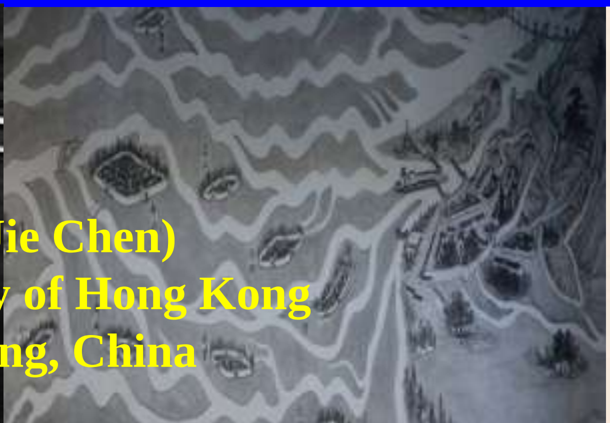
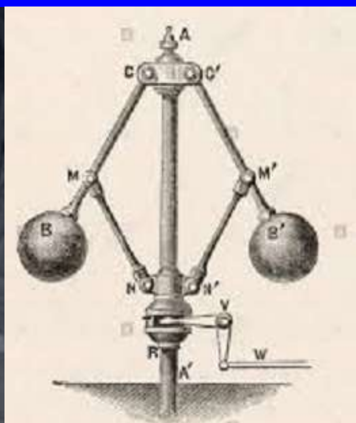
When is a Time-Delay System Stable and Stabilizable?

--A Third-Eye View

36th Chinese Control Conference
Dalian, July 2017





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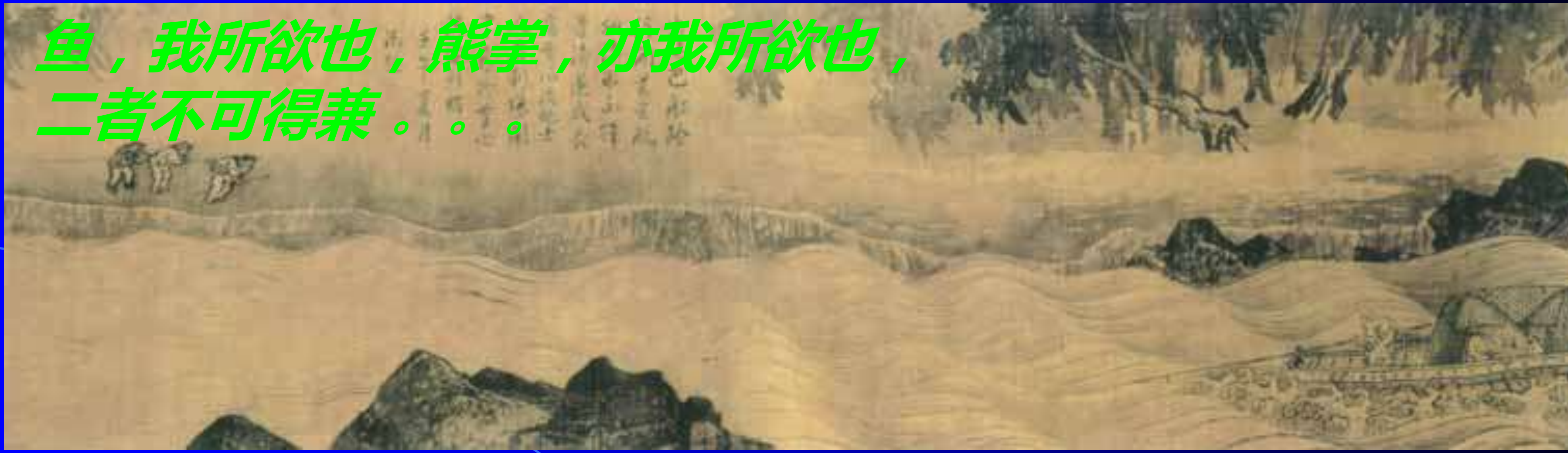
Those're such happy times not so long ago …

(CCC'03, Yichang)



Fundamental Limitation and Tradeoff of Feedback

CCC'03, Yichang

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When is a Time-Delay System Stable and Stabilizable?

--A Third-Eye View by an Amateur

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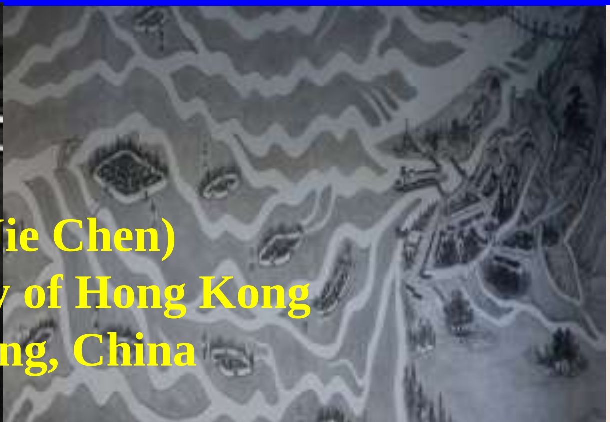
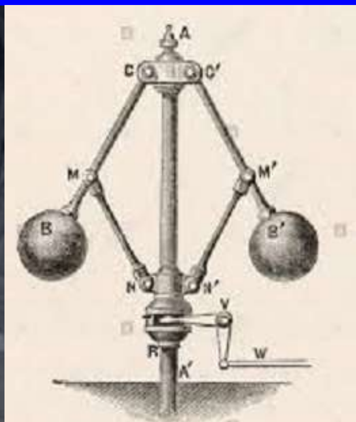

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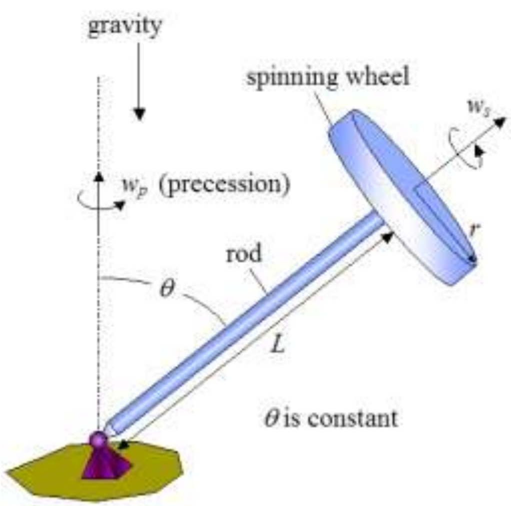
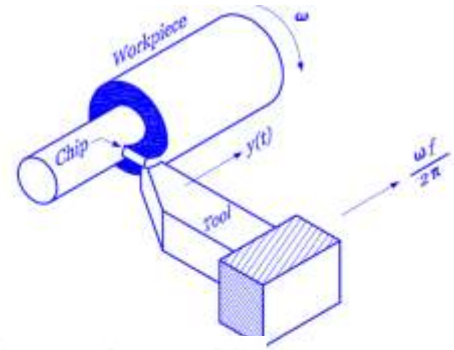
Why Delay? Classical Examples



