Output Group Consensus for Heterogeneous Linear Multi-Agent Systems Communicating over Switching Topology

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Abstract: In this paper, we aim to investigate the output group consensus problem for a switching network of heterogeneous linear systems such that outputs of the agents synchronize with each other in every cluster. From the internal model principle perspective, a necessary condition is first derived in terms of the system dynamics. With this necessary condition, a dynamic controller is then designed to solve the output group consensus problem in two steps, namely consensus of reference generators and output tracking for each agent. It is obtained that if every possible underlying topology of each cluster contains a directed spanning tree, then group consensus for the reference generators can be realized with sufficiently strong intra-cluster coupling strength. An appropriate controller is then designed to force the output of each agent to track the output of the reference generators. A simulation example is given at last to validate our theoretical findings.

Key Words: Output group consensus, switching topology, heterogeneous systems, internal model principle.

1 Introduction

Recently, inspired by the observation that in the engineering world, a multi-agent system may consist of agents governing by different system dynamics, especially with different dimensions, the output consensus problem for heterogeneous multi-agent systems has drawn attention of many researchers [4, 13]. As elaborated in [4, 13], in achieving output synchronization, there necessarily exists a reference generator which generates the nontrivial synchronized trajectory. In view of this, to realize the output consensus among multiple heterogeneous agents, a dynamic controller is broadly adopted. Intuitively, in leader-following consensus, the dynamic controller drives the output of the follower systems to that of the exo-system [11]. While in leaderless consensus, the dynamic controller generates the reference trajectory (function as *exo-system*) for each system to track [4, 13].

Real-world systems are usually composed of several interacting clusters of coupled agents [12]. Then, a more general consensus problem, termed group/cluster consensus problem which considers multiple clusters under general coupling topology involving possible negative couplings, has also received growing attentions recently [3, 7, 10, 14, 15]. It frequently arises when agents within the same cluster are cooperative while the agents from different clusters are repulsive and/or cooperative [7].

In the group consensus problem, the invariance of consensus manifold (a subspace in which the states of agents within the same cluster are identical) does not hold with just diffusive couplings. For this invariance problem of linearly coupled nonlinear systems, [1, 7] provide a necessary and sufficient *common inter-cluster coupling* condition, which refers to the scenario that the couplings each agent in the same cluster receives from any other cluster sum up equally, to guarantee the invariance of the group consensus manifold. Under this condition, intensive research concerning group consensus problem for homogeneous multi-agent systems has been conducted [3, 7], however for cooperative networks. Specifically, assuming that the couplings each agent receives from any other cluster sum up to zero, which naturally involve repulsive couplings and is termed *in-degree balanced condition*, group consensus for homogeneous multi-agent systems is investigated in [8, 9, 14]. To date, few research works have concentrated on group/cluster consensus for heterogeneous multi-agent systems, except for [6] where output cluster consensus for heterogeneous linear systems is studied under in-degree balanced condition based on the internal model principle.

Motivated by the above discussion, we aim to further address the output group consensus problem for heterogeneous linear multi-agent systems. The contributions of this paper are as follows. 1) The general common inter-cluster coupling condition, which allows for repulsive couplings between agents from different clusters, is investigated. 2) From the viewpoint of internal model principle, a necessary condition concerning system dynamics is derived. A distinct feature of such condition compared with that in [13] lies in that the influence brought by common inter-cluster couplings is explicitly involved. 3) We consider a general framework such that the linear multi-agent systems communicate over switching network topology in the presence of repulsive couplings. 4) To address the consensus problem over switching topology, a multiple Lyapunov function approach is applied, and structural conditions with respect to network structure and intra-cluster coupling strength are provided. An interesting and consistent conclusion with those elaborated in [8, 9] is finally made that if each cluster contains a directed spanning tree at each instant, and moreover the intra-cluster coupling strength is strong enough, then the reference generators are synchronized within each cluster.

The remainder of the paper is arranged into five sections. In Section 2, we introduce relevant graph notions and formu-

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late the problem. The main results concerning output group consensus are presented in Sections 3 and 4, followed by an illustrative example in Section 5. The paper is finally wrapped up with concluding remarks in Section 6.

