A Complete Solution to Mean Field Linear Quadratic Control

Bing-Chang Wang
School of Control Science and Engineering, Shandong University, Jinan 250061, P. R. China
E-mail: bcwang@sdu.edu.cn

Abstract: This paper studies uniform stabilization and social optimality for mean field linear quadratic control systems, where subsystems are coupled via dynamics and individual costs. For the finite horizon case, we first obtain a set of forward-backward stochastic differential equations (FBSDE) from the analysis of the social cost variation, and then design a feedback-type control by decoupling the FBSDE. The set of decentralized control laws is shown to have asymptotic social optimality. For the infinite horizon case, we design an asymptotically social optimal decentralized control using solutions of two Riccati equations. Two equivalent conditions are further given for uniform stabilization of the systems in different cases. Finally, we show that such decentralized control is a representation of the feedback control in previous works.

Key Words: Mean field game, open-loop control, social optimality, forward-backward stochastic differential equation

1 Introduction

Mean field games have drawn increasing attention in many fields including system control, applied mathematics and economics [6, 8, 13]. The mean field game involves a very large population of small interacting players with the feature that while the influence of each one is negligible, the impact of the overall population is significant. By combining mean field approximations and individual's best response, the dimensionality difficulty is overcome. Mean field games and control have found wide applications, including smart grids [10, 27], finance, economics [9, 14, 31], and social sciences [5], etc.

By now, mean field games have been intensively studied in the LQ (linear-quadratic) framework [7, 19, 20, 25, 29, 32]. Huang *et al.* developed the Nash certainty equivalence (NCE) based on the fixed-point method and designed an ϵ-Nash equilibrium for LQ mean field games with discount costs based on the NCE approach [19, 20]. The NCE approach was then applied to the cases with long run average costs [25] and with Markov jump parameters [32], respectively. Bensoussan *et al.* applied the adjoint equation approach and the fixed-point theorem to obtain a sufficient condition for the unique existence of the equilibrium strategy over a finite-time horizon [7]. For other aspects of mean field games, readers are referred to [11, 22, 24, 37] for nonlinear mean field games, [36] for oblivious equilibrium in dynamic games, [18, 33, 34] for mean field games with major players, [17, 29] for robust mean field games.

Besides noncooperative games, social optima in mean field models have also drawn much attention. The social optimum control refers to that all the players cooperate to optimize the common social cost—the sum of individual cost, which is usually regarded as a type of team decision problem [15, 30]. Huang *et al.* considered social optima in mean field LQ control, and provided an asymptotic team-optimal solution [21]. Wang and Zhang [35] investigated a mean field social optimal problem where a Markov jump parameter appears as a common source of randomness. For further literature, see [23] for social optima in mixed games, [3] for team-optimal control with finite population and partial information,

Most previous results on mean field games and control are given by using the fixed-point method [20, 21, 25, 33, 35]. However, the widely used tool in fixed-point analysis is contraction mapping, which is very conservative for the general systems. In this paper, we break away from the fixed-point method and solve the mean field control problem by tackling forward-backward stochastic differential equations (FBSDE). In recently years, some substantial progress for the optimal LQ control has been made by solving the FBSDE. See [38, 40] for details.

This paper investigates uniform stabilization and social optimality for mean field LQ control systems, where subsystems are coupled via dynamics and individual costs. For the finite horizon case, we first obtain a set of forward-backward stochastic differential equations (FBSDE) by examining the variation of the social cost, and then give a centralized feedback-type control laws by decoupling the FBSDE. With mean field approximations, we design a set of decentralized control laws, which is further shown to have asymptotic social optimality. For the infinite horizon case, we design a set of decentralized control laws using solutions of two Riccati equations, which is shown to be asymptotically social optimal. Two equivalent conditions are further given for uniform stabilization of the systems when the state weight Q is semi-positive definite or only symmetric. Finally, we show such set of decentralized control laws is a representation of the feedback control in previous works.

The main contributions of the paper are summarized as follows.

- From the analysis of the social cost variation we obtain a set of open-loop control laws, and then design a feedback-type decentralized control by tackling FBSDE with mean field approximations.
- In the case $Q \geq 0$, the necessary and sufficient conditions are given for uniform stabilization of the systems with the help of the Riccati equations' solutions and the system's observability.
- In the case that Q is only symmetric, the necessary and sufficient conditions are given for uniform stabilization of the systems using the Hamiltonian matrices.
- It is under nonconservative assumptions that we obtain an asymptotically social optimal decentralized control,

This author was supported by the National Natural Science Foundation of China under Grants 61773241 and 61503218.

and such control is shown to be equivalent to the feedback control given by the fixed-point method in previous works [21, 35].

The organization of the paper is as follows. Section 2 formulates the socially optimal control problem. In Section 3, we design a decentralized control by tackling BSDE for the finite horizon case, which is shown to be asymptotically social optimal. In Section 4, we design asymptotically social optimal control laws for the infinite horizon case and further give some equivalent conditions for uniform stabilization of the systems. Such set of decentralized control laws is compared with the feedback control of previous works in Section 5. In Section 6, two numerical examples are provided to show the effectiveness of the proposed control. Section 7 concludes the paper.

The following notation will be used throughout this paper. $\|\cdot\|$ denotes the Euclidean vector norm or matrix spectral norm. For a vector z and a matrix Q, $\|z\|_Q^2 = z^TQz$, and $Q > 0$ ($Q \geq 0$) means that Q is positive definite (semi-positive definite). For two vectors x, y, $\langle x, y\rangle = x^Ty$. $C([0,\infty),\mathbb{R}^n)$ is the space of all the n-dimensional continuous functions on $[0,\infty)$, and $C_{\rho/2}([0,\infty),\mathbb{R}^n)$ is a subspace of $C([0,\infty),\mathbb{R}^n)$ which is given by $\{f|\int_0^\infty e^{-\rho t}\|f(t)\|^2dt < \infty\}$. For convenience of presentation, we use $C, C_1, C_2, \cdots$ to denote generic positive constants, which may vary from place to place.

2 Problem Formulation

Consider a large population systems with N agents. Agent i evolves by the following stochastic differential equation:

$$\begin{aligned} dx_i(t) = [&Ax_i(t) + Bu_i(t) + Gx^{(N)}(t) + f(t)]dt \\ &+ \sigma(t)dW_i(t), \quad 1 \leq i \leq N, \end{aligned} \tag{1}$$

where $x_i \in \mathbb{R}^n$ and $u_i \in \mathbb{R}^r$ are the state and input of the ith agent. $x^{(N)}(t) = \frac{1}{N}\sum_{j=1}^N x_j(t)$, $f, \sigma \in C_{\rho/2}([0,\infty),\mathbb{R}^n)$. $\{W_i(t), 1 \leq i \leq N\}$ are a sequence of independent 1-dimensional Brownian motions. The cost function of agent i is given by

$$\begin{aligned} J_i(u) = \frac{1}{2}\mathbb{E}\int_0^\infty e^{-\rho t}\Big\{&\big\|x_i(t) - \Gamma x^{(N)}(t) - \eta\big\|_Q^2 \\ &+ \|u_i(t)\|_R^2\Big\}dt, \end{aligned} \tag{2}$$

where Q is symmetric and $R > 0$. $u = \{u_1, \ldots, u_N\}$. The admissible control set is given by

$$\mathcal{U}_{d,i} = \Big\{u_i \mid u_i(t) \in \sigma(x_i(s), 0 \leq s \leq t)\Big\}.$$

For comparison, define the centralized control set as

$$\mathcal{U}_{c,i} = \Big\{u_i \mid u_i(t) \in \sigma\{\bigcup_{i=1}^N \mathcal{F}_t^i\}\Big\},$$

where $\mathcal{F}_t^i = \sigma(x_i(0), W_i(s), 0 \leq s \leq t), i = 1, \cdots, N$.