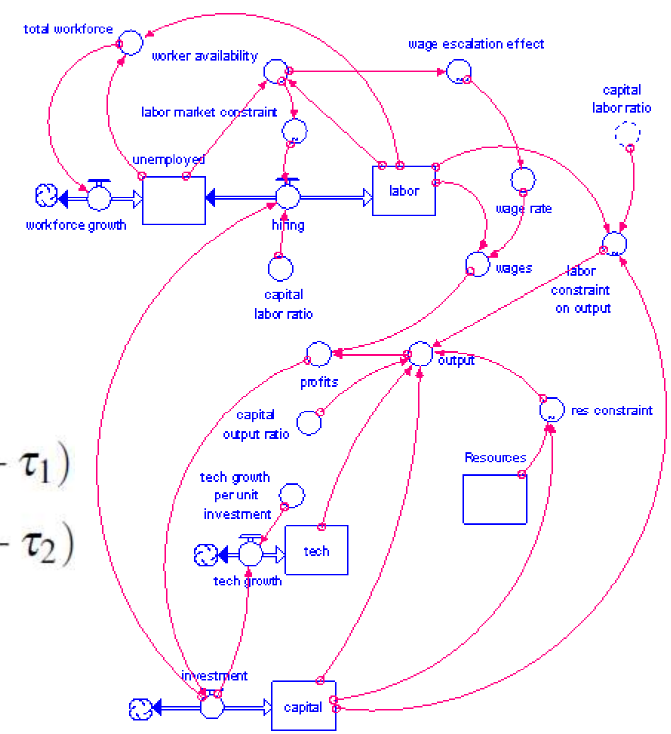


$$m\ddot{y}(t) + c\dot{y}(t) + ky(t) = -F_t(f + y(t) - y(t-\tau))$$

Cutting/Milling Process

$$M\ddot{y}(t) + G\dot{y}(t-\tau) + Ky(t) = 0$$

Classical Gyroscopic Systems

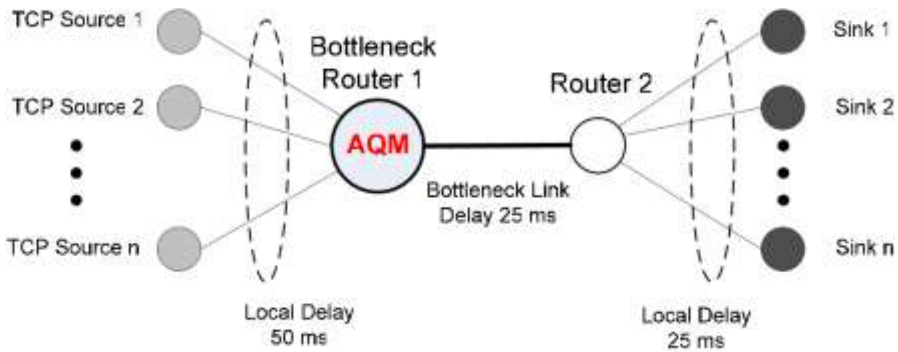


$$\dot{k}(t) = s_k k^{\alpha}(t-\tau_1)h^{\beta}(t-\tau_2) - (\delta + n + g)k(t-\tau_1)$$

$$\dot{h}(t) = s_h k^{\alpha}(t-\tau_1)h^{\beta}(t-\tau_2) - (\delta + n + g)h(t-\tau_2)$$

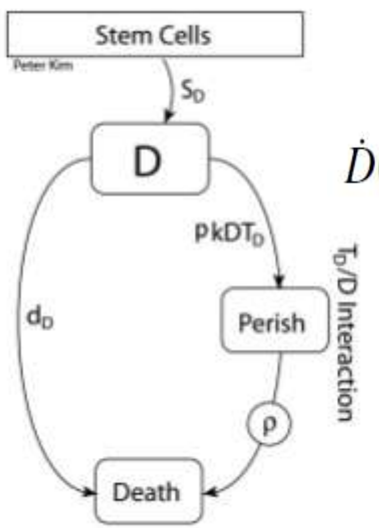
Classical Economic Growth Model

Why Delay? Contemporary Examples



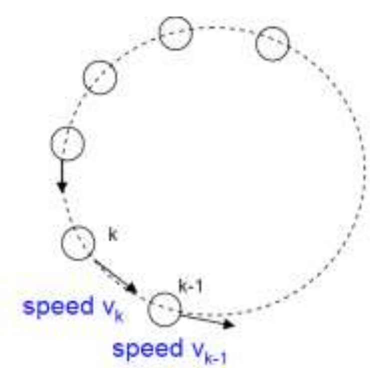
TCP/AQM Network

$$\dot{x}(t)=\begin{bmatrix}-\frac{N}{R_0^2C} & -\frac{1}{R_0^2}\\ \frac{N}{R_0} & -\frac{1}{R_0}\end{bmatrix}x(t)+\begin{bmatrix}-\frac{N}{R_0^2C} & -\frac{1}{R_0^2}\\ 0 & 0\end{bmatrix}x(t-R_0)+\begin{bmatrix}-\frac{R_0C^2}{2N^2}\\ 0\end{bmatrix}u(t-R_0)$$



