Simultaneous Social Cost Minimization and Nash Equilibrium Seeking in Non-cooperative Games

Maojiao Ye, Guoqiang Hu

School of Electrical and Electronic Engineering, Nanyang Technological University, 639798, Singapore

Abstract: An N-coalition non-cooperative game is formulated in this paper. In the considered game, there are N interacting coalitions and each of them includes a set of agents. Each coalition acts as a virtual player (VP) in the game that aims to minimize its own objective function, which is defined as the sum of the agents' local objective functions in the coalition. However, the actual decision-makers are not the coalitions but the agents therein. That is, the agents within each coalition collaboratively minimize the coalition's objective function while constituting an entity that serves as a self-interested player (i.e., the coalition) in the game among the interacting coalitions. A seeking strategy is designed for the agents to find the Nash equilibrium of the N-coalition non-cooperative games. The equilibrium seeking strategy is based on an adaptation of a dynamic average consensus protocol and the gradient play. The dynamic average consensus protocol is leveraged to estimate the averaged gradients of the coalitions' objective functions. The gradient play is then implemented by utilizing the estimated information to achieve the Nash equilibrium seeking. Convergence results are established by utilizing Lyapunov stability analysis. A numerical example is given in supportive of the theoretical results.

Key Words: Nash equilibrium seeking; social cost minimization; interacting coalitions; non-cooperative game

1 Introduction

The analysis of competition and cooperation across multiple interacting decision-makers has been extensively investigated in recent years (see e.g., [1]-[23]). In particular, Nash equilibrium seeking in non-cooperative games (see e.g., [1]-[11] and the references therein) and distributed optimization problems (see e.g., [12]-[23] and the references therein) are two of the main lines of research. The players in non-cooperative games are self-interested to minimize their own objective functions by adjusting their own actions and Nash equilibrium seeking deals with strategy design that can be adopted by the players to find the Nash equilibrium of the non-cooperative games. Distributed optimization concerns with a network of $M(M \geq 2)$ agents that cooperatively minimize $f(x)$, which is defined as

$$f(x) = \sum_{i=1}^{M} f_i(x), \tag{1}$$

where x denotes a vector of the decision variables and $f_i(x)$ is the local objective function of agent i.

This paper sublimes the non-cooperative games and distributed optimization problems to consider N-coalition non-cooperative games. In the N-coalition non-cooperative game, each coalition is considered as a virtual player (VP) that intends to minimize its own objective function. Nevertheless, the virtual players in the game are not the actual decision-makers and the minimization of the coalition's objective function is achieved by the agents in the corresponding coalition. The coalition's objective function is defined as the sum of the local objective functions associated with the agents in the corresponding coalition. The objective of this paper is to design a strategy to seek for the Nash equilibrium of the N-coalition non-cooperative games under the condition that the agents only have access into their own local objective functions.

Related Works: Two adversarial networks with opposing network objectives were considered in [1]. The networks' objective functions were defined as the sum of the local objective functions of the agents in the corresponding networks. The two-network zero-sum game was solved in a distributed manner by using the notion of saddle point. Nash equilibrium seeking for the two-network zero-sum game was further explored in [2] under switching communication topologies. The two-network zero-sum games can be considered as a special class of the considered N-coalition non-cooperative games by noticing that for the two-network zero-sum games, $N = 2$ and $f_1(\mathbf{x}_1, \mathbf{x}_2) = -f_2(\mathbf{x}_1, \mathbf{x}_2)$, where N is the number of the coalitions, $\mathbf{x}_i$ denotes the action of coalition i and $f_i(\cdot)$ denotes the objective function of coalition i.

It was reported in [1, 2] that adversaries in communication networks and sensor networks can be modeled by the aforementioned two-network zero-sum games. Similarly, the considered N-coalition non-cooperative game, which is highly motivated by the coexistence of cooperation and competition in many practical situations, is applicable to capture the cooperation and competition in economic markets and multi-agent networked systems. For example, in cloud computing, subsets of the cloud providers can form coalitions to establish resource pools to serve the users [24]. The service providers within the same coalition collaborate to maximize their total profit. This scenario basically falls into the formulated N-coalition non-cooperative games if the competition among different coalitions is further handled.

To solve the N-coalition non-cooperative games, consensus protocols are employed to disseminate local information among the agents in the same coalition via neighboring communication. Nash equilibrium seeking (see, e.g., [3, 4]) and social cost minimization (see, e.g., [21]-[23]) based on consensus methods have been investigated in several works. Average consensus protocols and leader-following consensus protocols were utilized in [3, 4] for Nash equilib-

This work was supported by Singapore Economic Development Board under EIRP grant S14-1172-NRF EIRP-IHL.

E-mail addresses: mjye@ntu.edu.sg (M. Ye); gqhu@ntu.edu.sg (G. Hu).

rium seeking in aggregative games and multi-agent games, respectively. Similarly, average consensus protocols were leveraged to develop extremum seeking schemes to solve a social cost minimization problem in [21] and [22]. Distributed economic dispatch in smart grids was studied in [23] by designing consensus-based primal-dual dynamics. However, to realize Nash equilibrium seeking for the N-coalition non-cooperative games, simultaneous social cost minimization and Nash equilibrium seeking for non-cooperative games should be achieved, which makes the problem more challenging compared with both social cost minimization problems and Nash equilibrium seeking for non-cooperative games. Despite the challenges, the main contributions of the paper are twofold.

- An N-coalition non-cooperative game is formulated and a Nash equilibrium seeking strategy is designed for the N-coalition non-cooperative games. The proposed Nash equilibrium seeking strategy is based on an adaptation of a dynamic average consensus protocol and the gradient play. The dynamic average consensus protocol is utilized to estimate the averaged gradient information of the coalitions' objective functions and the gradient play is implemented based on the estimated information to achieve Nash equilibrium seeking.
- It is analytically proven that under certain conditions, the Nash equilibrium of the N-coalition non-cooperative game is exponentially stable by utilizing the proposed seeking strategy. With stronger assumptions, non-local convergence results are derived.

The rest of the paper is structured as follows. Section 2 formulates the N-coalition non-cooperative games. The Nash equilibrium seeking strategy is proposed and analyzed in Section 3. A numerical example is given in Section 4 to verify the effectiveness of the proposed method. Section 5 concludes the paper.

Notations: The set of real numbers is denoted as $\mathbb{R}$ and $[\varphi_{ij}]_{vec}$ for $i \in \{1, 2, \cdots, N\}, j \in \{1, 2, \cdots, m_i\}$ is a $\sum_{i=1}^{N} m_i$ dimensional column vector defined as $[\varphi_{ij}]_{vec} = [\varphi_{11}, \varphi_{12}, \cdots, \varphi_{1m_1}, \varphi_{21}, \cdots, \varphi_{Nm_N}]^T$. We say that $\mathcal{H} \in \mathbb{R}^M \times \mathbb{R}^N$ if $\mathcal{H}$ is an $M \times N$ dimensional real matrix. Let $\mathcal{H} \in \mathbb{R}^N \times \mathbb{R}^N$ be a symmetric matrix, then, $\lambda_{min}(\mathcal{H}), \lambda_{max}(\mathcal{H})$ are the minimum and maximum eigenvalues of $\mathcal{H}$, respectively. Furthermore, an N dimensional column vector composed of $1(0)$ is denoted as $\mathbf{1}_N(\mathbf{0}_N)$, $\mathbf{0}_{M\times N}$ defines an $M \times N$ dimensional matrix composed of 0 and $I_{M\times M}$ denotes an $M \times M$ dimensional identity matrix. Furthermore, $\text{diag}\{k_{ij}\}$, where $k_{ij} \in \mathbb{R}$, $i \in \{1, 2, \cdots, N\}, j \in \{1, 2, \cdots, m_i\}$, is a diagonal matrix whose diagonal elements are $k_{11}, k_{12}, \cdots, k_{1m_1}, k_{21}, \cdots, k_{Nm_N}$, successively. Moreover, $\text{diag}\{H_i\}$, where $H_i \in \mathbb{R}^{N_i} \times \mathbb{R}^{M_i}$, $i \in \{1, 2, \cdots, k\}$, is a block diagonal matrix in which the ith diagonal block is H_i. The symbol $\otimes$ denotes the Kronecker product. The constants and parameters defined in the paper are real numbers.

