar{x}_i]}{dt} = \hat{A}\mathbb{E}[\bar{x}_i] - BR^{-1}B^T((\Pi - P)\bar{x} + s) + GE\bar{x}^{(N)} + f, \quad \mathbb{E}[\bar{x}_i(0)] = \bar{x}_0.
\]

(B.2)

It follows from A1) that

\[
\mathbb{E}[\bar{x}_i] = \mathbb{E}[\bar{x}_j] = \mathbb{E}[\bar{x}^{(N)}], \quad j \neq i.
\]

By comparing (18) and (B.2), we obtain \( \mathbb{E}[\bar{x}_i] = \bar{x} \). Note that \( \|\bar{x}\|^2 = \|\mathbb{E}[\bar{x}]\|^2 \leq \mathbb{E}[\|\bar{x}\|^2] \). (22) leads to

\[
\int_0^\infty e^{-\rho t} \|\dot{\bar{x}}(t)\|^2 dt < \infty.
\]

(B.3)

By (18), we have

\[
\bar{x}(t) = e^{(A + G - BR^{-1}B^T) t} \left[ \bar{x}_0 + \int_0^t e^{-(A + G - BR^{-1}B^T) t} h(\tau) d\tau \right],
\]

where \( h = -BR^{-1}B^T s + f \). By the arbitrariness of \( \bar{x}_0 \) with (B.3) we obtain that \( A + G - BR^{-1}B^T \Pi - \frac{\rho}{2} I \) is Hurwitz. That is, \( (A + G, B) \) is stabilizable. By [2], (16) admits a unique solution such that \( \Pi > 0 \). Note that \( \mathbb{E}[\bar{x}^{(N)}]^2 \leq \frac{1}{N} \sum_{i=1}^{N} \mathbb{E}[\bar{x}_i^2] \). Then from (22) we have

\[
\int_0^\infty e^{-\rho t} \|\bar{x}^{(N)}(t)\|^2 dt < \infty,
\]

(B.4)

which leads to \( \mathbb{E} \int_0^\infty e^{-\rho t} \|g(t)\|^2 dt < \infty \). Here \( g = -BR^{-1}B^T((\Pi - P)\bar{x} + s) + G\bar{x}^{(N)} + f \). By (B.1), we obtain

\[
\mathbb{E}[\|\dot{x}_i(t)\|^2] = \mathbb{E} \left[ e^{A\hat{t}} \left( x_{i0} + \int_0^t e^{-A\hat{t}} g(s) ds \right) \right]^2 + \mathbb{E} \int_0^t tr[\sigma(s)e^{(A + \hat{t})t}s\sigma(s)] ds.
\]

(B.5)

On the other hand, (20) gives

\[
\mathbb{E}[\|\bar{x}^{(N)}(t) - \bar{x}(t)\|^2]
\]

\[
= \mathbb{E}[\|e^{(A + G - \frac{\rho}{2} I)t}\bar{x}_0\|^2]
\]

\[
+ \frac{1}{N} \int_0^\infty tr[\sigma(s)e^{(A + G + \frac{\rho}{2} I)(t-s)s}] ds.
\]

By (B.5) and the arbitrariness of \( x_{i0} \), \( i = 1, \cdots, N \), we obtain that \( A + G - \frac{\rho}{2} I \) is Hurwitz.

(ii) \( \Rightarrow \) (iii). Define \( V(t) = e^{-\rho t} \mathbb{E}[\tilde{y}^T(t)\Pi \tilde{y}(t)] \), where \( \tilde{y} \) satisfies

\[
\frac{d\tilde{y}}{dt} = (A + G - BR^{-1}B^T \Pi)\tilde{y}, \quad \tilde{y}(0) = \tilde{y}_0.
\]

\[
\dot{V}(t) = \mathbb{E} \left[ \tilde{y}^T(t) \left[ -\rho \Pi + (A + G - BR^{-1}B^T \Pi)^T \Pi + \Pi (A + G - BR^{-1}B^T \Pi) \right] \tilde{y}(t) \right]
\]

\[
= \mathbb{E} \left[ \tilde{y}^T(t) \left[ -\Pi BR^{-1}B^T \Pi - (I - \Gamma)^T Q(I - \Gamma) \tilde{y}(t) \right] \right] \leq 0.
\]

By A3), we can prove that \( V(t) \to 0 \), which with \( \Pi > 0 \) further leads to \( \tilde{y} \to 0 \). This implies \( (A + G - \frac{\rho}{2} I, B) \) is stabilizable. Similarly, we can show \( (A - \frac{\rho}{2} I, B) \) is stabilizable.

(iii) \( \Rightarrow \) (i). This part has been proved in Theorem 4. \( \square \)
References