$$\dot{D}(t)=S_D-d_DD(t)-pkD(t-\rho)T_D(t-\rho)$$

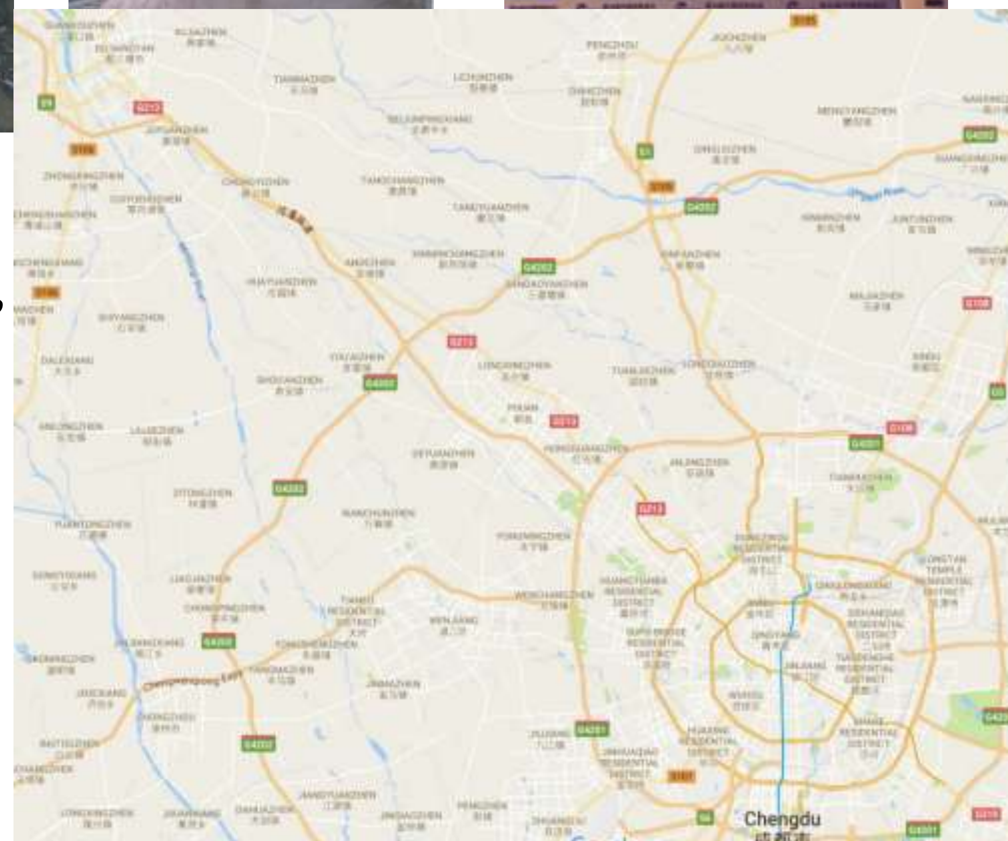
Donor Blood Cell Dynamics



Platoon

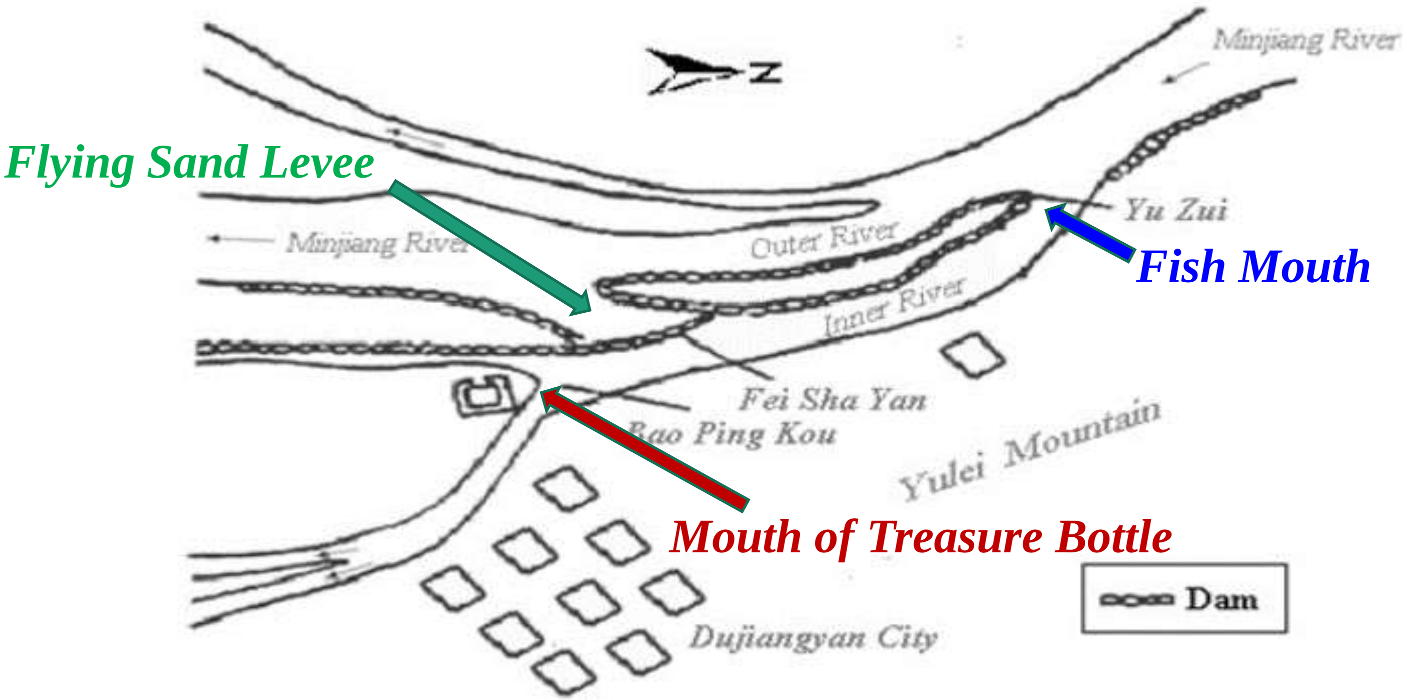
$$\dot{v}_k(t)=\alpha_k(v_{k-1}(t-\tau)-v_k(t-\tau))$$

The Dujiangyan (都江堰) Irrigation System



- **Built by Li Bing (李冰) in 256 BC, for flood control and water distribution.**
- **An isle constructed in the river center to divide the flow into two parts.**
- **Mountainside cut by fire and water to yield a water channel.**

Dujiangnyan as a Delay System



- **Fish Mouth (鱼嘴）divides the water. Deeper inner river and curvature leads to slower flow. Outer river flows much faster. This creates a *de facto* delay at Mouth of Treasure Bottle (宝瓶口）.**
- **Delay leads to accumulation of silt and sediment.**
- **Flying Sand Levee (飞沙堰) acts as an actuator, reducing the accumulation.**

Where are we? A Long Journey

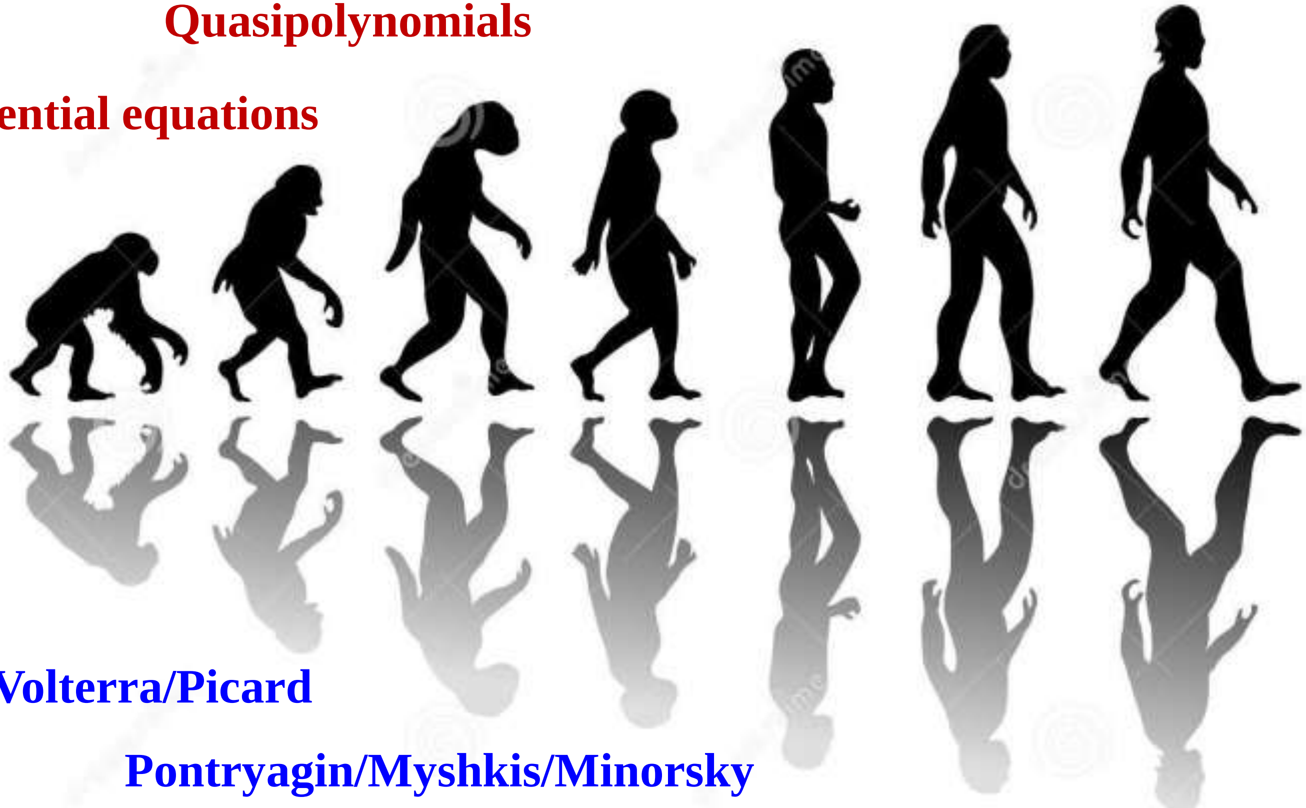
LMI

Robust control

Functional differential equations

Quasipolynomials

Differential equations



Euler/Volterra/Picard

Pontryagin/Myshkis/Minorsky

Bellman/Krasovskii/Razumikhim/Hale

← 1800 1950 1960 1970 1980 1990 →

Classical Methods and Rise of LMI

Classical Frequency Domain Methods

Two-Variable Criterion and many other variations: Elimination of one variable from two polynomial equations

$$a(s,\ z) = 0, \qquad z = e^{-\tau s}$$

$$a\left(-s, \frac{1}{z}\right) = 0$$

***Advantage*: Exact, necessary and sufficient stability conditions.**

***Disadvantage*: Require symbolic computation and hence are computationally inefficient; largely applicable to low-order systems with a small number (a couple) of delays.**

Contemporary Time Domain Method

LMI (Linear Matrix Inequality) conditions: Construct Lyapunov functionals

***Advantage*: Numerically efficient via convex optimization; can handle nonlinear time-varying systems, though essentially solve an augmented linear problem.**

***Disadvantage*: Inherently conservative, incremental improvements.**

A Third Eye

An Operator-Theoretic, Non-LMI Perspective

- **A mathematical problem: Eigenvalue series of matrix-valued functions.**

- **A control problem: Stability of delay systems.**

- **Stabilization and robust stabilization of delay systems.**

The Myth of the Third Eye

Hindu God: Shiva

Dao Deity: Erlang

GOT: Three-Eye Raven

Stability: Linear Delay Systems

- **Linear Delay Systems with Commensurate Delays**

$$\dot{x}(t) = A_0 x(t) + \sum_{k=0}^{q} A_k x(t - k\tau), \quad \tau \geq 0$$

- **Characteristic Function**

$$a(s, e^{-\tau s}) = \det\left(sI - \sum_{k=0}^{q} A_k e^{-k\tau s} \right)$$

- ***The system is stable if and only if***

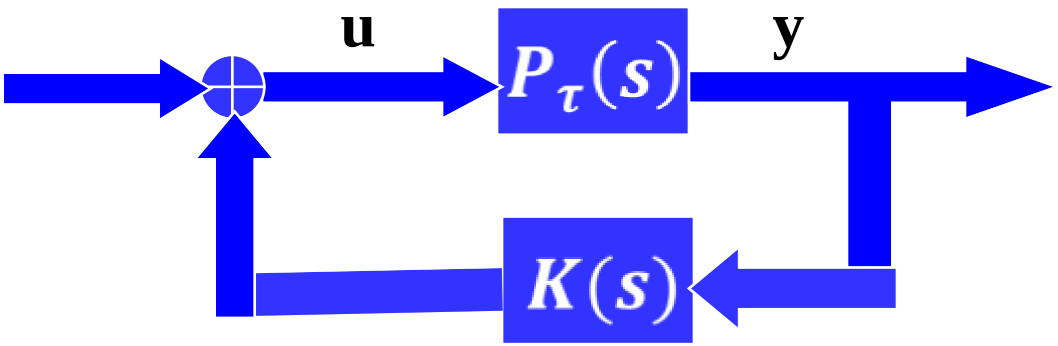
$$a(s, e^{-\tau s}) \neq 0, \quad \forall s \in \mathrm{CRHP}$$

Fundamental Issues-Stability

- **Delay Stability Margin**: Suppose that a system is stable at $\tau = 0$. What is the largest τ^* so that the system remains stable over $[0,\ \tau^*)$?