In this paper, we mainly study the following problem.

(P0). Seek a set of decentralized control strategies to optimize social cost for the system (1)-(2), i.e., $\inf_{u_i \in \mathcal{U}_i} J_{soc}$, where

$$J_{soc} = \sum_{i=1}^N J_i(u).$$

Assume

A1) $x_i(0), i = 1, ..., N$, are mutually independent and have the same mathematical expectation. $\mathbb{E}x_i(0) \equiv \bar{x}_0$, and there exists a constant C_0 (independent of N) such that $\max_{1\leq i\leq N}\mathbb{E}\|x_i(0)\|^2 < C_0$.

3 Finite Horizon Problem

For the convenience of design, we first consider the finite horizon problem **(P1)**.

$$\inf_{u_i \in \mathcal{U}_{d,i}} J_{\mathrm{soc}}^F(u),$$

where $J_{\mathrm{soc}}^F(u) = \inf_{u_i \in \mathcal{U}_{d,i}}\sum_{i=1}^N J_i^F(u)$ and

$$\begin{aligned} J_i^F(u) = \frac{1}{2}\mathbb{E}\int_0^T e^{-\rho t}\Big\{&\big\|x_i(t) - \Gamma x^{(N)}(t) - \eta\big\|_Q^2 \\ &+ \|u_i(t)\|_R^2\Big\}dt. \end{aligned} \tag{3}$$

We first give an equivalent condition for the convexity of Problem (P1).

Proposition 1 *Problem (P1) is convex in u if and only if for any $u_i \in \mathcal{U}_c$, $i = 1, \cdots, N$,*

$$\sum_{i=1}^N \int_0^T \Big\{\big\|y_i - \Gamma y^{(N)}\big\|_Q^2 + \|u_i\|_R^2\Big\}dt \geq 0,$$

where y_i satisfies

$$\begin{aligned} &dy_i = [Ay_i + Gy^{(N)} + Bu_i]dt, \\ &y_i(0) = 0, \ i = 1, 2, \cdots, N. \end{aligned} \tag{4}$$

By examining the variation of J_{soc}^F, we obtain the existence conditions of the centralized optimal control.

Theorem 1 *Suppose that (P1) is convex and $R > 0$. Then (P1) has a (unique) optimal control if and only if the following equation system admits a (unique) set of solutions $(x_i, p_i, i = 1, \cdots, N)$:*

$$\begin{cases} dx_i = \big(Ax_i - BR^{-1}B^Tp_i + Gx^{(N)} + f\big)dt + \sigma dW_i, \\ dp_i = -\big((A - \rho I)^Tp_i + G^Tp^{(N)}\big)dt \\ \qquad\quad - \big(Qx_i - \Sigma_1 x^{(N)} - \bar{\eta}\big)dt + \sum_{j=1}^N \beta_i^j dW_j, \\ x_i(0) = x_{i0}, \qquad p_i(T) = 0, \quad i = 1, \cdots, N, \end{cases} \tag{5}$$

where $\Sigma_1 \stackrel{\Delta}{=} \Gamma^TQ + Q\Gamma - \Gamma^TQ\Gamma$, $\bar{\eta} \stackrel{\Delta}{=} Q\eta - \Gamma^TQ\eta$, $p^{(N)} = \frac{1}{N}\sum_{i=1}^N p_i$, and furthermore the optimal control is given by $\check{u}_i = -R^{-1}B^Tp_i$.

Proof. See Appendix A. □

It follows from (5) that

$$\begin{cases} dx^{(N)} =\big((A+G)x^{(N)} - BR^{-1}B^Tp^{(N)} + f\big)dt \\ \qquad\qquad + \frac{1}{N}\sum_{i=1}^{N}\sigma dW_i, \\ dp^{(N)} = -\Big[(A+G-\rho I)^T p^{(N)} \\ \qquad\qquad - (I-\Gamma)^T Q(I-\Gamma)x^{(N)} + \bar\eta\Big]dt \\ \qquad\qquad + \frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}\beta_i^j dW_j, \\ x^{(N)}(0) = \frac{1}{N}\sum_{i=1}^{N} x_{i0}, \qquad p^{(N)}(T) = 0. \end{cases} \tag{6}$$

Let $p_i = Px_i + Kx^{(N)} + s$. Then by (5) and Itô's formula,

$$\begin{aligned} dp_i = &\dot{P}x_i + P\Big[\big(Ax_i - BR^{-1}B^T(Px_i + Kx^{(N)} + s) \\ &+ Gx^{(N)} + f\big)dt + \sigma dW_i\Big] + \dot{s} + \dot{K}x^{(N)} \\ &+ K\Big\{\big[(A+G)x^{(N)} - BR^{-1}B^T[(P+K)x^{(N)} \\ &+ s] + f\big]dt + \frac{1}{N}\sum_{i=1}^{N}\sigma dW_i\Big\} \\ = &-\big[(A-\rho I)^T(Px_i + Kx^{(N)} + s) \\ &+ G^T((P+K)x^{(N)} + s) \\ &+ Qx_i - \Sigma_1 x^{(N)} - \bar\eta\big]dt + \sum_{j=1}^{N}\beta_i^j dW_j. \end{aligned}$$

This implies that $\beta_i^i = \frac{1}{N}K\sigma + P\sigma$, $\beta_i^j = \frac{1}{N}K\sigma,\ j \neq i$,

$$\rho P = \dot{P} + A^TP + PA - PBR^{-1}B^TP + Q,\ P(T) = 0, \tag{7}$$

$$\begin{aligned} \rho K = &\dot{K} + (A+G)^TK + K(A+G) \\ &- PBR^{-1}B^TK - KBR^{-1}B^TP + G^TP \\ &+ PG - KBR^{-1}B^TK - \Sigma_1,\ K(T) = 0, \end{aligned} \tag{8}$$

$$\begin{aligned} \rho s = &\dot{s} + [A+G-BR^{-1}B^T(P+K)]^Ts \\ &+ (P+K)f - \bar\eta, \quad s(T) = 0. \end{aligned} \tag{9}$$

Then $\check{u}_i = -R^{-1}B^T(Px_i + Kx^{(N)} + s)$.

Theorem 2 *Let A1) hold and $Q \geq 0$. Then Problem (P1) has an optimal control*

$$\check{u}_i = -R^{-1}B^T(Px_i + Kx^{(N)} + s),$$

where P, K and s are determined by (7)-(9).