2 Problem Formulation

This paper formulates an N-coalition non-cooperative game. In the N-coalition non-cooperative game, there are $N(N \geq 1)$ interacting coalitions, each of which acts as a virtual player in a non-cooperative game and contains a set of agents, who are the actual decision-makers. Each agent is associated with a local objective function and the agents in the same coalition work collaboratively to minimize the sum of their local objective functions. More specifically, the objective of the agents in coalition i is

$$\min_{\mathbf{x}_i} \ f_i(\mathbf{x}_i, \mathbf{x}_{-i}), \tag{2}$$

with

$$f_i(\mathbf{x}_i, \mathbf{x}_{-i}) = \sum_{j=1}^{m_i} f_{ij}(\mathbf{x}_i, \mathbf{x}_{-i}), \tag{3}$$

where $m_i(m_i \geq 1)$ is the number of agents in coalition i and $f_{ij}(\mathbf{x}_i, \mathbf{x}_{-i})$ denotes the objective function of agent j in coalition i. Furthermore, $\mathbf{x}_i$ denotes the vector of the actions of the agents in coalition i and $\mathbf{x}_{-i}$ is the vector representing all the agents' actions other than the actions of the agents in coalition i, i.e., $\mathbf{x}_{-i} = [\mathbf{x}_1^T, \mathbf{x}_2^T, \cdots, \mathbf{x}_{i-1}^T, \mathbf{x}_{i+1}^T, \cdots, \mathbf{x}_N^T]^T$. In this paper, we suppose that $\mathbf{x}_i \in \mathbb{R}^{m_i} = [x_{i1}, x_{i2}, \cdots, x_{im_i}]^T$, where $x_{ij} \in \mathbb{R}$ denotes the action of agent j in coalition i for $i \in \{1, 2, \cdots, N\}$ and $j \in \{1, 2, \cdots, m_i\}$. For coalition $i, i \in \{1, 2, \cdots, N\}$, the agents therein can communicate with each other via a communication graph denoted as $\mathcal{G}_i = (\mathcal{V}_i, \mathcal{E}_i)$[1]. For notational convenience, let a_i^{jk} denote the element on the jth row and kth column of the adjacency matrix of $\mathcal{G}_i$, and L_i represent the Laplacian matrix of $\mathcal{G}_i$. Suppose that the Nash equilibrium of the N-coalition non-cooperative game exists and is finite, the objective of this paper is to design a strategy to seek for the Nash equilibrium of the N-coalition non-cooperative games under the condition that only $f_{ij}(\mathbf{x}_i, \mathbf{x}_{-i})$ [2] is available to agent j in coalition i.

Remark 1 *It's worth noting that if $N = 1, m_1 \geq 2$ the considered problem is reduced to a social cost minimization problem, in which a network of agents collaboratively minimize the sum of their local objective functions. Moreover, if $m_i = 1, \forall i \in \{1, 2, \cdots, N\}$, the considered model is reduced to a non-cooperative game. Hence, the N-coalition non-cooperative games formulated in this paper cover both the social cost minimization problem and the non-cooperative game as special cases. Note that in the subsequent analysis, we focus on the case where $N \geq 2, m_i \geq 2, \forall i \in \{1, 2, \cdots, N\}$.*

The following assumptions will be utilized in the rest of the paper.

Assumption 1 *The communication graphs $\mathcal{G}_i, \forall i \in \{1, 2, \cdots, N\}$ are undirected and connected.*

Assumption 2 *The agents' objective functions $f_{ij}(\mathbf{x}_i, \mathbf{x}_{-i}), \forall i \in \{1, 2, \cdots, N\}, j \in \{1, 2, \cdots, m_i\}$ are $\mathcal{C}^2$ functions.*

3 Nash Equilibrium Seeking for N-coalition Non-cooperative Games

In this section, a Nash equilibrium seeking strategy will be firstly designed for the N-coalition non-cooperative games.

[1] Related definitions on the graphs and games are given in the Section 6.1.

[2] Throughout this paper, $(\mathbf{x}_i, \mathbf{x}_{-i})$ and $(\mathbf{x}_i^*, \mathbf{x}_{-i}^*)$ might be alternatively written as $\mathbf{x}$ and $\mathbf{x}^*$, respectively, for notational convenience.

Then, discussions on the strategy design will be given to provide insights on how the strategy is designed. Lastly, convergence analysis of the Nash equilibrium under the proposed seeking strategy is conducted under certain conditions.

3.1 Nash Equilibrium Seeking Strategy Design

In the following, we design a novel Nash equilibrium seeking strategy for the N-coalition non-cooperative games. To search for the Nash equilibrium of the N-coalition non-cooperative games, the action of agent $j, j \in \{1, 2, \cdots, m_i\}$ in coalition $i, i \in \{1, 2, \cdots, N\}$ can be updated according to

$$\dot{x}_{ij} = -k_{ij} g_{ijj}, \tag{4}$$

where $k_{ij} = \delta \bar{k}_{ij}$, δ is a small positive parameter and $\bar{k}_{ij}$ is a fixed positive parameter. Furthermore, g_{ijk} for $j, k \in \{1, 2, \cdots, m_i\}$ are auxiliary variables governed by

$$\begin{aligned}\dot{g}_{ijk} =& - g_{ijk} - \sum_{l=1}^{m_i} a_i^{jl}(g_{ijk} - g_{ilk}) \\ &- \sum_{l=1}^{m_i} a_i^{jl}(z_{ijk} - z_{ilk}) + \frac{\partial f_{ij}(\mathbf{x})}{\partial x_{ik}} \\ \dot{z}_{ijk} =& \sum_{l=1}^{m_i} a_i^{jl}(g_{ijk} - g_{ilk}),\end{aligned} \tag{5}$$

where z_{ijk} for $i \in \{1, 2, \cdots, N\}, j, k \in \{1, 2, \cdots, m_i\}$ are auxiliary variables.

3.2 Discussions on the Nash Equilibrium Seeking Strategy

In the following, we provide some intuitions on how the seeking strategy is designed. The strategy in (5) is inspired by the average consensus protocol in Lemma 3 (see Section 6.1) and [21, 22]. It can be derived that at the equilibrium of (5), $g_{ijk} = \frac{1}{m_i}\sum_{j=1}^{m_i} \frac{\partial f_{ij}(\mathbf{x})}{\partial x_{ik}} = \frac{1}{m_i}\frac{\partial f_i(\mathbf{x})}{\partial x_{ik}}, \forall j \in \{1, 2, \cdots, m_i\}$ for fixed $\frac{\partial f_{ij}(\mathbf{x})}{\partial x_{ik}}, i \in \{1, 2, \cdots, N\}, j, k \in \{1, 2, \cdots, m_i\}$, by Lemma 3.

Let $g_i = [g_{i11}, g_{i12}, \cdots, g_{i1m_i}, g_{i21}, \cdots, g_{im_im_i}]^T \in \mathbb{R}^{m_i^2}$, $z_i = [z_{i11}, z_{i12}, \cdots, z_{i1m_i}, z_{i21}, \cdots, z_{im_im_i}]^T \in \mathbb{R}^{m_i^2}$, $p_i(\mathbf{x}) = [\frac{\partial f_{i1}(\mathbf{x})}{\partial x_{i1}}, \frac{\partial f_{i1}(\mathbf{x})}{\partial x_{i2}}, \cdots, \frac{\partial f_{i1}(\mathbf{x})}{\partial x_{im_i}}, \frac{\partial f_{i2}(\mathbf{x})}{\partial x_{i1}}, \cdots, \frac{\partial f_{im_i}(\mathbf{x})}{\partial x_{im_i}}]^T \in \mathbb{R}^{m_i^2}$. Then, for $j, k \in \{1, 2, \cdots, m_i\}$, (5) can be written as

$$\begin{aligned}\dot{g}_i =& - (I_{m_i^2 \times m_i^2} + L_i \otimes I_{m_i \times m_i}) g_i \\ &- (L_i \otimes I_{m_i \times m_i}) z_i + p_i(\mathbf{x}) \\ \dot{z}_i =& (L_i \otimes I_{m_i \times m_i}) g_i,\end{aligned} \tag{6}$$