- **Delay Stability Intervals** For what ranges of delay, is the system stable?

Fundamental Issues-Feedback Stabilization



- **Delay Stabilization Margin** **For a delay system**

$P_\tau(s) = e^{-\tau s} P_0(s)$, **or in state space form**

$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t-\tau)$$

what is the largest τ^* so that it is possible to find a *single LTI* feedback controller $K(s)$ to stabilize $P_\tau(s)$ for all $\tau \in [0, \tau^*)$?

Rethink the Delay Interval Problem

- **Quasipolynomial**

$$a(s, e^{-\tau s}) = \sum_{k=0}^{q} a_k(s) e^{-k\tau s}$$

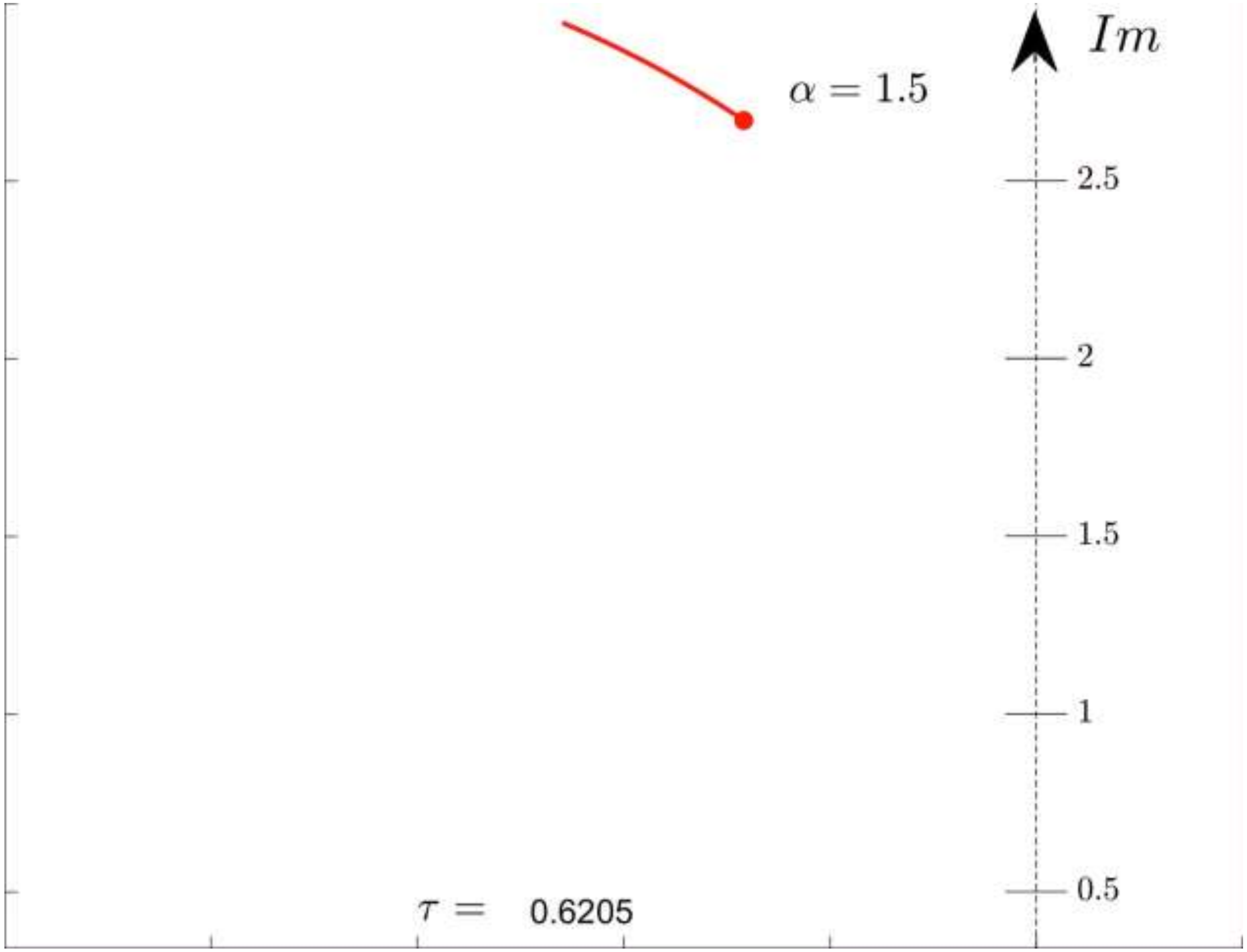
$$a_0(s) = s^n + \sum_{i=0}^{n-1} a_{0i} s^i, \quad a_k(s) = \sum_{i=0}^{n-1} a_{ki} s^i$$

- **Idea—Continuity and Analyticity**:

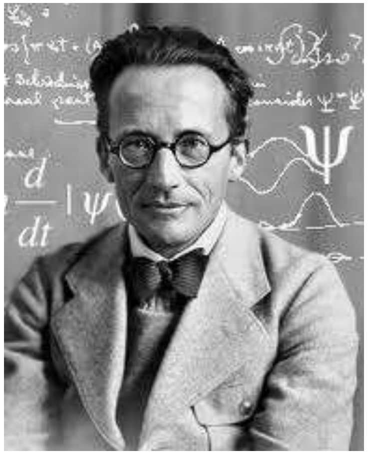
1. **To determine the critical parameter values, and the imaginary zeros, where the zeros may cross the imaginary axis.**
2. **To determine the asymptotic analytical behavior of critical characteristic zeros on the imaginary axis, and hence how the imaginary zeros may vary with the parameter.**

- **Question—Stability Switch Problem**: **When will an imaginary zero/eigenvalue cross from one half plane into another as the system parameter varies?**

Stability Switch: An Illustration

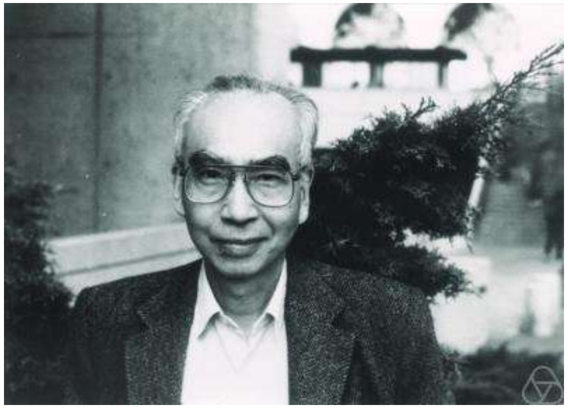


The Art of Operator Perturbation



E. Schrödinger

F. Rellich



T. Kato

...