Proof. Denote $\Pi = P + K$. Then from (8) and (9), Π satisfies

$$\begin{aligned} &\dot{\Pi} + (A+G)^T\Pi + \Pi(A+G) - \Pi BR^{-1}B^T\Pi \\ &+ (I-\Gamma)^TQ(I-\Gamma) = 0, \quad \Pi(T) = 0. \end{aligned} \tag{10}$$

Note that $Q \geq 0$ and $R > 0$. By [2, 39], (7) and (10) admit unique solutions $P \geq 0$ and $\Pi \geq 0$, respectively, which implies that (8) and (9) have unique solutions K and s, respectively. Then by [26, 40], FBSDE (5) admits a solution $(x_i, p_i, i = 1, \cdots, N)$. By Theorem 1, Problem (P1) has an optimal control given by $\check{u}_i = -R^{-1}B^T(Px_i + Kx^{(N)} + s)$, where P, K and s are determined by (7)-(9). □

As an approximation to $x^{(N)}$ in (6), we have

$$\frac{d\bar{x}}{dt} = (A+G)\bar{x} - BR^{-1}B^T(\Pi\bar{x} + s) + f,\ \bar{x}(0) = \bar{x}_0. \tag{11}$$

Then we may design a set of decentralized control laws as follows:

$$\begin{aligned} \hat{u}_i(t) = &-R^{-1}B^T(P\hat{x}_i(t) + K\bar{x}(t) + s(t)), \\ &0 \leq t \leq T,\ i = 1, \cdots, N, \end{aligned} \tag{12}$$

where P, K and s are determined by (7)-(9), and $\bar{x}$ and $\hat{x}_i$ satisfies (11) and

$$\begin{aligned} d\hat{x}_i = &\big[(A - BR^{-1}B^TP)\hat{x}_i - BR^{-1}B^T(K\bar{x} + s) \\ &+ G\hat{x}^{(N)} + f\big]dt + \sigma dW_i. \end{aligned} \tag{13}$$

We now give the result of asymptotic social optimality. Due to page limitations, the proof is omitted. Denote

$$\mathcal{U}_c = \Big\{(u_1, \cdots, u_N)\big|\ u_i(t) \in \sigma\{\bigcup_{i=1}^{N}\mathcal{F}_t^i\}\Big\}.$$

Theorem 3 *Let A1) hold and $Q \geq 0$. For (P1), the set of decentralized control laws $\{\hat{u}_1, \cdots, \hat{u}_N\}$ given by (12) has asymptotic social optimality, i.e.,*

$$\Big|\frac{1}{N}J_{soc}^F(\hat{u}) - \frac{1}{N}\inf_{u \in \mathcal{U}_c} J_{soc}^F(u)\Big| = O(\frac{1}{\sqrt{N}}).$$

4 Infinite Horizon Problem

Based the analysis in Section 3, we may design the following decentralized control for (P0):

$$\begin{aligned} \hat{u}_i(t) = &-R^{-1}B^T(P\hat{x}_i(t) + (\Pi - P)\bar{x}(t) + s(t)), \\ &t \geq 0,\ i = 1, \cdots, N, \end{aligned} \tag{14}$$

where P, Π are determined by

$$\rho P = A^TP + PA - PBR^{-1}B^TP + Q, \tag{15}$$

$$\begin{aligned} \rho\Pi = &(A+G)^T\Pi + \Pi(A+G) - \Pi BR^{-1}B^T\Pi \\ &+ (I-\Gamma)^TQ(I-\Gamma), \end{aligned} \tag{16}$$

and $s, \bar{x} \in C_{\rho/2}([0,\infty), \mathbb{R}^n)$ are determined by

$$\rho s = \dot{s} + [A+G-BR^{-1}B^T\Pi]^Ts + \Pi f - \bar\eta, \tag{17}$$

$$\frac{d\bar{x}}{dt} = (A+G)\bar{x} - BR^{-1}B^T(\Pi\bar{x} + s) + f,\ \bar{x}(0) = \bar{x}_0. \tag{18}$$

Here the existence conditions of P, Π, s and $\bar{x}$ need to be investigated further.

We provide the following assumptions:

A2) $(A - \frac{\rho}{2}I, B)$ is stabilizable, and $(A + G - \frac{\rho}{2}I, B)$ is stabilizable.

A3) $Q \geq 0$, $(A - \frac{\rho}{2}I, \sqrt{Q})$ is observable, and $(A + G - \frac{\rho}{2}I, \sqrt{Q}(I-\Gamma))$ is observable.

Lemma 1 *Under A1)-A3), (15) and (16) admit unique solutions $P > 0, \Pi > 0$, respectively, and (17)-(18) admits a set of unique solutions $s, \bar{x} \in C_{\rho/2}([0,\infty), \mathbb{R}^n)$.*

Proof. From A2)-A3) and [2], (15) and (16) admit unique solutions $P > 0, \Pi > 0$ such that $\bar{A} - \frac{\rho}{2}I$ and $A + G - BR^{-1}B^T\Pi - \frac{\rho}{2}I$ are Hurwitz. It is straightforward that $s, \bar{x} \in C_{\rho/2}([0,\infty), \mathbb{R}^n)$. □

We further introduce the following assumption.

A4) $\bar{A} + G - \frac{\rho}{2}I$ is Hurwitz.

Lemma 2 *Let A1)-A4) hold. Then for (P0)*

$$\mathbb{E}\int_0^\infty e^{-\rho t}\|\hat{x}^{(N)}(t) - \bar{x}(t)\|^2 dt = O(\frac{1}{N}). \tag{19}$$

Proof. We have

$$\begin{aligned}\hat{x}^{(N)}(t) - \bar{x}(t) = {} & e^{(\bar{A}+G)t}[\hat{x}^{(N)}(0) - \bar{x}(0)] \\ & + \frac{1}{N}\sum_{i=1}^N \int_0^t e^{(\bar{A}+G)(t-s)}\sigma dW_i(s).\end{aligned} \tag{20}$$

Thus,

$$\begin{aligned}
&\mathbb{E}\int_0^\infty e^{-\rho t}\left(\|\hat{x}^{(N)}(t) - \bar{x}(t)\|^2\right)dt \\
\le{}& 2\mathbb{E}\int_0^\infty \left\|e^{(\bar{A}+G-\frac{\rho}{2}I)t}\right\|^2 \|\hat{x}^{(N)}(0) - \bar{x}(0)\|^2 dt \\
&+ 2\mathbb{E}\int_0^\infty e^{-\rho t}\frac{1}{N}\left\|\int_0^t e^{(\bar{A}+G)(t-s)}\sigma dW_i(s)\right\|^2 dt \\
\le{}& 2\int_0^\infty \left\|e^{(\bar{A}+G-\frac{\rho}{2}I)t}\right\|^2 \mathbb{E}\|\hat{x}^{(N)}(0) - \bar{x}(0)\|^2 dt \\
&+\frac{2}{N}\mathbb{E}\int_0^\infty e^{-\rho t}\int_0^t tr\left[\sigma^T\sigma e^{(\bar{A}+G+\bar{A}^T+G^T)(t-s)}\right]dsdt \\
\le{}& \frac{2}{N}\int_0^\infty \left\|e^{(\bar{A}+G-\frac{\rho}{2}I)t}\right\|^2 \mathbb{E}\|\max_{1\le i\le N}\hat{x}_i(0)\|^2 dt \\
&+ \frac{2}{N}\mathbb{E}\int_0^\infty e^{-\rho t}C\Big|\frac{1-e^{2\delta t}}{\delta}\Big|dt \le O(\frac{1}{N}),
\end{aligned}$$

where $\delta < \rho/2$. □

The systems are shown to be uniformly stabilized.