$\forall i \in \{1, 2, \cdots, N\}$. Let $U_i = [U_{i1} \quad U_{i2}]$, where $U_{i1} \in \mathbb{R}^{m_i^2} \times \mathbb{R}^{m_i^2 - m_i}$ and $U_{i2} \in \mathbb{R}^{m_i^2} \times \mathbb{R}^{m_i}$, be an orthogonal matrix such that $U_{i1}^T(L_i \otimes I_{m_i \times m_i})$ is full row rank and $U_{i2}^T(L_i \otimes I_{m_i \times m_i}) = \mathbf{0}_{m_i \times m_i^2}$. Furthermore, let $z_i = U_i[y_i^T \quad \bar{y}_i^T]^T$, where $y_i \in \mathbb{R}^{m_i^2 - m_i}$ and $\bar{y}_i \in \mathbb{R}^{m_i}$ are column vectors [3]. Then,

$$\begin{aligned}\dot{g}_i =& - (I_{m_i^2 \times m_i^2} + L_i \otimes I_{m_i \times m_i}) g_i \\ &- (L_i \otimes I_{m_i \times m_i}) U_{i1} y_i + p_i(\mathbf{x}) \\ \dot{y}_i =& U_{i1}^T (L_i \otimes I_{m_i \times m_i}) g_i,\end{aligned} \tag{7}$$

and $\dot{\bar{y}}_i = \mathbf{0}_{m_i}$, by which $\bar{y}_i(t) = \bar{y}_i(0), i \in \{1, 2, \cdots, N\}$. By the equivalence of (6) and (7), it can be derived that at the equilibrium of (7), $g_{ijk} = \frac{1}{m_i}\frac{\partial f_i(\mathbf{x})}{\partial x_{ik}}, \forall i \in \{1, 2, \cdots, N\}, j, k \in \{1, 2, \cdots, m_i\}$ for fixed $\frac{\partial f_{ij}(\mathbf{x})}{\partial x_{ik}}, i \in \{1, 2, \cdots, N\}, j, k \in \{1, 2, \cdots, m_i\}$.

Define $\tau = \delta t$. Then, in the τ-time scale,

$$\frac{dx_{ij}}{d\tau} = -\bar{k}_{ij} g_{ijj}, \tag{8}$$

and

$$\begin{aligned}\delta \frac{dg_i}{d\tau} =& - (I_{m_i^2 \times m_i^2} + L_i \otimes I_{m_i \times m_i}) g_i \\ &- (L_i \otimes I_{m_i \times m_i}) U_{i1} y_i + p_i(\mathbf{x}) \\ \delta \frac{dy_i}{d\tau} =& U_{i1}^T (L_i \otimes I_{m_i \times m_i}) g_i.\end{aligned} \tag{9}$$

Define $\bar{g}_i = [\bar{g}_{i11}, \bar{g}_{i12}, \cdots, \bar{g}_{i1m_i}, \bar{g}_{i21}, \cdots, \bar{g}_{im_im_i}]^T = g_i - g_i^q(\mathbf{x}), \bar{y}_i = y_i - y_i^q(\mathbf{x})$, where $g_i^q(\mathbf{x}) = [g_{i11}^q, g_{i12}^q, \cdots, g_{i1m_i}^q, g_{i21}^q, \cdots, g_{im_im_i}^q]^T, y_i^q(\mathbf{x})$ are the quasi-steady states of g_i, y_i, respectively, that satisfy

$$H_i \begin{bmatrix} g_i^q(\mathbf{x}) \\ y_i^q(\mathbf{x}) \end{bmatrix} + \begin{bmatrix} p_i(\mathbf{x}) \\ \mathbf{0}_{m_i^2 - m_i} \end{bmatrix} = \mathbf{0}_{2m_i^2 - m_i}, \tag{10}$$

where
$H_i = - \begin{bmatrix} I_{m_i^2 \times m_i^2} + L_i \otimes I_{m_i \times m_i} & (L_i \otimes I_{m_i \times m_i}) U_{i1} \\ -U_{i1}^T (L_i \otimes I_{m_i \times m_i}) & \mathbf{0}_{(m_i^2 - m_i) \times (m_i^2 - m_i)} \end{bmatrix}$, is Hurwitz by Assumption 1 and Lemma 4.

Hence,

$$\begin{aligned}\delta \frac{d\bar{g}_i}{d\tau} =& - (I_{m_i^2 \times m_i^2} + L_i \otimes I_{m_i \times m_i}) \bar{g}_i \\ &- (L_i \otimes I_{m_i \times m_i}) U_{i1} \bar{y}_i - \delta \left(\frac{\partial (g_i^q(\mathbf{x}))^T}{\partial \mathbf{x}} \right)^T \frac{d\mathbf{x}}{d\tau} \\ \delta \frac{d\bar{y}_i}{d\tau} =& U_{i1}^T (L_i \otimes I_{m_i \times m_i}) \bar{g}_i - \delta \left(\frac{\partial (y_i^q(\mathbf{x}))^T}{\partial \mathbf{x}} \right)^T \frac{d\mathbf{x}}{d\tau}.\end{aligned} \tag{11}$$

Letting $\delta = 0$ "freezes" $\bar{g}_i, \bar{y}_i$ to zero which gives g_i, y_i on the quasi-steady states and

$$\frac{dx_{ij}}{d\tau} = - \frac{\bar{k}_{ij}}{m_i} \frac{\partial f_i(\mathbf{x})}{\partial x_{ij}}, \tag{12}$$

$\forall i \in \{1, 2, \cdots, N\}$, $j \in \{1, 2, \cdots, m_i\}$ by which the convergence to the Nash equilibrium can be achieved under certain conditions (see e.g., [3, 4, 7]).

3.3 Convergence Results

In the following, local convergence results are firstly presented under certain conditions, followed by non-local convergence results under stronger conditions. To facilitate the subsequent analysis, the following assumptions are made.

Assumption 3 *There exists at least one, possibly multiple isolated Nash equilibria $\mathbf{x}^*$ on which*

$$\frac{\partial f_i}{\partial \mathbf{x}_i}(\mathbf{x}^*) = \mathbf{0}_{m_i}, \forall i \in \{1, 2, \cdots, N\}, \tag{13}$$

and $\frac{\partial^2 f_i}{\partial \mathbf{x}_i^2}(\mathbf{x}^*), \forall i \in \{1, 2, \cdots, N\}$ are symmetric positive definite.

Assumption 4 *The matrix*

$$B=\begin{bmatrix} \frac{\partial^2 f_1}{\partial \mathbf{x}_1^2}(\mathbf{x}^*) & \frac{\partial^2 f_1}{\partial \mathbf{x}_1\partial \mathbf{x}_2}(\mathbf{x}^*) & \cdots & \frac{\partial^2 f_1}{\partial \mathbf{x}_1\partial \mathbf{x}_N}(\mathbf{x}^*) \\ \frac{\partial^2 f_2}{\partial \mathbf{x}_2\partial \mathbf{x}_1}(\mathbf{x}^*) & \frac{\partial^2 f_2}{\partial \mathbf{x}_2^2}(\mathbf{x}^*) & & \vdots \\ \vdots & & \ddots & \\ \frac{\partial^2 f_N}{\partial \mathbf{x}_N\partial \mathbf{x}_1}(\mathbf{x}^*) & \cdots & & \frac{\partial^2 f_N}{\partial \mathbf{x}_N^2}(\mathbf{x}^*) \end{bmatrix}, \tag{14}$$

is strictly diagonally dominant.

In Assumptions 3-4,

$$\frac{\partial^2 f_i(\mathbf{x})}{\partial \mathbf{x}_i\partial \mathbf{x}_j}=\begin{bmatrix} \frac{\partial^2 f_i(\mathbf{x})}{\partial x_{i1}\partial x_{j1}} & \frac{\partial^2 f_i(\mathbf{x})}{\partial x_{i1}\partial x_{j2}} & \cdots & \frac{\partial^2 f_i(\mathbf{x})}{\partial x_{i1}\partial x_{jm_j}} \\ \frac{\partial^2 f_i(\mathbf{x})}{\partial x_{i2}\partial x_{j1}} & \frac{\partial^2 f_i(\mathbf{x})}{\partial x_{i2}\partial x_{j2}} & \cdots & \frac{\partial^2 f_i(\mathbf{x})}{\partial x_{i2}\partial x_{jm_j}} \\ \vdots & & \ddots & \vdots \\ \frac{\partial^2 f_i(\mathbf{x})}{\partial x_{im_i}\partial x_{j1}} & \frac{\partial^2 f_i(\mathbf{x})}{\partial x_{im_i}\partial x_{j2}} & \cdots & \frac{\partial^2 f_i(\mathbf{x})}{\partial x_{im_i}\partial x_{jm_j}} \end{bmatrix}.$$

Remark 2 *Assumptions 3-4 are adapted from [3]-[4] and [7] to ensure the convergence to the Nash equilibrium by utilizing the gradient play as indicated in Lemma 2. Moreover, it's worth noting that Assumptions 3-4 only characterize local Nash equilibria without requiring the set of the Nash equilibria to be a singleton.*

For notational convenience, define $\mathcal{P}(\mathbf{x})=\frac{\partial F(\mathbf{x})}{\partial \mathbf{x}}=[(\frac{\partial f_1(\mathbf{x})}{\partial \mathbf{x}_1})^T,(\frac{\partial f_2(\mathbf{x})}{\partial \mathbf{x}_2})^T,\cdots,(\frac{\partial f_N(\mathbf{x})}{\partial \mathbf{x}_N})^T]^T$. Furthermore, let $g^q(\mathbf{x}),y^q(\mathbf{x})$ be the concatenated vectors of $g_i^q(\mathbf{x}),y_i^q(\mathbf{x}),i\in\{1,2,\cdots,N\}$, respectively.