Eigenvalue Perturbation

- **Matrix Operator $T(x)$ of a real variable x .**

- $T(x)$ **is analytic in the neighborhood of $x = 0$, namely,**

$$T(x) = T(0) + xT'(0) + x^2 \frac{T''(0)}{2!} + \ldots$$

- **Eigenvalue of $T(x)$: Can be expanded in a power series or a Puiseux series.**

Kato's Theory

Semisimple Eigenvalues

Let $\lambda^{(0)}$ be a semisimple eigenvalue of $T(0)$ with multiplicity m. Then the corresponding eigenvalues of $T(x)$ are analytic in x and have the form

$$\mu_i(x) = \lambda^{(0)} + \lambda_i^{(1)} x + o(x^2), \quad i = 1, \ldots, m,$$

where $\lambda_i^{(1)}$ are the eigenvalues of $PT'(0)P$

$$P = \frac{1}{2\pi j} \oint_{\Gamma} (\xi I - T(0))^{-1} \mathrm{d}\xi$$

Kato's Reduction Procedure-Semisimple

Define the reduced resolvent $R(\xi,\ x) = (T(x) - \xi I)^{-1}$

$$\tilde{T}(x) = -\left(\frac{1}{x}\right)\frac{1}{2\pi j}\oint_{\Gamma}(\xi - \lambda^{(0)})R(\xi, x)\mathrm{d}\xi$$

$$\tilde{T}(x) = \tilde{T}(0) + x\tilde{T}'(0) + o(x)$$

An iterative procedure

$$\mu_i(x) = \lambda^{(0)} + x\lambda_i[\tilde{T}(x)]$$

Non-Semisimple Eigenvalues

$$T(x) = \begin{pmatrix} 0 & 1 \\ x & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + x \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

$$\lambda[T(x)] = \pm x^{1/2}$$

- **Singularity occurs. The eigenvalue is multi-valued, and hence not analytic.**
- **No longer expandable in a power series, but instead in a *fractional* Puiseux series.**

Reduction Procedure: Non-Semisimple Eigenvalues

A doubly iterative procedure

$$\mu_i(x) = \lambda^{(0)} + x^{\frac{1}{m}} \lambda_i^{\frac{1}{m}}[T_1(x)]$$

$$\lambda_i[T_k(x)] = \lambda_{k-1}^{(1)} + x^{\frac{1}{m}} \lambda_i^{\frac{1}{m}}[T_{k+1}(x)]$$

- **A non-semisimple eigenvalue can be expanded via an iteration of continuous *m-th* roots of unity, in a fractional Puiseux series.**
- **The result provides a complete answer!**

First-Order Analysis

Semisimple

$$\mu_i(x) = \lambda^{(0)} + \lambda_i^{(1)} x + o\left(x^2\right), i = 1, \ldots, m$$

$\lambda_i^{(1)}$: Eigenvalues of $R_1 T'(0) Q_1$

Non-Semisimple

$$\mu_i(x) = \lambda^{(0)} + (\lambda_i^{(1)})^{1/m} x^{1/m} + o\left(x^{2/m}\right), i = 1, \ldots, m$$

$$\lambda_i^{(1)} = r_m T'(0) q_1$$

Eigen-Decomposition

Let $\lambda^{(0)}$ be an eigenvalue of $T(0)$ with multiplicity *m*.

Let $\lambda^{(0)}$ be ordered as the first eigenvalue.

$T(0)$ can be decomposed as

$$T(0) = Q\Sigma R = \begin{bmatrix} Q_1 & Q_2 \end{bmatrix} \begin{bmatrix} \Sigma_1 & 0 \\ 0 & \Sigma_2 \end{bmatrix} \begin{bmatrix} R_1 \\ R_2 \end{bmatrix}$$

$\lambda^{(0)}$ ***semisimple***

$$\Sigma_1 = \begin{bmatrix} \lambda^{(0)} & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & \cdots & \lambda^{(0)} \end{bmatrix}$$

$\lambda^{(0)}$ ***non-semisimple***

$$\Sigma_1 = \begin{bmatrix} \lambda^{(0)} & 1 & \cdots & 0 \\ 0 & \ddots & \ddots & 0 \\ \vdots & & \ddots & 1 \\ 0 & \cdots & \cdots & \lambda^{(0)} \end{bmatrix}$$

$R = Q^{-1} = \begin{bmatrix} r_1^T & \cdots & r_n^T \end{bmatrix}^T$, and $Q = \begin{bmatrix} q_1 & \cdots & q_n \end{bmatrix}$ **consist of eigenvectors and generalized eigenvectors of** $T(0)$ **.**

Second-Order Series: Semisimple Case

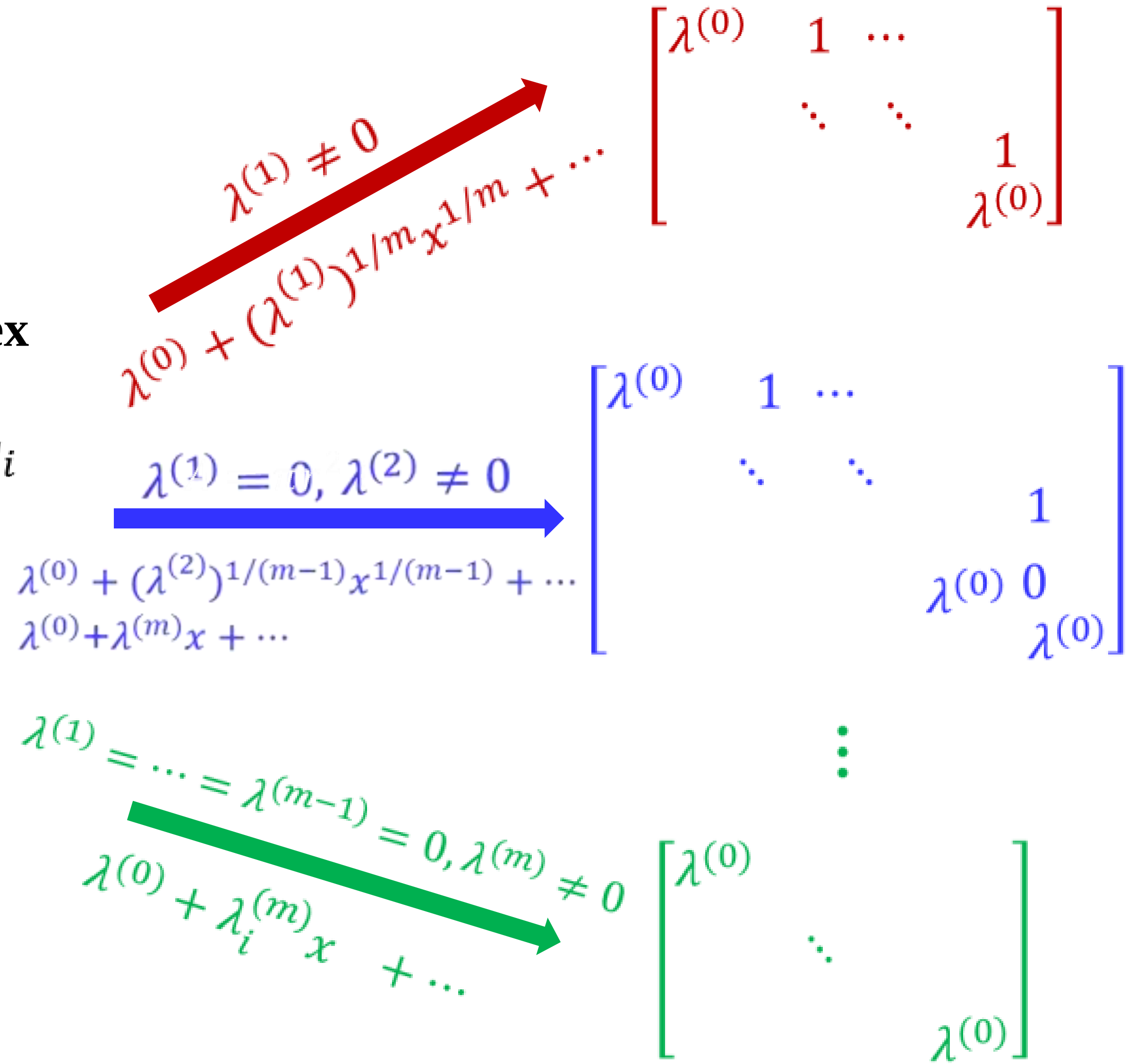
$$\mu_i(x) = \lambda^{(0)} + \lambda_i^{(1)}x + \mu_{ij}x^2 + o(x^2),$$
$$i = 1, \dots, m, j = 1, \dots, d$$

- **It takes another eigen-decomposition to compute the second-order coefficients.**
- **The second-order coefficients depend on the second-order derivative $T''(0)$.**

Second-Order Analysis

Non-Semisimplicity Index

$$\lambda^{(k)} = \sum_{i=1}^{k} r_{m-k+i} T'(0) q_i$$
$$k = 1, \ldots, m$$



The Stability Problem

$$\dot{x}(t) = A_0 x(t) + \sum_{k=0}^{q} A_k x(t - k\tau),$$

$$a(s, e^{-\tau s}) = \det\left(sI - \sum_{k=0}^{q} A_k e^{-k\tau s} \right)$$

- **Step 1—Continuity: To determine the critical delay values, and the imaginary zeros.**