Theorem 4 *Let A1)-A4) hold. Then for any N,*

$$\sum_{i=1}^N \mathbb{E}\int_0^\infty e^{-\rho t}\left(\|\hat{x}_i(t)\|^2 + \|\hat{u}_i(t)\|^2\right)dt < \infty. \tag{21}$$

Proof. See Appendix B. □

We now give two equivalent conditions for uniform stabilization of the systems.

Theorem 5 *Let A3) hold. Then for (P0) the following statements are equivalent:*

(i) For any initial condition $(x_1(0), \cdots, x_N(0))$ satisfying A1),

$$\sum_{i=1}^N \mathbb{E}\int_0^\infty e^{-\rho t}\left(\|\hat{x}_i(t)\|^2 + \|\hat{u}_i(t)\|^2\right)dt < \infty. \tag{22}$$

(ii) (15) and (16) admit unique solutions $P > 0, \Pi > 0$, respectively, and $\bar{A} + G - \frac{\rho}{2}I$ is Hurwitz.

(iii) A2) and A4) hold.

Proof. See Appendix B. □

For the more general case that Q are only symmetric, we have the following equivalent conditions for the stabilization of the systems.

Denote

$$M_1 = \begin{bmatrix} A - \frac{\rho}{2}I & BR^{-1}B^T \\ Q & -A^T + \frac{\rho}{2}I \end{bmatrix},$$

$$M_2 = \begin{bmatrix} A + G - \frac{\rho}{2}I & BR^{-1}B^T \\ \hat{Q} & -(A+G)^T + \frac{\rho}{2}I \end{bmatrix}.$$

Theorem 6 *Assume that both M_1 and M_2 have no eigenvalues on the imaginary axis. Then for (PS) the following statements are equivalent:*

(i) For any $(x_1(0), \cdots, x_N(0))$ satisfying A1),

$$\sum_{i=1}^N \mathbb{E}\int_0^\infty e^{-\rho t}\left(\|\hat{x}_i(t)\|^2 + \|\hat{u}_i(t)\|^2\right)dt < \infty.$$

(ii) (15) and (16) admit stabilizing solutions[1], respectively, and $\bar{A} + G$ is Hurwitz.

(iii) A2) and A4) hold.

Remark 1 *M_1 and M_2 are Hamilton matrices. The Hamilton matrix plays a significant role in studying general algebraic Riccati equations. See more details of the property of Hamilton matrices in [1, 28].*

To show Theorem 6, we need two lemmas.

Lemma 3 *Let A1) hold. Assume that (15) and (16) admit stabilizing solutions, respectively, and $\bar{A} + G$ is Hurwitz. Then*

$$\sum_{i=1}^N \mathbb{E}\int_0^\infty e^{-\rho t}\left(\|\hat{x}_i(t)\|^2 + \|\hat{u}_i(t)\|^2\right)dt < \infty.$$

Proof. From the definition of stabilizing solutions, $A - BR^{-1}B^TP$ and $A + G - BR^{-1}B^T\Pi$ are Hurwitz. By the argument in the proof of Theorem 4, the lemma follows. □

Lemma 4 *[28] Equations (15) and (16) admit stabilizing solutions if and only if A2) holds and both M_1 and M_2 have no eigenvalues on the imaginary axis.*

The Proof of Theorem 6. By a similar argument in the proof of Theorem 4 combined with Lemmas 3 and 4, the Theorem follows. □

Example 1 *Consider a scalar system with $A = a$, $B = b$, $G = g$, $Q = q$, $\Gamma = \gamma$, $R = r > 0$. Then*

$$M_1 = \begin{bmatrix} a - \rho/2 & b^2/r \\ q & -a + \rho/2 \end{bmatrix},$$

$$M_2 = \begin{bmatrix} a + g - \rho/2 & b^2/r \\ q(1-\gamma)^2 & -(a + g - \rho/2) \end{bmatrix}.$$

By direct computations, neither M_1 nor M_2 has eigenvalues in imaginary axis if and only if

$$(a - \frac{\rho}{2})^2 + \frac{b^2}{r}q > 0, \tag{23}$$

$$(a + g - \frac{\rho}{2})^2 + \frac{b^2}{r}(1-\gamma)^2 q > 0. \tag{24}$$

[1] For a Riccati equation (15), P is called a stabilizing solution if P satisfies (15) and all the eigenvalues of $A - BR^{-1}B^TP$ are in left half-plane.

Note that if $q > 0$, then (23) holds, and if $(1-\gamma)^2 q > 0$, then (24) holds.

For this model, the Riccati equation (15) is written as

$$\frac{b^2}{r}p^2 - (2a-\rho)p - q = 0. \tag{25}$$

Let $\Delta = 4[(a-\rho/2)^2 + b^2 q/r]$. If (23) holds then $\Delta > 0$, which implies (25) admits two solutions. If $q > 0$ then (25) has a unique positive solution such that $a - b^2 p/r - \rho/2 = -\sqrt{\Delta}/2 < 0$.

Assume that (23) and (24) hold. By Theorem 6, the system is uniformly stable if and only if $(a-\rho/2, b)$ is stabilizable (i.e., $b \neq 0$ or $a - \rho/2 < 0$), and $a - b^2 p/r - \rho/2 + g < 0$. Note that $a - b^2 p/r - \rho/2 < 0$. When $g \leq 0$, we have $a - b^2 p/r - \rho/2 + g < 0$.

We are in a position to state the asymptotic optimality of the decentralized control. Due to page limitations, the proof is omitted.

Theorem 7 *Let A1)-A4) hold. The set of decentralized control laws $\{\hat{u}_1, \cdots, \hat{u}_N\}$ given by (14) has asymptotic social optimality, i.e.,*

$$\left|\frac{1}{N}J_{soc}(\hat{u}) - \frac{1}{N}\inf_{u \in \mathcal{U}_c} J_{soc}(u)\right| = O(\frac{1}{\sqrt{N}}).$$

5 Comparison of Different Solutions

In this section, we compare the decentralized control laws in (14) with the feedback decentralized control in previous works [21].

We first introduce a definition from [4].

Definition 1 *For a control problem with an admissible control set $\mathcal{U}$, a control law $u \in \mathcal{U}$ is said to be a representation of another control $u^* \in \mathcal{U}$ if*

(i) they both generate the same unique state trajectory, and

(ii) they both have the same open-loop value on this trajectory.