The following results are given in supportive of the closed-loop system analysis.

Lemma 1 *Suppose that Assumptions 1-2 are satisfied. Then, for each $\mathbf{x}^*$ that satisfies Assumption 3, $(\mathbf{x}^*,g^q(\mathbf{x}^*),y^q(\mathbf{x}^*))$ is the equilibrium of the system in* (4) *and* (7) *with the equilibrium of $\begin{bmatrix} g_i \\ y_i\end{bmatrix}$, $i\in\{1,2,\cdots,N\}$ being unique. Furthermore, $g_i^q(\mathbf{x})$, $y_i^q(\mathbf{x}),\forall i\in\{1,2,\cdots,N\}$ are linear combinations of the elements in $p_i(\mathbf{x})$.*

Proof: See Section 6.2 for the proof. □

Lemma 2 *Suppose that Assumption 2 holds. The agents update their actions according to*

$$\dot{x}_{ij}=-\frac{\bar{k}_{ij}}{m_i}\frac{\partial f_i(\mathbf{x})}{\partial x_{ij}}, \tag{15}$$

$\forall i\in\{1,2,\cdots,N\},j\in\{1,2,\cdots,m_i\}$. Then, each Nash equilibrium that satisfies Assumptions 3-4 is exponentially stable.

Proof: The proof follows the proof of Theorem 2 in [3]. □

Based on the above results, we are now ready to conduct the convergence analysis.

Theorem 1 *Suppose that Assumptions 1-2 hold. Then, for each $\mathbf{x}^*$ that satisfies Assumptions 3-4, there exists a positive constant δ^* such that for each $\delta\in(0,\delta^*)$, $(\mathbf{x}^*,g^q(\mathbf{x}^*),y^q(\mathbf{x}^*))$ is exponentially stable under* (4) *and* (7).

Proof: See Section 6.3 for the proof. □

In the following, we present a non-local convergence result under a stronger condition.

Assumption 5 *The cost functions $f_i(\mathbf{x}_i,\mathbf{x}_{-i})$ are convex in $\mathbf{x}_i,\forall i\in\{1,2,\cdots,N\}$, for any fixed $\mathbf{x}_{-i}$. Moreover, there exists a positive constant m such that*

$$(\mathbf{x}-\mathbf{x}')^T(\mathcal{P}(\mathbf{x})-\mathcal{P}(\mathbf{x}'))\geq m||\mathbf{x}-\mathbf{x}'||^2, \tag{16}$$

for all $\mathbf{x},\mathbf{x}'\in\mathbb{R}^{\sum_{i=1}^N m_i}$.

Remark 3 *To derive a non-local convergence result, Assumption 5 is enforced to ensure the uniqueness of the Nash equilibrium. Note that this assumption is a commonly adopted assumption in the existing literature that concerns with Nash equilibrium seeking in non-cooperative games (see, e.g., [4] [11][27] and the references therein).*

Define g,y as the concatenated vectors of g_i,y_i for $i\in\{1,2,\cdots,N\}$, respectively. Then, the following result can be derived.

Theorem 2 *Suppose that Assumptions 1, 2 and 5 hold. Then, for each positive constant Δ, there exists a positive constant $\delta^*(\Delta)$ such that for each $\delta\in(0,\delta^*)$, $(\mathbf{x}(t),g(t),y(t))$ generated by* (4) *and* (7) *converges exponentially to $(\mathbf{x}^*,g^q(\mathbf{x}^*),y^q(\mathbf{x}^*))$ for $||[\mathbf{x}(0)^T,g(0)^T,y(0)^T]^T||<\Delta$.*

Proof: See Section 6.4 for the proof. □

Remark 4 *As $(\mathbf{x}_i,g_i,y_i)$ converges to $(\mathbf{x}_i^*,g_i^q(\mathbf{x}^*),y_i^q(\mathbf{x}^*))$ for $i\in\{1,2,\cdots,N\}$ as $t\to\infty$, $(\mathbf{x}_i,g_i,z_i)$ generated by* (4) *and* (5) *converges to $(\mathbf{x}_i^*,g_i^q(\mathbf{x}^*),U_i[(y_i^q(\mathbf{x}^*))^T\quad(\bar{y}_i(0))^T]^T)$ as $t\to\infty$ under the given conditions.*

Remark 5 *Note that if there are coalitions with only one agent, the corresponding agent can update its own action according to the gradient play (see* (12)*). If this is the case and the agents in coalitions that have more than one agent update their actions according to the proposed seeking strategy in* (4) *and* (5)*, then, similar results as Theorems 1-2 can be derived.*

4 A Numerical Example

In this section, three interacting coalitions labeled as 1, 2 and 3, respectively, are considered. Furthermore, there are 5, 6 and 4 agents in coalitions 1-3, respectively (i.e., $m_1=5,m_2=6,m_3=4$). The communication graphs for the coalitions are depicted in Fig. 1.

The coalitions' objective functions are

$$f_1(\mathbf{x})=\sum_{j=1}^5 f_{1j}(\mathbf{x}),f_2(\mathbf{x})=\sum_{j=1}^6 f_{2j}(\mathbf{x}),$$
$$f_3(\mathbf{x})=\sum_{j=1}^4 f_{3j}(\mathbf{x}),$$

for coalitions 1-3, respectively. Furthermore, in coalition 1, the local objective functions of agents 1-5 are respectively

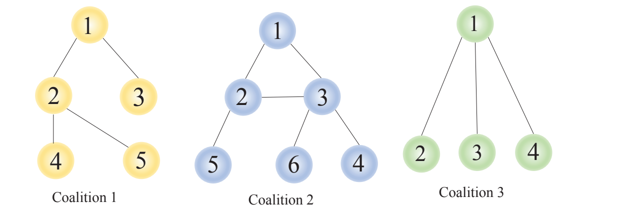

Fig. 1: Communication graphs for the agents in the coalitions.

defined as

$$f_{11} = 10x_{11}^2 + x_{11}\sum_{i=1}^{3}\sum_{j=1}^{m_i} x_{ij}, f_{12} = 9x_{12}^2 + x_{12}\sum_{i=1}^{3}\sum_{j=1}^{m_i} x_{ij},$$

$$f_{13} = 10x_{13}^2 + x_{13}\sum_{i=1}^{3}\sum_{j=1}^{m_i} x_{ij}, f_{14} = 12x_{14}^2 + x_{14}\sum_{i=1}^{3}\sum_{j=1}^{m_i} x_{ij},$$

$$f_{15} = 14x_{15}^2 + x_{15}\sum_{i=1}^{3}\sum_{j=1}^{m_i} x_{ij}.$$

In coalition 2, the local objective functions are defined as

$$f_{21} = 15x_{21}^2 + x_{21}\sum_{i=1}^{3}\sum_{j=1}^{m_i} x_{ij}, f_{22} = 20x_{22}^2 + x_{22}\sum_{i=1}^{3}\sum_{j=1}^{m_i} x_{ij},$$

$$f_{23} = 25x_{23}^2 + x_{23}\sum_{i=1}^{3}\sum_{j=1}^{m_i} x_{ij}, f_{24} = 16x_{24}^2 + x_{24}\sum_{i=1}^{3}\sum_{j=1}^{m_i} x_{ij},$$

$$f_{25} = 27x_{25}^2 + x_{25}\sum_{i=1}^{3}\sum_{j=1}^{m_i} x_{ij}, f_{26} = 18x_{26}^2 + x_{26}\sum_{i=1}^{3}\sum_{j=1}^{m_i} x_{ij},$$

for agents 1-6, respectively. Furthermore, the local objective functions of agents 1-4 in coalition 3 are

$$f_{31} = 5x_{31}^2 + x_{31}x_{11}, f_{32} = 10x_{32}^2 + x_{32}x_{12},$$
$$f_{33} = 9x_{33}^2 + x_{33}x_{21}, f_{34} = 15x_{34}^2 + x_{34}x_{22},$$

respectively.