- **Two-Variable Criterion:**

$$a(s, e^{-\tau s}) = \sum_{k=0}^{q} a_k(s) e^{-k\tau s}$$

$$a(s, e^{-\tau s}) = a(s, z), \; z = e^{-\tau s}$$

Schur-Cohen Criterion

- **A complex polynomial**:

$$p(z) = \sum_{k=0}^{q} p_k z^k$$

- **Schur-Cohen matrix:**

$$C = \begin{pmatrix} p_q & p_{q-1} & \cdots & p_1 \\ & p_q & \cdots & p_2 \\ & & \ddots & \vdots \\ & & & p_q \end{pmatrix}^H \begin{pmatrix} p_q & p_{q-1} & \cdots & p_1 \\ & p_q & \cdots & p_2 \\ & & \ddots & \vdots \\ & & & p_q \end{pmatrix} - \begin{pmatrix} p_0 & & & \\ p_1 & p_0 & & \\ \vdots & \ddots & \ddots & \\ p_{q-1} & \cdots & p_1 & p_0 \end{pmatrix} \begin{pmatrix} p_0 & & & \\ p_1 & p_0 & & \\ \vdots & \ddots & \ddots & \\ p_{q-1} & \cdots & p_1 & p_0 \end{pmatrix}^H$$

- **Schur-Cohen Theorem**: **The complex polynomial $p(z)$ is Schur stable if and only if the Schur-Cohen matrix is positive definite.**

Step 1: The Matrix Pencil Solution

- **Bivariate Polynomial:**

$$a(s, z) = \sum_{k=0}^{q} a_k(s) z^k$$

- **Schur-Cohen Criterion: Consider the frequency-dependent polynomial** $a(j\omega, z)$ **and construct the Schur-Cohen matrix, which is a matrix polynomial in** ω **. The matrix is positive definite iff the polynomial in z has no root outside the unit disc.**

- **Matrix Pencil Solution: Set the determinant of the Schur-Cohen matrix to be zero (a polynomial equation), whose positive roots** ω_k **correspond to unitary solutions**

$z_k = e^{-j\theta_k}$

Critical Delays $\tau_k = \frac{\theta_k}{\omega_k}$ **Critical Zeros** $j\omega_k$

All critical delays and imaginary zeros/eigenvalues can be computed efficiently by solving a matrix pencil problem.

Step 2: Compute Eigenvalue Series

- **Reformulation as an Operator Perturbation Problem: At each critical pair $(\tau^*, j\omega^*)$, expand**

$$T(\tau) = \sum_{k=0}^{q} A_k e^{-jk\tau\omega}$$

$$T(\tau) = T(\tau^*) + T'(\tau^*)(\tau - \tau^*) + \frac{T''(\tau^*)}{2}(\tau - \tau^*)^2 + \cdots$$

- **The coefficient matrices can be determined readily; e.g.,**

$$T'(\tau^*) = -j\omega^* \sum_{k=1}^{q} kA_k e^{-jk\tau^*\omega^*}$$

- **The eigenvectors required are computed in Step I when solving $(\tau^*, j\omega^*)$.**
- **When conducting eigen-decomposition, use eigenvalues to determine the critical pairs, and use eigenvectors to determine the crossing behavior, all in one shot!**

A General Semisimple Result

$$j\omega^* + \lambda^{(1)}(\tau - \tau^*) + \lambda_j^{(2)}(\tau - \tau^*)^2$$

$Re\{\lambda^{(1)}\} > 0\ (< 0)$

The imaginary eigenvalue enters RHP (LHP).

$$\lambda^{(1)} = \lambda(R_1 T'(\tau^*) Q_1)$$

$Re\{\lambda^{(1)}\} = 0$

Inconclusive!

$$\lambda_j^{(2)} = \lambda\left(R_1^{(2)} R_1 (\tfrac{1}{2} T''(\tau^*) - T'(\tau^*) S T'(\tau^*)) Q_1 Q_1^{(2)}\right)$$

$Re\{\lambda_j^{(2)}\} > 0\ (< 0)$

The imaginary eigenvalue enters RHP (LHP).

Simple Imaginary Eigenvalue

$(\boldsymbol{\tau}^*, \boldsymbol{j\omega}^*)$

$Re\{r_1 T'(\tau^*) q_1\} > 0\ (< 0)$

The imaginary eigenvalue enters RHP (LHP).

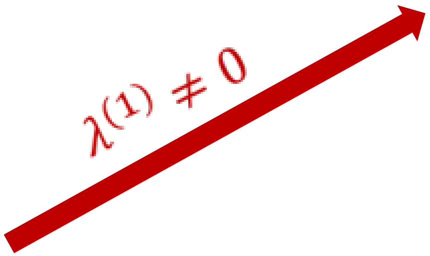
$Re\{r_1 T'(\tau^*) q_1\} = 0$

Inconclusive!

$Re\{r_1 (\frac{1}{2} T''(\tau^*) - T'(\tau^*) S T'(\tau^*)) q_1\} > 0\ (< 0)$

The imaginary eigenvalue enters RHP (LHP).

A Non-Semisimple Result



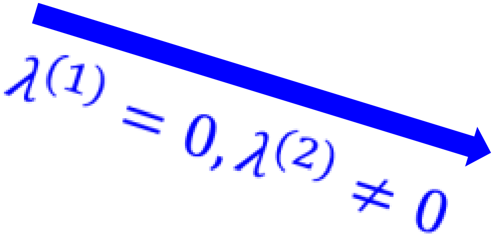
The critical eigenvalue radiates at equi-distant angles

$$\frac{2(i-1)\pi + \measuredangle\lambda^{(1)}}{m}$$

$$\lambda^{(1)} = r_m T'(\tau^*) q_1$$

$$\lambda^{(2)} = r_m T'(\tau^*) q_2 + r_{m-1} T'(\tau^*) q_1$$

$$\frac{2(i-1)\pi + \measuredangle\lambda^{(2)}}{m-1}$$



The critical eigenvalue radiates at equi-distant angles.

Example 1

$$A_0 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -2 & -3 & -5 & -2 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -2 & 0 & -1 & 0 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -2 & -1 & -1 & 0 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -5 & 0 & -2 \end{bmatrix}$$

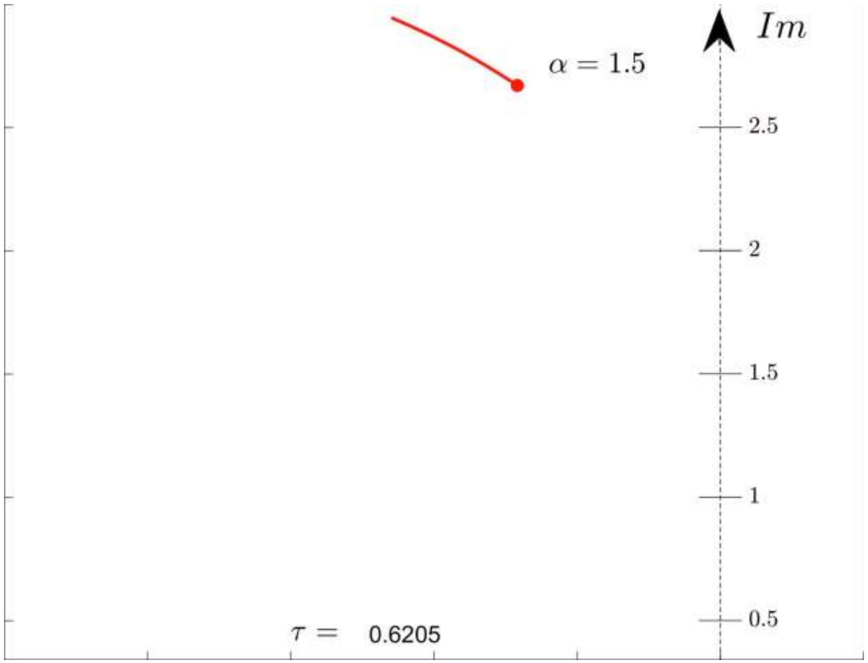
τ^*	ω^*	r_1	q_1	$r_1 T'(0) q_1$
0.3786 5.2002 $\cdots$	1.3026	$\begin{bmatrix} 1 \\ 1.3026j \\ -1.6967j \\ -2.2102j \end{bmatrix}$	$\begin{bmatrix} 0.2343 - 0.2347j \\ -0.00507 - 0.5835j \\ -0.1009 - 0.1478j \\ -0.0226 - 0.00749j \end{bmatrix}$	0.9203-0.1331j

Example 1: Zero Locus

Example 2: First-Order Condition Fails!