Notations: Let I_n be the identity matrix and $0_{n\times n}$ the zero matrix in $\mathbb{R}^{n\times n}$. diag$\{a_1, \ldots, a_q\}$ denotes the diagonal matrix with a_i being the i-th diagonal element. The *spectrum* of a square matrix A, denoted by $\sigma(A)$, is the set of all eigenvalues of A. The imaginary axis is denoted by $j\mathbb{R}$.

2 Preliminary

2.1 Graph Notions

The interaction topology is represented by a directed graph $\mathsf{G} = (\mathsf{V}, \mathsf{E}, \mathsf{A})$ of order N with a finite nonempty set of nodes $\mathsf{V} = \{1, 2, \ldots, N\}$, a set of edges $\mathsf{E} \subset \mathsf{V} \times \mathsf{V}$, and a weighted *adjacency matrix* $\mathsf{A} = \left[a_{ij}\right] \in \mathbb{R}^{N\times N}$, where a_{ij} is the weight, also called coupling strength in this work, of the directed edge (j, i) satisfying $a_{ij} \neq 0$ if (j, i) is an edge of G and $a_{ij} = 0$ otherwise. Moreover, assume $a_{ii} = 0$ for all $i \in \mathsf{V}$ to avoid self-loops. Note that a_{ij} for inter-cluster couplings can be either positive or negative corresponding respectively to the cooperative and competitive interactions. The Laplacian matrix L of $\mathsf{G} = (\mathsf{V}, \mathsf{E}, \mathsf{A})$ is defined as $\mathsf{L} = \text{diag}\{\Delta_1, \ldots, \Delta_N\} - \mathsf{A}$, where $\Delta_i = \sum_{j=1}^{N} a_{ij}$, $i = 1, \ldots, N$, [2]. A *directed path* is a sequence of edges in a directed graph of the form $(i_1, i_2), (i_2, i_3), \ldots, (i_{q-1}, i_q)$. A digraph *has a directed spanning tree* if there exists at least one node, called the root, having a directed path to every other node.

The interaction graph G is switching among finite given digraphs. Given an infinite sequence of consecutive non-overlapping time interval $[t_k, t_{k+1})$, $k \in \mathbb{N}$ with $t_0 = 0$, $t_{k+1} - t_k > \tau_0$, where τ_0 is called the dwell time and across which the interaction topology is fixed. The time sequence $t_1, t_2, \ldots$, is called the switching sequence, at which the interaction topology changes. Let $\mathsf{G}^{\sigma(t)}$ be the interaction graph at time t, with $\sigma(t) : [0, +\infty) \to \{1, 2, \ldots, p\}$. Hence, it is assumed that $\mathsf{G}(t)$ switches among p different interaction graphs $\left\{\mathsf{G}^1, \ldots, \mathsf{G}^p\right\}$.

2.2 System Model and Problem of Interest

Consider a group of N agents governed by heterogeneous linear system dynamics:

$$\begin{aligned} \dot{x}_i(t) &= A_i x_i(t) + B_i u_i(t), \\ y_i(t) &= C_i x_i(t),\ i = 1, \ldots, N, \end{aligned} \tag{1}$$

where $x_i = [x_i^1, \ldots, x_i^{n_i}]^{\mathrm{T}} \in \mathbb{R}^{n_i}$, $u_i \in \mathbb{R}^{m_i}$, and $y_i \in \mathbb{R}^p$ are respectively the state, input, and output of agent i, $A_i \in \mathbb{R}^{n_i\times n_i}$, $B_i \in \mathbb{R}^{n_i\times m_i}$, $C_i \in \mathbb{R}^{p\times n_i}$.

In what follows, at no loss of generality, assume that the N nodes, each representing a heterogeneous linear system, are divided into q ($q > 1$) disjoint clusters, namely $\mathsf{V}_1, \ldots, \mathsf{V}_q$, such that $\cup_{\ell=1}^{q} \mathsf{V}_\ell = \mathsf{V}$, and the number of nodes in a cluster, say V_ℓ, is N_ℓ, $1 \le \ell \le q$. These N nodes can be labeled in such a way that they are indexed as $\sum_{j=0}^{\ell-1} N_j + 1, \ldots, \sum_{j=0}^{\ell} N_j$, where $N_0 = 0$, i.e., $\mathsf{V}_\ell = \{\sum_{j=0}^{\ell-1} N_j + 1, \ldots, \sum_{j=0}^{\ell} N_j\}$. Let $\bar{i}$ denote the subscript of the cluster, which node i belongs to, i.e. $i \in \mathsf{V}_{\bar{i}}$, and G_ℓ be the underlying topology of cluster V_ℓ, $\ell = 1, \ldots, q$, i.e., $\mathsf{V}(\mathsf{G}_\ell) = \mathsf{V}_\ell$. For later use, define $\kappa_0 = 0$, $\kappa_\ell = \sum_{p=1}^{\ell} N_p$.