By direct calculation, it can be derived that the Nash equilibrium of the game is $\mathbf{x}^* = \mathbf{0}_{\sum_{i=1}^{3} m_i}$. The initial conditions are set as $x_{11}(0) = 5, x_{12}(0) = -2, x_{13}(0) = 3, x_{14}(0) = 2, x_{15}(0) = 3, x_{21}(0) = -5, x_{22}(0) = 5, x_{23}(0) = -1, x_{24}(0) = 4, x_{25}(0) = 4, x_{26}(0) = 2, x_{31}(0) = 2, x_{32}(0) = 4, x_{33}(0) = 3, x_{34}(0) = 6$ in the simulation and $g_{ijk}(0) = 0, z_{ijk}(0) = 0$ for $i = \{1, 2, 3\}, j, k = \{1, 2, \cdots, m_i\}$.

The agents' actions generated by the proposed method in (4) and (5) are shown in Fig. 2. From the simulation result, it can be seen that the agents' actions generated by the proposed method in (4) and (5) converge to zero, which is the Nash equilibrium of the non-cooperative game among the 3 coalitions, thus verifying the effectiveness of the proposed method.

5 Conclusion

In this paper, an N-coalition non-cooperative game is considered. The coalitions are modeled as VPs in a non-cooperative game that aim to minimize their own objective

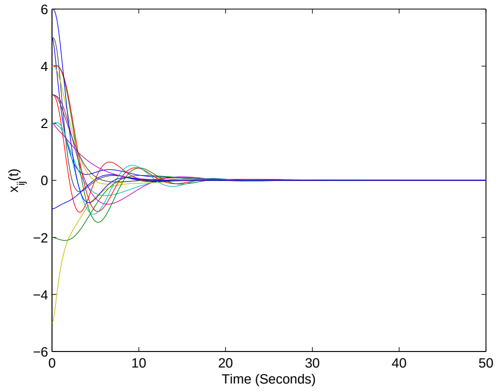

Fig. 2: The agents' actions $x_{ij}(t), i \in \{1, 2, 3\}, j \in \{1, 2, \cdots, m_i\}$ generated by the proposed method in (4) and (5).

functions. Each coalition's objective function is defined as the sum of the agents' local objective functions in the corresponding coalition. The actions of the coalitions are determined by the agents in the coalition. Through communication among the agents in each coalition via an undirected and connected communication topology, a Nash equilibrium seeking strategy is proposed. By Lyapunov stability analysis, convergence results are obtained under certain conditions.

References

[1] B. Gharesifard and J. Cortes, "Distributed convergence to Nash equilibria in two-network zero-sum games," *Automatica,* vol. 49, pp. 1683-1692, 2013.

[2] Y. Lou, Y. Hong, L. Xie, G. Shi and K. Johansson, "Nash equilibrium computation in subnetwork zero-sum games with switching communications," *IEEE Transactions on Automatic Control,* vol. 61, no. 10, pp. 2920-2935, 2016.

[3] M. Ye and G. Hu, "Game design and analysis for price-based demand response: an aggregate game approach," *IEEE Transactions on Cybernetics,* vol. 47, no. 3, pp. 720-730, 2017.

[4] M. Ye and G. Hu, "Distributed Nash equilibrium seeking by a consensus based approach," *IEEE Transactions on Automatic Control,* accepted, DOI: 10.1109/TAC.2017.2688452.

[5] J. Shamma and G. Arslan, "Dynamic fictitious play, dynamic gradient play, and distributed convergence to Nash equilibria," *IEEE Transactions on Automatic Control*, vol. 50, pp. 312-327, 2005.

[6] D. Fudenberg and D. Levine, *The theory of learning in games*, Cambridge, MA: MIT Press, 1998.

[7] P. Frihauf, M. Krstic and T. Basar, "Nash equilibrium seeking in non-coopearive games," *IEEE Transactions on Automatic Control,* vol. 57, pp. 1192-1207, 2012.

[8] M. Stankovi, K. Johansson and D. Stipanovi, "Distributed seeking of Nash equilibria with applications to mobile sensor networks," *IEEE Transactions on Automatic Control,* vol. 57, pp. 904-919, 2012.

[9] S. Liu and M. Krstic, "Stochastic Nash equilibrium seeking for games with general nonlinear payoffs," *SIAM Journal on Control and Optimization,* vol. 49, pp. 1659-1679, 2011.

[10] K. Vamvoudakis, F. Lewis and G. Hudas, "Multi-agent differential graphical games: online adaptive learning solution for synchronization with optimality," *Automatica*, vol. 48, pp. 1598-1611, 2012.

[11] M. Zhu, E. Frazzoli, "Distributed robust adaptive equilibrium computation for generalized convex games," *Automatica*, vol. 63, pp. 82-91, 2016.

[12] M. Ye and G. Hu, "Distributed extremum seeking for constrained networked optimization and its application to energy consumption control in smart grid," *IEEE Transactions on Control Systems Technology,* vol. 24, pp. 2048-2058, 2016.

[13] C. Sun, M. Ye, G. Hu, "Distributed time-varying quadratic optimization for multiple agents under undirected graphs," *IEEE Transactions on Automatic Control,* accepted, DOI: 10.1109/TAC.2017.2673240.

[14] A. Nedic and A. Olshevsky, "Distributed optimization over time-varying directed graphs," *IEEE Transactions on Automatic Control*, vol. 60, pp. 601-615, 2015.

[15] S. Boyd, N. Parikh, E. Chu, B. Peleato and J. Eckstein, "Distributed optimization and statistical learning via the alternating direction method of multiplier," *Foundations and Trends in Machine Learning*, vol. 3, pp. 1-122, 2011.

[16] S. Kia, J. Cortes and S. Martinez, "Distributed convex optimization via continuous-time coordination algorithmns with discrete-time communication," *Automatica,* vol. 55, pp. 254-264, 2015.

[17] N. Li and J. Marden, "Designing games for distributed optimization," *IEEE Journal of Selected Topics in Signal Processing,* vol. 7, pp. 230-242, 2013.

[18] M. Ye and G. Hu, "Distributed optimization for systems with time-varying quadratic objective functions," *IEEE Conference on Decision and Control,* pp. 3285-3290, 2015.

[19] P. Lin, W. Ren and Y. Song, "Distributed multi-agent optimization subject to nonidentical constraints and communication delays," *Automatica,* vol. 65, pp. 120-131, 2016.

[20] G. Droge, H. Kawashima and M. Egerstedt, "Continuous-time proportional-integral distributed optimization for networked systems," *Journal of Decision and Control,* vol. 1, pp. 191-213, 2014.

[21] A. Menon and J. Baras, "Collaborative extremum seeking for welfare optimization," *IEEE Conference on Decision and Control*, pp. 346-351, 2014.

[22] S. Dougherty and M. Guay, "Extremum-seeking control of distributed systems using consensus estimation," *IEEE Conference on Decision and Control*, pp. 3450-3455, 2014.

[23] L. Bai, M. Ye, C. Sun and G. Hu, "Distributed control for optimal economic dispatch via saddle point dynamics and consensus algorithms," *IEEE Conference on Decision and Control,* pp. 6934-6939, 2016.

[24] D. Niyato, A. Vasilakos and Z. Kun, "Resource and revenue sharing with coalition formation of cloud providers: game theoretic approach," *IEEE/ACM International Symposium on Cluster, Cloud and Grid Computing,* pp. 215-224, 2011.

[25] R. Freeman, P. Yang and K. Lynch, "Stability and convergence properties of dynamic average consensus estimators," *IEEE Conference on Decision and Control*, pp. 398-403, 2006.

[26] H. Khailil, Nonlinear System, Prentice Hall, 3rd edtion, 2002.

[27] J. Rosen, "Existence and uniquness of equilibrium points for concave N-person games," *Econometrica,* vol. 33, no. 3, p. 520, Jul. 1965.