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} x(t) + \begin{bmatrix} 0 & 0 \\ 0 & -\alpha \end{bmatrix} x(t-\tau)$$

- **For $\alpha = 1$, the system has a critical pair at (π, j).**
- **The first-order test is inconclusive. The second-order condition indicates that the critical eigenvalue bounces back into LHP.**

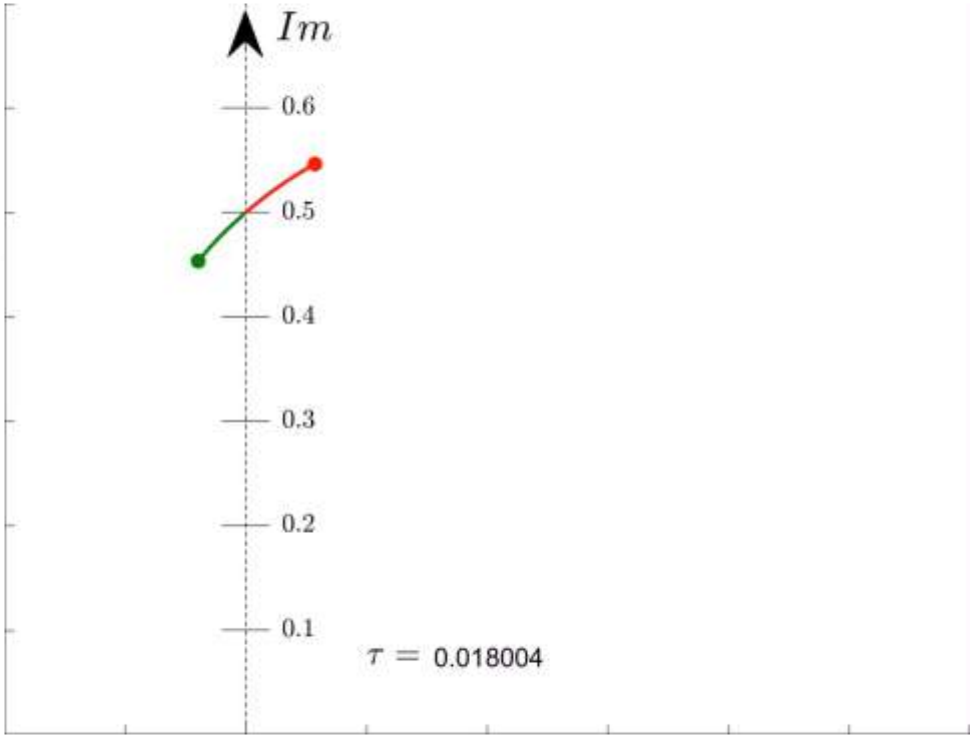


Example 3: Non-Semisimple Eigenvalue

$$A_0 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1/4 & 0 & 0 & 0 \\ 0 & 1/4 & 0 & 0 \end{bmatrix}$$

$$A_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

- **The system has a critical pair at $(0, j0.5)$.**
- **The first-order test indicates that the critical eigenvalue departs at the angles $\pi/4$ and $5\pi/4$.**



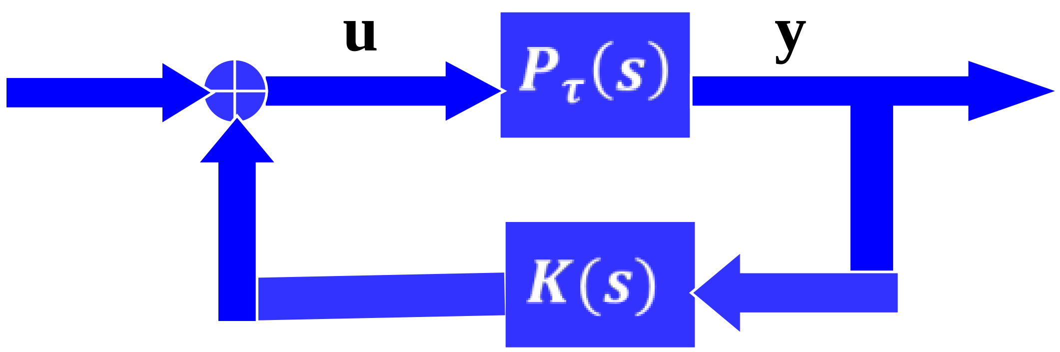
Comparisons

- **To conventional algebraic methods**:
  Numerical **vs Symbolic computation**

- **To LMI tests**:
  Necessary & Sufficient **vs Sufficient conditions;**
  Linear Algebra problem **vs LMI Optimization.**

- **Complexity in problem size:**
  $O(n^2q)$ **(state space) &** $O(nq)$ **(quasipolynomial) vs**
  $O(n^4q^2)$ **(non-conservative LMI)**

Feedback Stabilization



- **Delay Stabilization Margin** **For a delay system**

$P_\tau(s) = e^{-\tau s} P_0(s)$, **or in state space form**

$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t-\tau)$$

what is the largest τ^* so that it is possible to find a *single LTI* feedback controller $K(s)$ to stabilize $P_\tau(s)$ for all $\tau \in [0, \tau^*)$?

The Delay Stabilization Margin Problem

Delay Stabilization Margin

$$\tau^* = \sup\{\mu: \exists K(s) \text{ stabilizes } e^{-\tau s}P(s), \forall \tau \in [0, \mu)\}$$

Gain Margin:

$$k^* = \sup\{\mu: \exists K(s) \text{ stabilizes } kP(s), \forall k \in [1, \mu)\}$$

Phase Margin

$$\theta^* = \sup\{\mu: \exists K(s) \text{ stabilizes } e^{-j\theta}P(s), \forall \theta \in (-\mu, \mu)\}$$

- **The delay stabilization problem is fundamentally more difficult. No viable necessary and sufficient condition exists. The problem remains open.**
- **Suggested as an unresolved challenge in *Unsolved Problem in Mathematical Systems and Control Theory*.**

Bounds and Sufficient Conditions

- **Bound for guaranteed stabilization: There exists a controller $K(s)$ that stabilizes $P_\tau(s)$ for all $\tau \in [0, \tau_{\min})$,**
  - Fourth-order, 3 delays

$$\tau_{\min} = \sup\{\tau:\ \inf ||T_0(S)(e^{-\tau s}-1)||_\infty < 1\}$$

$$T_0(s) = P_0(s)K(s)[1 + P_0(s)K(s)]^{-1}$$

- **Solving the weighted H_∞ probem**

- **Rational approximation: A wide variety of approximations including bilinear transformation are available for $e^{-\tau s}$-1.**

- **Solution via Nevanlinna-Pick interpolation.**

Rational Approximation

$$w_\tau(s) = \frac{b_\tau(s)}{a_\tau(s)} = \frac{b_q(\tau s)^q + \cdots + b_1(\tau s) + b_0}{a_q(\tau s)^q + \cdots + a_1(\tau s) + a_0}$$

$$g(\omega) = \left|e^{-j\omega\tau} - 1\right| = \begin{cases} 2\sin(\omega\tau/2) & |\omega\tau| \le \pi \\ 2 & \text{otherwis} \end{cases}$$