For Problem (P0), let $f = 0, G = 0$. In [21, Theorem 4.3], the decentralized control law is given by

$$\breve{u}_i = -BR^{-1}(Px_i + \bar{s}), \quad i = 1, \cdots, N, \tag{26}$$

where P is the semi-positive solution of (15), and $\bar{s} = \bar{K}\bar{x} + \phi$. Here $\bar{K}, \bar{x}$ and $\phi \in C_{\rho/2}([0,\infty), \mathbb{R}^n)$ determined by

$$\begin{aligned}
\rho\bar{K} &= \bar{K}(A - BR^{-1}B^T P) + (A - BR^{-1}B^T P)^T \bar{K} \\
&\quad - \bar{K}BR^{-1}B^T\bar{K}^T - \Sigma_1, \\
\frac{d\bar{x}^\dagger}{dt} &= (A - BR^{-1}B^T P)\bar{x}^\dagger - BR^{-1}B^T(\bar{K}\bar{x}^\dagger + \phi), \\
\frac{d\phi}{dt} &= -[A - BR^{-1}B^T(P + \bar{K}) - \rho I]\phi + \bar{\eta}.
\end{aligned}$$

By comparing this with (16)-(18), one can obtain that $\bar{K} = \Pi - P$, $\bar{x} = \bar{x}^\dagger$ and $\phi = s$. From the analysis above, we have the equivalence of the two decentralized control laws.

Proposition 2 *The set of decentralized control laws $\{\hat{u}_1, \cdots, \hat{u}_N\}$ in (14) is a representation of $\{\breve{u}_1, \cdots, \breve{u}_N\}$ given in (26).*

6 Numerical Examples

In this section, two numerical examples are provided to illustrate the effectiveness of the proposed decentralized control laws.

We first consider a scalar system with 50 agents in Problem (P0). Take $B = 1, G = -0.2, f(t) = 1, \sigma(t) = 0.1, \rho = 0.6, \Gamma = -0.2, \eta = 5, Q = 1, R = 1$ in $(1)-(2)$. The initial states of 50 agents are taken independently from a normal distribution $N(5, 0.5)$. Then, under the control law (14), the state trajectories of agents for the case with $A = 0.2$ and $A = 1$ are shown in Fig. 1 and Fig. 2, respectively. After the transient phase, the states of agents behave similarly and achieve agreement roughly.

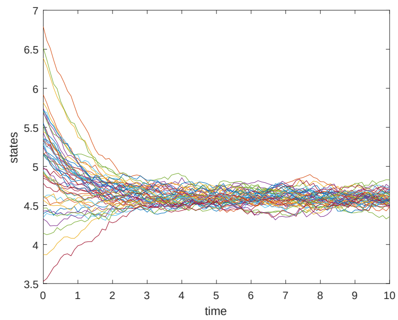

Fig. 1: State trajectories of 50 agents with $A = 0.2$.

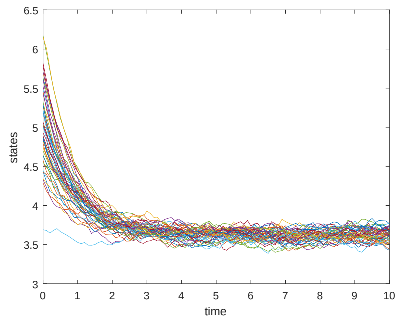

Fig. 2: State trajectories of 50 agents with $A = 1$.

For the case $A = 1$, the trajectories of $\bar{x}$ and $\hat{x}^{(N)}$ in Problems (P0) is shown in Fig. 3. It can be seen that $\bar{x}$ and $\hat{x}^{(N)}$ coincide well, which illustrate the consistency of mean field approximations.

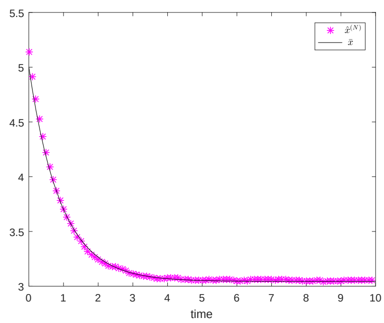

Fig. 3: State trajectories of 50 agents with $A = 1$.

Next, we consider the 2-dimensional case of (P0). Take parameters as follows: $A = \begin{bmatrix} 0.1 & 0 \\ -1 & 0.2 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $G = \begin{bmatrix} -0.5 & 0 \\ 0 & -0.3 \end{bmatrix}$, $B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $\Gamma = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$, $R = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $\eta = \begin{bmatrix} 0 \\ 0.5 \end{bmatrix}$ and $\sigma = 0.5$. Denote $\hat{x}_i(t) = \begin{bmatrix} \hat{x}_i^1(t) \\ \hat{x}_i^2(t) \end{bmatrix}$, where both of $\hat{x}_i^1(0)$ and $\hat{x}_i^2(0)$ are taken independently from a normal distribution $N(5, 0.5)$. Under the control (14), the trajectories of $\hat{x}_i^1$ and $\hat{x}_i^2$ are shown in Fig. 4 and Fig. 5, respectively.

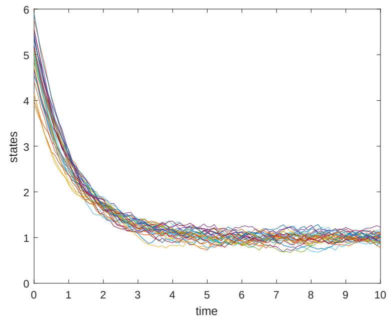

Fig. 4: Trajectories of $\hat{x}_i^1$, $i = 1, \cdots, 50$.

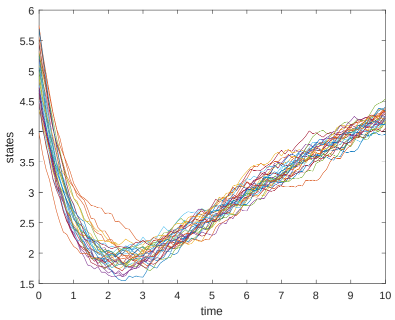

Fig. 5: Trajectories of $\hat{x}_i^2$, $i = 1, \cdots, 50$.

7 Concluding Remarks

In this paper, we considered uniform stabilization and social control for mean field LQ control systems, where subsystems are coupled via dynamics and individual costs. For the finite and infinite horizon cases, we design open-loop decentralized control laws by using solutions of Riccati equations, respectively, which are further shown to be asymptotically social optimal. Two equivalent conditions are further given for uniform stabilization of the systems. Finally, we show such decentralized control is equivalent to the feedback control in previous works.