Output Group Consensus Problem: Design appropriate control laws u_i for $i = 1, \ldots, N$, as follows

$$\left\{ \begin{aligned} \dot{\zeta}_i &= F_i \zeta_i + O_i y_i + G_{1i}\left(e_\zeta^i + e_y^i\right), && \text{(2a)} \\ u_i &= K_{1i} x_i(t) + K_{2i} \zeta_i(t) + G_{2i}\left(e_\zeta^i + e_y^i\right), && \text{(2b)} \end{aligned} \right.$$

where $\zeta_i \in \mathbb{R}^m$, $e_\zeta^i = \sum_{j=1}^{N} a_{ij}^{\sigma(t)}(\zeta_j - \zeta_i)$, $e_y^i = \sum_{j=1}^{N} a_{ij}^{\sigma(t)}(y_j - y_i)$, such that for any initial states of the heterogeneous system (1), there holds $\lim_{t\to\infty} \|y_i(t) - y_j(t)\| = 0$, $\forall \bar{i} = \bar{j}$, $i, j = 1, \ldots, N$.

In this paper, our aim is to introduce appropriate controllers and derive sufficient conditions to solve the above output group consensus problem. To this purpose, a prerequisite requirement is that the group consensus manifold

$$\mathcal{S} = \left\{[x_1^{\mathrm{T}}, \ldots, x_N^{\mathrm{T}}]^{\mathrm{T}} : C_i x_i = C_j x_j, \forall \bar{i} = \bar{j}\right\}$$

should be invariant for heterogeneous system (1) coupled through (2a) and (2b). As elaborated in [1] and [7], a necessary and sufficient condition for $\mathcal{S}$ to be invariant through linearly coupled ordinary differential equation is that the *common inter-cluster coupling condition* is satisfied, i.e.,

$$\sum_{j\in \mathsf{V}_\ell} a_{ij}^{\sigma(t)} = d_{k\ell}^{\sigma(t)},\ \ \forall i \in \mathsf{V}_k,\ \ k, \ell = 1, \ldots, q,\ \ k \neq \ell, \tag{3}$$

where $d_{k\ell}^{\sigma(t)}$ is a constant irrelevant to the choice of i and j in clusters V_k and V_ℓ, respectively. This means for agents within the same cluster, the sums of the weight of the incoming couplings from any of the other cluster are the same. Throughout this paper, this common inter-cluster condition is adopted.

Under the common inter-cluster coupling condition, the Laplacian matrix $\mathsf{L}^{\sigma(t)} = [l_{ij}^{\sigma(t)}]_{N\times N}$ of $\mathsf{G}^{\sigma(t)}$ is written as follows

$$\mathsf{L}^{\sigma(t)} = \begin{bmatrix} \mathsf{L}_{11}^{\sigma(t)} + D_1^{\sigma(t)} & \cdots & \mathsf{L}_{1q}^{\sigma(t)} \\ \vdots & \ddots & \vdots \\ \mathsf{L}_{q1}^{\sigma(t)} & \cdots & \mathsf{L}_{qq}^{\sigma(t)} + D_q^{\sigma(t)} \end{bmatrix},$$

where $\mathsf{L}_{\ell k}^{\sigma(t)}$ ($\ell, k = 1, \ldots, q$, $\ell \neq k$) specifies the inter-cluster couplings from cluster V_k to cluster V_ℓ, $D_\ell^{\sigma(t)} = d_\ell^{\sigma(t)} I_{N_\ell} = \sum_{j=1, j\neq\ell}^{q} d_{\ell j}^{\sigma(t)} I_{N_\ell}$ for $\ell = 1, \ldots, q$. Note that $\mathsf{L}_{\ell\ell}^{\sigma(t)}$ represents the Laplacian matrix of $\mathsf{G}_\ell^{\sigma(t)}$.