$$g(\omega) \le |w_\tau(j\omega)|$$

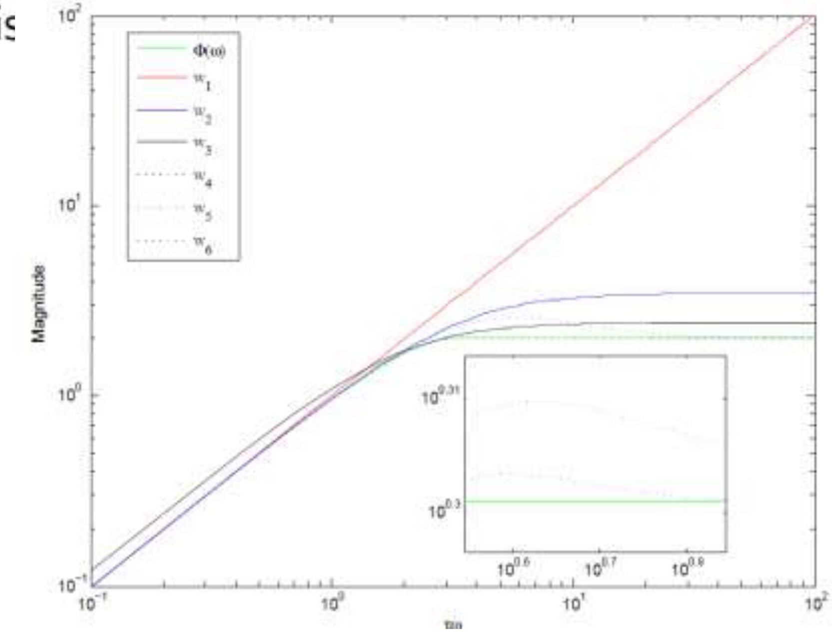
$$w_{1\tau}(s) = \tau s,$$

$$w_{2\tau}(s) = \frac{\tau s}{1+\tau s/3.465},$$

$$w_{3\tau}(s) = \frac{1.216\tau s}{1+\tau s/2},$$

$$w_{4\tau}(s) = \frac{\tau s(2\times 0.2152^2\tau s+1)}{(0.2152\tau s+1)^2}$$

$$w_{5\tau}(s) = \frac{\tau s}{1+\tau s/2} \times \frac{0.1791(\tau s)^2+0.7093\tau s+1}{0.1791(\tau s)^2+0.5798\tau s+1}$$



$$w_{6\tau}(s) = \frac{\tau s}{1+\tau s/2}\,\frac{0.03061(\tau s)^4+0.2102(\tau s)^3+0.7087(\tau s)^2+1.203\tau s+1}{0.03061(\tau s)^4+0.1918(\tau s)^3+0.6457(\tau s)^2+1.104\tau s+1}$$

Guaranteeing Stabilization: Compute Eigenvalues

$$\underline{\tau} = \overline{\lambda}^{-1}\left(\begin{bmatrix} -\Phi_0^{-1}\Phi_1 & \cdots & -\Phi_0^{-1}\Phi_{2q-1} & -\Phi_0^{-1}\Phi_{2q} \\ I & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & I & 0 \end{bmatrix}\right)$$

$$\underline{\tau} \leq \tau_{\min} \leq \tau^*$$

- **$\bar{\lambda}$: The largest positive eigenvalue.**
- **Φ_k: Matrices depending on the coefficients of the rational approximation and the unstable poles/nonminimum phase zeros of $P_0(s)$.**
- **Applicable to systems with an arbitrary number of unstable poles/nonminimum phase zeros.**

Analytical Bounds

A Single Pole p

$$\underline{\tau} = \frac{1.722}{p}$$

Exact Margin $\tau^* = \frac{2}{p}$

A Single Pole p with NMP Zero z

$$\underline{\tau} = \frac{1}{p}\left|\frac{p-z}{p+z}\right|$$

Generalizations:

- **The approach can be generalized to systems with time-varying delays.**
- **It can also be generalized to MIMO systems, to find estimates on the so-called delay radius.**
- **Similar results can be obtained using PID controllers.**
- **Extensions to networked delay systems, multi-agent delayed communications.**

Summary

- **Main Results**:
  1. ***Eigenvalue Perturbation Approach:***: **A general mathematical result providing a computationally efficient stability analysis approach, requiring only the solution of a matrix pencil problem.**
  2. ***Linear Delay Systems***: **A full characterization of stability for all possible delay parameter values.**
  3. ***Stabilization:*** **Bounds on the fundamental delay stabilization margin.**

- **Highlights**:
  1. ***Full Leverage on Computation:*** **Joint and judicious use of eigenvalues and eigenvectors.**
  2. ***Analytical vs Continuity Properties:*** **Much like continuity, the analyticity of eigenvalues plays an important role.**
  3. **A unified operator-theoretic approach for stability and stabilization of linear time-delay systems, generalizable to networked control and multi-agent robustness problems.**

The Amazing Dujiangyan … for Real!

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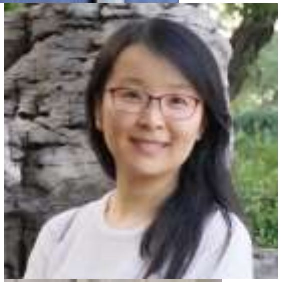

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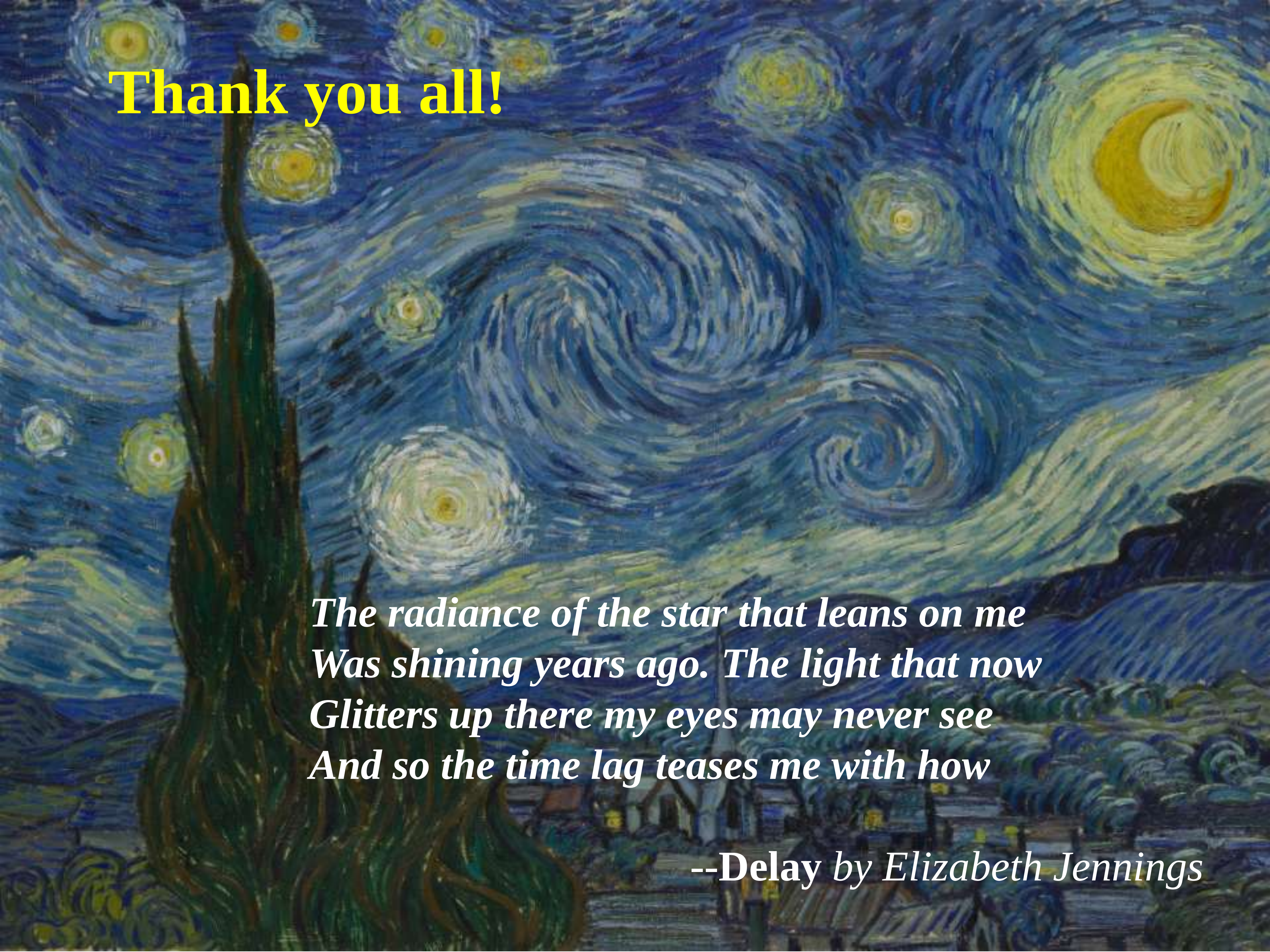

Thank you all!

The radiance of the star that leans on me
Was shining years ago. The light that now
Glitters up there my eyes may never see
And so the time lag teases me with how

--**Delay** *by Elizabeth Jennings*