6 Appendix

6.1 Preliminaries

A graph is defined as $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ where $\mathcal{E}$ is the edge set satisfying $\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$ with $\mathcal{V} = \{1, 2, \cdots, M\}$ being the set of nodes in the network. It is undirected if for every $(i, j) \in \mathcal{E}$, $(j, i) \in \mathcal{E}$. An undirected graph is connected if there exists a path between any pair of distinct vertices. The elements in the adjacency matrix $\mathcal{A}$ are defined as $a_{ij} = 1$ if node j is connected with node i, else, $a_{ij} = 0$. Furthermore, $a_{ii} = 0$. The neighboring set of agent i is defined as $\mathcal{N}_i = \{j \in \mathcal{V} | (j, i) \in \mathcal{E}\}$. The Laplacian matrix L is defined as $L = \mathcal{D} - \mathcal{A}$, where $\mathcal{D}$ is a diagonal matrix whose ith diagonal element is equal to the out degree of node i, represented by $\sum_{j=1}^{M} a_{ij}$ [12].

A game in a normal form is defined as a triple $\Gamma \triangleq \{\mathcal{N}, X, f\}$ where $\mathcal{N} = \{1, 2, \cdots, N\}$ is the set of N players, $X = X_1 \times \cdots \times X_N$, $X_i \subseteq \mathbb{R}^{m_i}$ is the set of actions for player i, and $f = (f_1, f_2, \cdots, f_N)$ where f_i is the cost function of player i [3].

Nash equilibrium is an action profile on which no player can reduce its cost by unilaterally changing its own action, i.e., an action profile $\mathbf{x}^* = (\mathbf{x}_i^*, \mathbf{x}_{-i}^*) \in X$ is the Nash equilibrium if

$$f_i(\mathbf{x}_i^*, \mathbf{x}_{-i}^*) \leq f_i(\mathbf{x}_i, \mathbf{x}_{-i}^*), \forall i \in \mathcal{N}, \tag{17}$$

for all $\mathbf{x}_i \in X_i$ [3].

Lemma 3 *[25] Let $\mathcal{G}$ be an undirected and connected graph, L be the Laplacian matrix of $\mathcal{G}$. Then, for any constant vector $\mathbf{u} \in \mathbb{R}^M$, the state of the following system*

$$\begin{bmatrix} \dot{\mathbf{x}} \\ \dot{\mathbf{y}} \end{bmatrix} = \begin{bmatrix} -I - L & -L \\ L & \mathbf{0}_{M\times M} \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{y} \end{bmatrix} + \begin{bmatrix} \mathbf{u} \\ \mathbf{0}_M \end{bmatrix}, \tag{18}$$

with arbitrary initial conditions $\mathbf{x}(0), \mathbf{y}(0) \in \mathbb{R}^M$ remains bounded and $\mathbf{x}(t)$ converges exponentially to $\frac{1}{M}\mathbf{1}_M^T \mathbf{u} \mathbf{1}_M$ as $t \to \infty$.

Lemma 4 *[21] Let $W \in \mathbb{R}^N \times \mathbb{R}^N$ and $Q \in \mathbb{R}^N \times \mathbb{R}^M$. If W is Hurwitz and rank$(Q) = M$, then $\begin{bmatrix} W & Q \\ -Q^T & \mathbf{0}_{M\times M} \end{bmatrix}$ is Hurwitz.*

6.2 Proof of Lemma 1

Setting the right-hand sides of (4) and (7) to zero gives

$$\begin{aligned} -(I_{m_i^2\times m_i^2} + & L_i \otimes I_{m_i\times m_i}) g_i \\ & - (L_i \otimes I_{m_i\times m_i}) U_{i1} y_i + p_i(\mathbf{x}) = \mathbf{0}_{m_i^2}, \\ & U_{i1}^T (L_i \otimes I_{m_i\times m_i}) g_i = \mathbf{0}_{m_i^2 - m_i}, \end{aligned} \tag{19}$$

and

$$- k_{ij} g_{ijj} = 0, i \in \{1, 2, \cdots, N\}, j \in \{1, 2, \cdots, m_i\}. \tag{20}$$

By (19), $g_{i1j} = g_{i2j} = \cdots = g_{im_ij} = \frac{1}{m_i}\frac{\partial f_i(\mathbf{x})}{\partial x_{ij}}$ at the equilibrium for $i \in \{1, 2, \cdots, N\}, j \in \{1, 2, \cdots, m_i\}$. Furthermore, by (20), $g_{ijj} = 0$ at the equilibrium. Hence, $\frac{1}{m_i}\frac{\partial f_i}{\partial x_{ij}}(\mathbf{x}) = 0$ for all $i \in \{1, 2, \cdots, N\}$ and $j \in \{1, 2, \cdots, m_i\}$, which indicates that $(\mathbf{x}^*, g^q(\mathbf{x}^*), y^q(\mathbf{x}^*))$ is the equilibrium of the system in (4) and (7). As the matrix H_i is Hurwitz by Assumption 1 and Lemma 4, it is invertible and $\begin{bmatrix} g_i^q(\mathbf{x}) \\ y_i^q(\mathbf{x}) \end{bmatrix} = -H_i^{-1} \begin{bmatrix} p_i(\mathbf{x}) \\ \mathbf{0}_{m_i^2 - m_i} \end{bmatrix}$, which indicates that $g_i^q(\mathbf{x}), y_i^q(\mathbf{x})$ are linear combinations of the elements in $p_i(\mathbf{x})$. In particular, $g_{ijj}^q = \frac{1}{m_i}\frac{\partial f_i(\mathbf{x})}{\partial x_{ij}}$, where g_{ijj}^q denotes the quasi-steady state of g_{ijj}. Hence, for each $\mathbf{x} = \mathbf{x}^*$,

$$\left[\begin{array}{c} g_i \\ y_i \end{array}\right] = \left[\begin{array}{c} g_i^q(\mathbf{x}^*) \\ y_i^q(\mathbf{x}^*) \end{array}\right] = -H_i^{-1}\left[\begin{array}{c} p_i(\mathbf{x}^*) \\ \mathbf{0}_{m_i^2-m_i} \end{array}\right]$$

at the equilibrium of (4) and (7). Therefore, for each $\mathbf{x} = \mathbf{x}^*$, the equilibrium of $\left[\begin{array}{c} g_i \\ y_i \end{array}\right]$ is unique and is $-H_i^{-1}\left[\begin{array}{c} p_i(\mathbf{x}^*) \\ \mathbf{0}_{m_i^2-m_i} \end{array}\right]$.

6.3 Proof of Theorem 1

Letting $\tau = \delta t$ gives

$$\delta\left[\begin{array}{c} \frac{dg_1}{d\tau} \\ \frac{dy_1}{d\tau} \\ \vdots \\ \frac{dg_N}{d\tau} \\ \frac{dy_N}{d\tau} \end{array}\right] = H\left[\begin{array}{c} g_1 \\ y_1 \\ \vdots \\ g_N \\ y_N \end{array}\right] + \left[\begin{array}{c} p_1(\mathbf{x}) \\ \mathbf{0}_{m_1^2-m_1} \\ \vdots \\ p_N(\mathbf{x}) \\ \mathbf{0}_{m_N^2-m_N} \end{array}\right]$$
$$\frac{d\mathbf{x}}{d\tau} = -\mathrm{diag}\{\bar{k}_{ij}\}[g_{ijj}]_{vec},$$
$$i \in \{1,2,\cdots,N\}, j \in \{1,2,\cdots,m_i\}, \tag{21}$$

where $H = \mathrm{diag}\{H_i\}$.