A Proof of Theorem 1

Necessity. Suppose that $(\breve{u}_1, \cdots, \breve{u}_N)$ is a centralized optimal control to Problem (P1). $\breve{x}_i$ is the state of agent i under the control $\breve{u}_i$. Let $\delta u_i = u_i - \breve{u}_i$, $i = 1, 2, \cdots, N$, where $u_i, \breve{u}_i \in \mathcal{U}_{c,i}$, $\mathbb{E}\int_0^T \|u_i\|^2 dt < \infty$ and $\mathbb{E}\int_0^T \|\breve{u}_i\|^2 dt < \infty$. Denote $\delta x_i = x_i - \breve{x}_i, i = 1, 2, \cdots, N$, and $\delta x^{(N)} = \frac{1}{N}\sum_{j=1}^N \delta x_j$. Let $\delta J_{soc}^F(\breve{u}_i)$ be the first variation of J_{soc}^F with respect to $(\delta u_1, \cdots, \delta u_N)$. By (1),

$$\begin{aligned} &d(\delta x_i) = [A(\delta x_i) + B(\delta u_i) + G(\delta x^{(N)})]dt, \\ &\delta x_i(0) = 0, \quad i = 1, 2, \cdots, N. \end{aligned} \tag{A.1}$$

Then we have

$$\begin{aligned} 0 &= \delta J_{soc}^F(\breve{u}_i) \\ &= \sum_{i=1}^N \mathbb{E}\int_0^T e^{-\rho t}\Big[\big\langle Q\big(\breve{x}_i - (\Gamma \breve{x}^{(N)} + \eta)\big), \\ &\qquad \delta x_i - \Gamma \delta x^{(N)}\big\rangle + \langle R_2 \breve{u}_i, \delta u_i\rangle\Big]dt. \end{aligned} \tag{A.2}$$

Assume

$$\begin{aligned} &dp_i = \alpha_i dt + \beta_i^i dW_i + \sum_{j\neq i}\beta_i^j dW_j, \\ &p_i(T) = 0, \quad i = 1, \cdots, N, \end{aligned} \tag{A.3}$$

where α_i and β_i^j are to be determined. Then by Itô's formula,

$$\begin{aligned} 0 &= \mathbb{E}[e^{-\rho T}\langle p_i(T), \delta x_i(T)\rangle - \langle p_i(0), \delta x_i(0)\rangle] \\ &= \mathbb{E}\int_0^T \big[\langle \alpha_i, \delta x_i\rangle + \langle p_i, (A - \rho I)\delta x_i + B\delta u_i\rangle\big]dt. \end{aligned} \tag{A.4}$$

Note that

$$\begin{aligned} &\sum_{i=1}^N \mathbb{E}\int_0^T \big\langle Q\big(\breve{x}_i - (\Gamma \breve{x}^{(N)} + \eta)\big), -\Gamma \delta x^{(N)}\big\rangle dt \\ =&\mathbb{E}\int_0^T \big\langle -\Gamma^T Q \sum_{i=1}^N \big(\breve{x}_i - (\Gamma \breve{x}^{(N)} + \eta)\big), \frac{1}{N}\sum_{j=1}^N \delta x_j\big\rangle dt \\ =&\sum_{j=1}^N \mathbb{E}\int_0^T \big\langle -\frac{\Gamma^T Q}{N}\sum_{i=1}^N \big(\breve{x}_i - (\Gamma \breve{x}^{(N)} + \eta)\big), \delta x_j\big\rangle dt \\ =&\sum_{j=1}^N \mathbb{E}\int_0^T \big\langle -\Gamma^T Q\big((I - \Gamma)\breve{x}^{(N)} - \eta\big), \delta x_j\big\rangle dt. \end{aligned}$$

It follows by (A.2)-(A.4) that

$$\begin{aligned} 0 =&\mathbb{E}\sum_{i=1}^N \int_0^T e^{-\rho t}\Big[\big\langle Q\big(\breve{x}_i - (\Gamma \breve{x}^{(N)} + \eta)\big), \\ &\delta x_i - \Gamma \delta x^{(N)}\big\rangle + \langle R_0 \breve{u}_i + B^T p_i, \delta u_i\rangle\Big]dt \\ =&\sum_{i=1}^N \mathbb{E}\int_0^T e^{-\rho t}\big[\langle \alpha_i + (A - \rho I)^T p_i + G^T p^{(N)}, \delta x_i\rangle \\ =&\sum_{i=1}^N \mathbb{E}\int_0^T e^{-\rho t}\Big\langle R\breve{u}_i + B^T p_i, \delta u_i\Big\rangle dt \\ &+ \sum_{i=1}^N \mathbb{E}\int_0^T e^{-\rho t}\Big\langle Q\big(\breve{x}_i - (\Gamma \breve{x}^{(N)} + \eta)\big) \\ &- \Gamma^T Q\big((I - \Gamma)\breve{x}^{(N)} - \eta\big) + \alpha_i \\ &+ (A - \rho I)^T p_i + G^T p^{(N)}, \delta x_i\Big\rangle dt, \end{aligned}$$

which leads to

$$\begin{aligned} \alpha_i =& -\big[(A - \rho I)^T p_i - \Gamma^T Q\big((I - \Gamma)\breve{x}^{(N)} - \eta\big) \\ &+ Q\big(\breve{x}_i - (\Gamma \breve{x}^{(N)} + \eta)\big) + G^T p^{(N)}\big], \\ \breve{u}_i =& -R^{-1}B^T p_i. \end{aligned}$$

Thus, we have the following optimality system:

$$\begin{cases} d\check{x}_i = (A\check{x}_i - BR^{-1}B^T p_i + G\check{x}^{(N)} + f)dt + \sigma dW_i, \\ d\check{p}_i = -((A-\rho I)^T\check{p}_i + G^T\check{p}^{(N)} + Q\check{x}_i - \Sigma_1\check{x}^{(N)} \\ \qquad + \bar{\eta})dt + \sum_{j=1}^N \beta_i^j dW_j, \\ \check{x}_i(0) = x_{i0}, \quad \check{p}_i(T) = 0, \quad i = 1, \cdots, N, \end{cases} \tag{A.5}$$

such that $\check{u}_i = -R^{-1}B^T p_i$. This implies that the equation systems (5) admits a solution $(\check{x}_i, \check{p}_i)$.

Sufficiency. Suppose (5) admits a (unique) solution $(\check{x}_i, \check{p}_i)$. Let $\check{u}_i = -R^{-1}B^T p_i$. By (A.2)-(A.4), $\delta J^F_{\mathrm{soc}}(\check{u}) = 0$. Since Problem (P1) is convex in u, then for any $u = (u_1, \cdots, u_N), u_i \in \mathcal{U}_c$,

$$J^F_{soc}(u) \geq J^F_{soc}(\check{u}).$$

□

B Proofs of Theorems 4 and 5

Proof of Theorem 4. By A1)-A4), Lemmas 1 and 2, we obtain that $\bar{x} \in C_{\rho/2}([0,\infty), \mathbb{R}^n)$ and

$$\mathbb{E}\int_0^\infty e^{-\rho t}\left(\|\hat{x}^{(N)}(t) - \bar{x}(t)\|^2\right)dt = O(\frac{1}{N}),$$

which further gives that $x^{(N)} \in C_{\rho/2}([0,\infty), \mathbb{R}^n)$. Denote $g \triangleq -BR^{-1}B^T((\Pi - P)\bar{x} + s) + Gx^{(N)} + f$. Then $g \in C_{\rho/2}([0,\infty), \mathbb{R}^n)$ and

$$\hat{x}_i(t) = e^{\bar{A}t}\hat{x}_{i0} + \int_0^t e^{\bar{A}(t-s)}g(s)ds + \int_0^t e^{\bar{A}(t-s)}\sigma dW_i(s). \tag{B.1}$$