Then,

$$\left[\begin{array}{c} \frac{d\bar{g}_1}{d\tau} \\ \frac{d\bar{y}_1}{d\tau} \\ \vdots \\ \frac{d\bar{g}_N}{d\tau} \\ \frac{d\bar{y}_N}{d\tau} \end{array}\right] = \left[\begin{array}{c} \frac{dg_1}{d\tau} \\ \frac{dy_1}{d\tau} \\ \vdots \\ \frac{dg_N}{d\tau} \\ \frac{dy_N}{d\tau} \end{array}\right] - \left[\begin{array}{c} \left(\frac{\partial\left(g_1^q(\mathbf{x})\right)^T}{\partial\mathbf{x}}\right)^T \\ \left(\frac{\partial\left(y_1^q(\mathbf{x})\right)^T}{\partial\mathbf{x}}\right)^T \\ \vdots \\ \left(\frac{\partial\left(g_N^q(\mathbf{x})\right)^T}{\partial\mathbf{x}}\right)^T \\ \left(\frac{\partial\left(y_N^q(\mathbf{x})\right)^T}{\partial\mathbf{x}}\right)^T \end{array}\right]\frac{d\mathbf{x}}{d\tau}$$

$$= \frac{H}{\delta}\left[\begin{array}{c} \bar{g}_1 \\ \bar{y}_1 \\ \vdots \\ \bar{g}_N \\ \bar{y}_N \end{array}\right] - \left[\begin{array}{c} \left(\frac{\partial\left(g_1^q(\mathbf{x})\right)^T}{\partial\mathbf{x}}\right)^T \\ \left(\frac{\partial\left(y_1^q(\mathbf{x})\right)^T}{\partial\mathbf{x}}\right)^T \\ \vdots \\ \left(\frac{\partial\left(g_N^q(\mathbf{x})\right)^T}{\partial\mathbf{x}}\right)^T \\ \left(\frac{\partial\left(y_N^q(\mathbf{x})\right)^T}{\partial\mathbf{x}}\right)^T \end{array}\right]\frac{d\mathbf{x}}{d\tau}, \tag{22}$$

and

$$\frac{d\mathbf{x}}{d\tau} = -\mathrm{diag}\{\bar{k}_{ij}\}\left[\begin{array}{c} \bar{g}_{ijj} + \frac{1}{m_i}\frac{\partial f_i(\mathbf{x})}{\partial x_{ij}} \end{array}\right]_{vec},$$
$$i \in \{1,2,\cdots,N\}, j \in \{1,2,\cdots,m_i\}, \tag{23}$$

where we have utilized the conclusion that the quasi-steady state of g_{ijj} is $\frac{1}{m_i}\frac{\partial f_i(\mathbf{x})}{\partial x_{ij}}, i \in \{1,2,\cdots,N\}, j \in \{1,2,\cdots,m_i\}$, i.e., $g_{ijj}^q = \frac{1}{m_i}\frac{\partial f_i(\mathbf{x})}{\partial x_{ij}}$.

According to Lemma 2, the Nash equilibrium is exponentially stable under the gradient play in (15). Hence, there exists a function $W : D_0 \to \mathbb{R}$ where $D_0 = \{\mathbf{x} \in \mathbb{R}^{\sum_{i=1}^N m_i} | ||\mathbf{x} - \mathbf{x}^*|| \leq r_0\}$ for some positive constant r_0 such that

$$c_1||\mathbf{x}-\mathbf{x}^*||^2 \leq W(\mathbf{x}) \leq c_2||\mathbf{x}-\mathbf{x}^*||^2,$$
$$\frac{\partial W(\mathbf{x})}{\partial\mathbf{x}}^T\left(-\mathrm{diag}\left\{\frac{\bar{k}_{ij}}{m_i}\right\}\frac{\partial F(\mathbf{x})}{\partial\mathbf{x}}\right) \leq -c_3||\mathbf{x}-\mathbf{x}^*||^2,$$
$$\left|\left|\frac{\partial W(\mathbf{x})}{\partial\mathbf{x}}\right|\right| \leq c_4||\mathbf{x}-\mathbf{x}^*||, \tag{24}$$

for some positive constants c_1, c_2, c_3, c_4 by Theorem 4.14 of [26]. The following analysis is conducted for the domain in which the inequalities in (24) are satisfied.

Define the Lyapunov candidate function as

$$V = cW(\mathbf{x}) + (1-c)\eta^T P\eta, \tag{25}$$

where $c \in (0,1)$ is a constant, $\eta = [\bar{g}_1^T, \bar{y}_1^T, \cdots, \bar{g}_N^T, \bar{y}_N^T]^T$, and P is a symmetric positive definite matrix that satisfies $PH + H^TP = -Q$, for some symmetric positive definite matrix Q as H is Hurwitz by utilizing Assumption 1 and the result in Lemma 4.

Define $\chi = [(\mathbf{x}-\mathbf{x}^*)^T, \eta^T]^T$. Then, there exists a domain $D_1 = \{\chi | ||\chi|| \leq r_1\}$, for some positive constant r_1 such that the time-derivative of the Lyapunov candidate function along the given trajectory in (22)-(23) satisfies

$$\begin{aligned}\frac{dV}{d\tau} =& -c\frac{\partial W(\mathbf{x})}{\partial\mathbf{x}}^T\mathrm{diag}\{\bar{k}_{ij}\}\left[\begin{array}{c} \bar{g}_{ijj} + \frac{1}{m_i}\frac{\partial f_i(\mathbf{x})}{\partial x_{ij}} \end{array}\right]_{vec} \\ &+ (1-c)\left(\frac{d\eta}{d\tau}^T P\eta + \eta^T P\frac{d\eta}{d\tau}\right) \\ =& -c\frac{\partial W(\mathbf{x})}{\partial\mathbf{x}}^T\mathrm{diag}\left\{\frac{\bar{k}_{ij}}{m_i}\right\}\frac{\partial F(\mathbf{x})}{\partial\mathbf{x}} \\ &- c\frac{\partial W(\mathbf{x})}{\partial\mathbf{x}}^T\mathrm{diag}\{\bar{k}_{ij}\}[\bar{g}_{ijj}]_{vec} \\ &+ (1-c)\left[\left(\frac{1}{\delta}\eta^T H^T - G\left(\mathbf{x}, \frac{d\mathbf{x}}{d\tau}\right)^T\right)P\eta\right. \\ &\left.+\eta^T P\left(\frac{1}{\delta}H\eta - G\left(\mathbf{x}, \frac{d\mathbf{x}}{d\tau}\right)\right)\right] \\ =& -c\frac{\partial W(\mathbf{x})}{\partial\mathbf{x}}^T\mathrm{diag}\left\{\frac{\bar{k}_{ij}}{m_i}\right\}\frac{\partial F(\mathbf{x})}{\partial\mathbf{x}} \\ &+ \frac{1-c}{\delta}\eta^T(H^TP + PH)\eta \\ &- c\frac{\partial W(\mathbf{x})}{\partial\mathbf{x}}^T\mathrm{diag}\{\bar{k}_{ij}\}[\bar{g}_{ijj}]_{vec} \\ &- 2(1-c)\eta^T PG\left(\mathbf{x}, \frac{d\mathbf{x}}{d\tau}\right),\end{aligned} \tag{26}$$

where $G(\mathbf{x}, \frac{d\mathbf{x}}{d\tau}) = \left[\begin{array}{c} \left(\frac{\partial\left(g_1^q(\mathbf{x})\right)^T}{\partial\mathbf{x}}\right)^T \\ \left(\frac{\partial\left(y_1^q(\mathbf{x})\right)^T}{\partial\mathbf{x}}\right)^T \\ \vdots \\ \left(\frac{\partial\left(g_N^q(\mathbf{x})\right)^T}{\partial\mathbf{x}}\right)^T \\ \left(\frac{\partial\left(y_N^q(\mathbf{x})\right)^T}{\partial\mathbf{x}}\right)^T \end{array}\right]\frac{d\mathbf{x}}{d\tau}$.

Hence, there exists a positive constant β_1 such that for $\chi \in D_1$,

$$\frac{dV}{d\tau} \leq - cc_3||\mathbf{x}-\mathbf{x}^*||^2 - \frac{1-c}{\delta}\lambda_{min}(Q)||\eta||^2 + c\beta_1||\mathbf{x}-\mathbf{x}^*||||\eta|| + 2(1-c)\eta^T P \begin{bmatrix} (\frac{\partial(g_1^q(\mathbf{x}))^T}{\partial \mathbf{x}})^T \\ (\frac{\partial(y_1^q(\mathbf{x}))^T}{\partial \mathbf{x}})^T \\ \vdots \\ (\frac{\partial(g_N^q(\mathbf{x}))^T}{\partial \mathbf{x}})^T \\ (\frac{\partial(y_N^q(\mathbf{x}))^T}{\partial \mathbf{x}})^T \end{bmatrix} \text{diag}\{\bar{k}_{ij}\}[\bar{g}_{ijj}]_{vec} + 2(1-c)\eta^T P \begin{bmatrix} (\frac{\partial(g_1^q(\mathbf{x}))^T}{\partial \mathbf{x}})^T \\ (\frac{\partial(y_1^q(\mathbf{x}))^T}{\partial \mathbf{x}})^T \\ \vdots \\ (\frac{\partial(g_N^q(\mathbf{x}))^T}{\partial \mathbf{x}})^T \\ (\frac{\partial(y_N^q(\mathbf{x}))^T}{\partial \mathbf{x}})^T \end{bmatrix} \text{diag}\left\{\frac{\bar{k}_{ij}}{m_i}\right\}\left[\frac{\partial f_i(\mathbf{x})}{\partial x_{ij}}\right]_{vec}. \quad (27)$$

Since the second partial derivatives of $f_{ij}(\mathbf{x})$ for $i \in \{1,2,\cdots,N\}, j \in \{1,2,\cdots,m_i\}$ are bounded for $\chi \in D_1$, $\left|\left|\frac{\partial(g_i^q(\mathbf{x}))^T}{\partial \mathbf{x}}\right|\right|$ and $\left|\left|\frac{\partial(y_i^q(\mathbf{x}))^T}{\partial \mathbf{x}}\right|\right|$, for $i \in \{1,2,\cdots,N\}$ are bounded for $\chi \in D_1$. Moreover, $\frac{\partial f_i(\mathbf{x})}{\partial x_{ij}}$ for $i \in \{1,2,\cdots,N\}$ and $j \in \{1,2,\cdots,m_i\}$ are Lipschitz by Lemma 3.2 in [26]. Hence, it can be derived that there exist positive constants β_2, β_3 such that for $\chi \in D_1$,

$$\begin{aligned} \frac{dV}{d\tau} \leq & - cc_3||\mathbf{x}-\mathbf{x}^*||^2 - \frac{1-c}{\delta}\lambda_{min}(Q)||\eta||^2 \\ & + c\beta_1||\mathbf{x}-\mathbf{x}^*||||\eta|| + 2(1-c)\beta_2||\eta||^2 \\ & + 2(1-c)\beta_3||\eta||||\mathbf{x}-\mathbf{x}^*||. \end{aligned} \quad (28)$$

Define

$$\bar{A} = \begin{bmatrix} cc_3 & -\frac{c\beta_1}{2} - (1-c)\beta_3 \\ -\frac{c\beta_1}{2} - (1-c)\beta_3 & \frac{(1-c)\lambda_{min}(Q)}{\delta} - 2(1-c)\beta_2 \end{bmatrix}.$$

Then, $\bar{A}$ is symmetric positive definite if $\delta < \frac{4(1-c)cc_3\lambda_{min}(Q)}{(c\beta_1+2(1-c)\beta_3)^2+8cc_3(1-c)\beta_2}$. Let

$$\delta^* = \frac{4(1-c)cc_3\lambda_{min}(Q)}{(c\beta_1+2(1-c)\beta_3)^2+8cc_3(1-c)\beta_2}. \quad (29)$$

Then, for each $\delta \in (0, \delta^*)$,

$$\frac{dV}{d\tau} \leq -\lambda_{min}(\bar{A})||\chi||^2, \quad (30)$$

where $\lambda_{min}(\bar{A})$ is positive.

Since,

$$V = cW(\mathbf{x}) + (1-c)\eta^T P\eta, \quad (31)$$

it can be derived that there exist positive constants μ_1 and $\mu_2 (\mu_2 \geq \mu_1)$ such that $\mu_1||\chi(\tau)||^2 \leq V(\chi) \leq \mu_2||\chi(\tau)||^2$. Hence, by utilizing the Comparison Lemma [26], it can be derived that

$$\begin{aligned} \mu_1||\chi(\tau)||^2 \leq V(\tau) \leq & e^{-\frac{\lambda_{min}(\bar{A})}{\mu_2}\tau}V(0) \\ \leq & e^{-\frac{\lambda_{min}(\bar{A})}{\mu_2}\tau}\mu_2||\chi(0)||^2. \end{aligned} \quad (32)$$

Therefore,

$$||\chi(\tau)||^2 \leq \frac{\mu_2}{\mu_1}e^{-\frac{\lambda_{min}(\bar{A})}{\mu_2}\tau}||\chi(0)||^2, \quad (33)$$

i.e.,

$$||\chi(\tau)|| \leq \sqrt{\frac{\mu_2}{\mu_1}}e^{-\frac{\lambda_{min}(\bar{A})}{2\mu_2}\tau}||\chi(0)||. \quad (34)$$

Since $\chi(\tau) = [(\mathbf{x}-\mathbf{x}^*)^T, \eta^T]^T$, it can be derived that for $\chi \in D_1$,

$$\left|\left|\begin{bmatrix} \mathbf{x}-\mathbf{x}^* \\ g_1 - g_1^q(\mathbf{x}^*) \\ y_1 - y_1^q(\mathbf{x}^*) \\ \vdots \\ g_N - g_N^q(\mathbf{x}^*) \\ y_N - y_N^q(\mathbf{x}^*) \end{bmatrix}\right|\right| \leq \left|\left|\begin{bmatrix} \mathbf{x}-\mathbf{x}^* \\ \eta \end{bmatrix}\right|\right| + \left|\left|\begin{bmatrix} \mathbf{0}_{\sum_{k=1}^N m_k} \\ g_1^q(\mathbf{x}) - g_1^q(\mathbf{x}^*) \\ y_1^q(\mathbf{x}) - y_1^q(\mathbf{x}^*) \\ \vdots \\ g_N^q(\mathbf{x}) - g_N^q(\mathbf{x}^*) \\ y_N^q(\mathbf{x}) - y_N^q(\mathbf{x}^*) \end{bmatrix}\right|\right| \leq K_1\left|\left|\begin{bmatrix} \mathbf{x}-\mathbf{x}^* \\ \eta \end{bmatrix}\right|\right| \leq$$

$$K_1\sqrt{\frac{\mu_2}{\mu_1}}e^{-\frac{\lambda_{min}(\bar{A})}{2\mu_2}\tau}\left|\left|\begin{bmatrix} \mathbf{x}(0)-\mathbf{x}^* \\ g_1(0) - g_1^q(\mathbf{x}^*) \\ y_1(0) - y_1^q(\mathbf{x}^*) \\ \vdots \\ g_N(0) - g_N^q(\mathbf{x}^*) \\ y_N(0) - y_N^q(\mathbf{x}^*) \end{bmatrix}\right|\right| + K_1\sqrt{\frac{\mu_2}{\mu_1}}e^{-\frac{\lambda_{min}(\bar{A})}{2\mu_2}\tau}\left|\left|\begin{bmatrix} \mathbf{0}_{\sum_{k=1}^N m_k} \\ g_1^q(\mathbf{x}^*) - g_1^q(\mathbf{x}(0)) \\ y_1^q(\mathbf{x}^*) - y_1^q(\mathbf{x}(0)) \\ \vdots \\ g_N^q(\mathbf{x}^*) - g_N^q(\mathbf{x}(0)) \\ y_N^q(\mathbf{x}^*) - y_N^q(\mathbf{x}(0)) \end{bmatrix}\right|\right| \leq$$

$$K_2\sqrt{\frac{\mu_2}{\mu_1}}e^{-\frac{\lambda_{min}(\bar{A})}{2\mu_2}\tau}\left|\left|\begin{bmatrix} \mathbf{x}(0)-\mathbf{x}^* \\ g_1(0) - g_1^q(\mathbf{x}^*) \\ y_1(0) - y_1^q(\mathbf{x}^*) \\ \vdots \\ g_N(0) - g_N^q(\mathbf{x}^*) \\ y_N(0) - y_N^q(\mathbf{x}^*) \end{bmatrix}\right|\right|,$$

for some positive constants K_1 and K_2. Hence, in the τ-time scale, $(\mathbf{x}^*, g^q(\mathbf{x}^*), y^q(\mathbf{x}^*))$ is exponentially stable. Converting it back to t-time scale, the conclusion is derived.

6.4 Proof of Theorem 2

Following the proof of Theorem 1, it can be derived that the system in (4) and (7) can be rewritten as (22)-(23) in the τ-time scale. Define the Lyapunov candidate function as

$$\begin{aligned} V = & \frac{c}{2}(\mathbf{x}-\mathbf{x}^*)^T\left(\text{diag}\left\{\frac{\bar{k}_{ij}}{m_i}\right\}\right)^{-1}(\mathbf{x}-\mathbf{x}^*) \\ & + (1-c)\eta^T P\eta, \end{aligned} \quad (35)$$

where $c \in (0,1)$ is a constant. Then, the rest of proof follows the steps in the proof of Theorem 1 and the details are omitted due to space limitation.