Note that $\bar{A} - \frac{\rho}{2}I$ is Hurwitz. By Schwarz's inequality,

$$\begin{aligned} &\mathbb{E}\int_0^\infty e^{-\rho t}\|\hat{x}_i(t)\|^2 dt \\ \leq& C + 3\mathbb{E}\int_0^\infty e^{-\rho s}\|g(s)\|^2 \int_s^\infty t\big\|e^{(\bar{A}-\frac{\rho}{2}I)(t-s)}\big\|^2 dtds \\ &+ 3C\mathbb{E}\int_0^\infty e^{-\rho s}\|\sigma(s)\|^2 \int_s^\infty \big\|e^{(\bar{A}-\frac{\rho}{2}I)(t-s)}\big\|^2 dtds \\ \leq& C + 3C\mathbb{E}\int_0^\infty e^{-\rho s}\|g(s)\|^2 ds \\ &+ 3C\mathbb{E}\int_0^\infty e^{-\rho s}\|\sigma(s)\|^2 ds \leq C_1. \end{aligned}$$

□

Proof of Theorem 4. i)⇒ ii). By (13),

$$\begin{aligned} \frac{d\mathbb{E}[\hat{x}_i]}{dt} =& \bar{A}\mathbb{E}[\hat{x}_i] - BR^{-1}B^T((\Pi - P)\bar{x} + s) \\ &+ G\mathbb{E}[\hat{x}^{(N)}] + f, \quad \mathbb{E}[\hat{x}_i(0)] = \bar{x}_0. \end{aligned} \tag{B.2}$$

It follows from A1) that

$$\mathbb{E}[\hat{x}_i] = \mathbb{E}[\hat{x}_j] = \mathbb{E}[\hat{x}^{(N)}],\ j \neq i.$$

By comparing (18) and (B.2), we obtain $\mathbb{E}[\hat{x}_i] = \bar{x}$. Note that $\|\bar{x}\|^2 = \big\|\mathbb{E}\hat{x}_i\big\|^2 \leq \mathbb{E}\|\hat{x}_i\|^2$. (22) leads to

$$\int_0^\infty e^{-\rho t}\|\bar{x}(t)\|^2 dt < \infty. \tag{B.3}$$

By (18), we have

$$\begin{aligned} \bar{x}(t) =& e^{(A+G-BR^{-1}B^T\Pi)t}\Big[\bar{x}_0 \\ &+ \int_0^t e^{-(A+G-BR^{-1}B^T\Pi)\tau}h(\tau)d\tau\Big], \end{aligned}$$

where $h = -BR^{-1}B^T s + f$. By the arbitrariness of $\bar{x}_0$ with (B.3) we obtain that $A + G - BR^{-1}B^T\Pi - \frac{\rho}{2}I$ is Hurwitz. That is, $(A + G, B)$ is stabilizable. By [2], (16) admits a unique solution such that $\Pi > 0$. Note that $\mathbb{E}[x^{(N)}]^2 \leq \frac{1}{N}\sum_{i=1}^N \mathbb{E}[\hat{x}_i^2]$. Then from (22) we have

$$\mathbb{E}\int_0^\infty e^{-\rho t}\big\|\hat{x}^{(N)}(t)\big\|^2 dt < \infty, \tag{B.4}$$

which leads to $\mathbb{E}\int_0^\infty e^{-\rho t}\|g(t)\|^2 dt < \infty$. Here $g = -BR^{-1}B^T((\Pi - P)\bar{x} + s) + G\hat{x}^{(N)} + f$. By (B.1), we obtain

$$\begin{aligned} \mathbb{E}\|\hat{x}_i(t)\|^2 =& \mathbb{E}\left\|e^{\bar{A}t}\left(x_{i0} + \int_0^t e^{-\bar{A}s}g(s)ds\right)\right\|^2 \\ &+ \mathbb{E}\int_0^t tr\big[\sigma^T(s)e^{(\bar{A}^T+\bar{A})(t-s)}\sigma(s)\big]ds. \end{aligned}$$

By (22) and the arbitrariness of x_{i0} we obtain that $\bar{A} - \frac{\rho}{2}I$ is Hurwitz, i.e., (A, B) is stabilizable. By [2], (15) admits a unique solution such that $P > 0$.

From (B.3) and (B.4),

$$\mathbb{E}\int_0^\infty e^{-\rho t}\big\|\hat{x}^{(N)}(t) - \bar{x}(t)\big\|^2 dt < \infty. \tag{B.5}$$

On the other hand, (20) gives

$$\begin{aligned} &\mathbb{E}\big\|\hat{x}^{(N)}(t) - \bar{x}(t)\big\|^2 \\ =& \mathbb{E}\Big\|e^{(\bar{A}+G)t}[\hat{x}^{(N)}(0) - \bar{x}_0]\Big\|^2 \\ &+ \frac{1}{N}\int_0^t tr\big[\sigma^T(s)e^{(\bar{A}^T+G^T+\bar{A}+G)(t-s)}\sigma(s)\big]ds. \end{aligned}$$

By (B.5) and the arbitrariness of $x_{i0}, i = 1, \cdots, N$, we obtain that $\bar{A} + G - \frac{\rho}{2}I$ is Hurwitz.

(ii)⇒(iii). Define $V(t) = e^{-\rho t}\mathbb{E}[\bar{y}^T(t)\Pi\bar{y}(t)]$, where $\bar{y}$ satisfies

$$\frac{d\bar{y}}{dt} = (A + G - BR^{-1}B^T\Pi)\bar{y}, \quad \bar{y}(0) = \bar{y}_0.$$

$$\begin{aligned} \dot{V}(t) =& \mathbb{E}\Big\{\bar{y}^T(t)\big[-\rho\Pi + (A + G - BR^{-1}B^T\Pi)^T\Pi \\ &+ \Pi(A + G - BR^{-1}B^T\Pi)\big]\bar{y}(t)\Big\} \\ =& \mathbb{E}\Big\{\bar{y}^T(t)\big[-\Pi BR^{-1}B^T\Pi \\ &- (I - \Gamma)^T Q(I - \Gamma)\big]\bar{y}(t)\Big\} \leq 0. \end{aligned}$$

By A3), we can prove that $V(t) \to 0$, which with $\Pi > 0$ further leads to $\bar{y} \to 0$. This implies $(A + G - \frac{\rho}{2}I, B)$ is stabilizable. Similarly, we can show $(A - \frac{\rho}{2}I, B)$ is stabilizable.

(iii)⇒(i). This part has been proved in Theorem 4. □

References

[1] H. Abou-Kandil, G. Freiling, V. Ionescu, and G. Jank, *Matrix Riccati Equations in Control and Systems Theory*. Birkhiiuser Verlag, 2003.

[2] B. D. O. Anderson, J. B. Moore. Optimal Control: Linear Quadratic Methods. Englewood Cliffs, NJ: Prentice Hall, 1990.

[3] J. Arabneydi and A. Mahajan. Team-optimal solution of finite number of mean-field coupled LQG subsystems. *Proc. 54th IEEE CDC*, Osaka, Japan, pp. 5308-5313, Dec. 2015.

[4] T. Basar and G. J. Olsder, *Dynamic Noncooperative Game Theory*, Academic Press, London, 1982.

[5] D. Bauso, H. Tembine, and T. Basar, Opinion dynamics in social networks through mean-field games. *SIAM Journal on Control and Optimization, 54*(6), 3225-3257, 2016.

[6] A. Bensoussan, J. Frehse, and P. Yam. *Mean Field Games and Mean Field Type Control Theory*. Springer, New York, 2013.

[7] A. Bensoussan, K.C. Sung, S.C. Yam, and S. P. Yung, "Linear-quadratic mean field games," *J. Optimization Theory & Applications*, vol. 169, no. 2, pp. 496-529, 2016.

[8] P. E. Caines, M. Huang, and R. P. Malhame, Mean field games. In *Handbook of Dynamic Game Theory*, T. Basar and G. Zaccour Eds., Springer, Berlin, 2017.

[9] P. Chan, and R. Sircar, Bertrand and Cournot mean field games, *Applied Mathematics & Optimization, 71*(3), 533-569, 2015.

[10] Y. Chen, A. Busic, and S. Meyn, State estimation and mean field control with application to demand dispatch. *Proc. 54nd IEEE CDC*, Osaka (pp. 6548-6555), 2015.

[11] R. Carmona and F. Delarue. Probabilistic analysis of mean-field games. *SIAM J. Control Optim.*, vol. 51, no. 4, pp. 2705-2734, 2013.

[12] B. Djehiche, and M. Huang. A characterization of sub-game perfect equilibria for SDEs of mean field type. *Dynamic Games Appl.*, vol. 6, no. 1, pp. 55-81, 2016.

[13] D. A. Gomes and J. Saude. Mean field games models–a brief survey. *Dyn. Games Appl.*, vol. 4, no. 2, pp. 110-154, 2014.

[14] Guéant, O., Lasry, J.-M., & Lions, P.-L. Mean field games and applications. In Paris- Princeton Lectures on Mathematical Finance, pp. 205-266, Springer-Verlag: Heidelberg, Germany, 2011.

[15] Y. C. Ho, Team decision theory and information structures, *Proc. IEEE*, vol. 68, pp. 644-654, 1980.

[16] J. Huang and M. Huang. Mean field LQG games with model uncertainty. *Proc. 52nd IEEE CDC*, Florence, Italy, pp. 3103-3108, December 2013.

[17] J. Huang and M. Huang. Robust mean field linear-quadratic-Gaussian games with model uncertainty, *SIAM J. Control Optim., 55*(5), 2811-2840.

[18] M. Huang. Large-population LQG games involving a major player: the Nash certainty equivalence principle. *SIAM J. Control Optim.*, vol. 48, pp. 3318-3353, 2010.

[19] M. Huang, P. E. Caines, and R. P. Malhamé. Individual and mass behaviour in large population stochastic wireless power control problems: Centralized and Nash equilibrium solutions. *Proc. 42nd IEEE CDC*, Maui, HI, pp. 98-103, Dec. 2003.

[20] M. Huang, P. E. Caines and R. P. Malhamé. Large-population cost-coupled LQG problems with non-uniform agents: individual-mass behavior and decentralized ε-Nash equilibria. *IEEE Trans. Automat. Control*, vol. 52, pp. 1560-1571, 2007.

[21] M. Huang, P. Caines and R. Malhame, Social optima in mean field LQG control: centralized and decentralized strategies, *IEEE Trans. Automat. Control*, vol. 57, no. 7, pp. 1736-1751, 2012.

[22] M. Huang, R. P. Malhamé, and P. E. Caines. Large population stochastic dynamic games: closed-loop McKean-Vlasov systems and the Nash certainty equivalence principle. *Communication in Information and Systems*, vol. 6, pp. 221-251, 2006.

[23] M. Huang and L. Nguyen, Linear-quadratic mean field teams with a major agent, *Proc. 55th IEEE CDC*, pp. 6958-6963, 2016.

[24] J. M. Lasry and P. L. Lions. Mean field games. *Japan J. Math.*, vol. 2, pp. 229-260, 2007.

[25] T. Li and J.-F. Zhang. Asymptotically optimal decentralized control for large population stochastic multiagent systems. *IEEE Trans. Automat. Control*, vol. 53, no. 7, pp. 1643-1660, August 2008.

[26] J. Ma and J. Yong. *Forward-backward Stochastic Differential Equations and their Applications*, Lecture Notes in Math. 1702. Springer-Verlag, New York, 1999.

[27] Z. Ma, D. Callaway, and I. Hiskens, Decentralized charging control for large populations of plug-in electric vehicles. *IEEE Transactions on Control Systems Technology*, vol. 21, no. 1, pp. 67-78, 2013.

[28] B. P. Molinari, "The time-invariant linear-quadratic optimal control problem," *Automatica*, vol. 13, no. 4, pp. 347-357, 1977.

[29] J. Moon, and T. Basar. Linear quadratic risk-sensitive and robust mean field games, *IEEE Trans. Automat. Control*, vol. 62, no. 3, 2017.

[30] R. Radner, Team decision problems, *Ann. Math. Stat.*, vol. 33, pp. 857-881, 1962.

[31] B. Wang and M. Huang, Dynamic production output adjustment with sticky prices: A mean field game approach, in *Proceedings of the 45th IEEE Conference on Decision and Control*, Osaka, Japan, 2015, pp. 4438-4443.

[32] B.-C. Wang and J.-F. Zhang. Mean field games for large-population multiagent systems with Markov jump parameters. *SIAM J. Control Optim.*, vol. 50, no. 4, pp. 2308-2334, 2012.

[33] B.-C. Wang, and J.-F. Zhang. Distributed control of multi-agent systems with random parameters and a major agent, *Automatica*, vol. 48, no. 9, pp. 2093-2106, 2012.

[34] B.-C. Wang, and J.-F. Zhang. Hierarchical mean field games for multiagent systems with tracking-type costs: Distributed ε-Stackelberg equilibria. *IEEE Trans. Automat. Control*, vol. 59, no. 8, pp. 2241-2247, 2014.

[35] B.-C. Wang and J.-F. Zhang. Social optima in mean field linear-quadratic-Gaussian models with Markov jump parameters, *SIAM J. Control Optim.*, vol. 55, no. 1, pp. 429-456, 2017.

[36] G. Weintraub, C. Benkard, and B. Van Roy, Markov perfect industry dynamics with many firms, *Econometrica*, vol. 76, no. 6, pp. 1375–1411, 2008.

[37] H. Yin, P. G. Mehta, S. P. Meyn, and U. V. Shanbhag. Synchronization of coupled oscillators is a game. *IEEE Trans. Autom. Control*, vol. 57, no. 4, pp. 920-935, April 2012.

[38] J. Yong. Linear-quadratic optimal control problems for mean-field stochastic differential equations. *SIAM J. Control Optim.*, vol. 51, no. 4, pp. 2809-2838, 2013.

[39] J. Yong, and X. Y. Zhou. *Stochastic Controls: Hamiltonian Systems and HJB Equations*. Springer-Verlag, New York, 1999.

[40] H. Zhang, and J. Xu. Control for Itô stochastic systems with input delay. *IEEE Transactions on Automatic Control*, 2017, vol. 62, no. 1, pp. 350-365.