3 Internal Model Principle

In this section, we first present a preliminary result, which is the fundamental ingredient of our main result. This preliminary result extends that presented in [13] for complete consensus to the case where group consensus is taken into consideration. Inserting (2a) and (2b) into (1) yields

$$\left\{ \begin{aligned} \dot{x}_i(t) &= A_i x_i + B_i K_{1i} x_i + B_i K_{2i} \zeta_i + B_i G_{2i}\left(e_\zeta^i + e_y^i\right), && \text{(4a)} \\ \dot{\zeta}_i(t) &= F_i \zeta_i + O_i y_i + G_{1i}\left(e_\zeta^i + e_y^i\right), && \text{(4b)} \end{aligned} \right.$$

which can be equivalently transformed into the following compact form

$$\left\{ \begin{aligned} \dot{\hat{x}} &= \hat{A}\hat{x} + \hat{B} e_{\hat{x}}, && \text{(5a)} \\ \hat{y} &= \hat{C}\hat{x}, && \text{(5b)} \end{aligned} \right.$$

where $\hat{x} = [\hat{x}_1^{\mathrm{T}}, \ldots, \hat{x}_N^{\mathrm{T}}]^{\mathrm{T}}$, $\hat{x}_i = [x_i^{\mathrm{T}}, \zeta_i^{\mathrm{T}}]^{\mathrm{T}}$, $\hat{A}$, $\hat{B}$, and $\hat{C}$ are block diagonal matrices with each block being

$$\hat{A}_i = \begin{bmatrix} A_i + B_iK_{1i} & B_iK_{2i} \\ O_iC_i & F_i \end{bmatrix}, \ \hat{B}_i = \begin{bmatrix} B_iG_{2i} \\ G_{1i} \end{bmatrix}, \ \hat{C}_i = \begin{bmatrix} C_i & 0 \end{bmatrix}.$$

In terms of group consensus for (5a) and (5b), the following result extends the internal model principle proposed in [13]. For the sake of brevity, the network topology is assumed to be fixed in the next theorem, i.e., we use l_{ij} instead of $l_{ij}^{\sigma(t)}$.

Theorem 1 *Consider N linear state-space models* (1) *coupled through dynamic controllers* (2a) *and* (2b). *If* $(y_i - y_j) \to 0$ *and* $(\zeta_i - \zeta_j) \to 0$, *as* $t \to \infty$, *for* $\bar{i} = \bar{j}$, $i, j = 1, \ldots, N$, *then there exist a scalar m, matrices* $S \in \mathbb{R}^{m\times m}$, *and* $R_\ell \in \mathbb{R}^{p\times m}$, $\ell = 1, \ldots, q$, *where* $\sigma(S) \subset \bar{\mathbb{C}}_+$ *and* (S, R_ℓ) *is observable, and matrices* $\Pi_i \in \mathbb{R}^{n_i\times m}$, $\Gamma_i \in \mathbb{R}^{p_i\times m}$, *and* $\Lambda_i \in \mathbb{R}^{m_i\times m}$, *such that*

$$\begin{cases} A_i\Pi_i + B_i\left(\Gamma_i + \sum_{j=1}^{N} l_{ij}\Lambda_j\right) = \Pi_iS, & \text{(6a)} \\ C_i\Pi_i = R_\ell. & \text{(6b)} \end{cases}$$

Furthermore, there exists $z_\ell(0) \in \mathbb{R}^m$ *such that*

$$\lim_{t\to\infty} \left\| y_i(t) - R_\ell e^{St} z_\ell(0) \right\| = 0, \tag{7}$$

$\forall i \in \mathsf{V}_\ell$, $\ell = 1, \ldots, q$.

Proof. Since $(y_i - y_j) \to 0$ and $(\zeta_i - \zeta_j) \to 0$ as $t \to \infty$, system (5a) has an attractive invariant subspace M where $C_jx_j = C_ix_i$ and $\zeta_j = \zeta_i$ for $i, j \in \mathsf{V}_\ell$, $\ell = 1, \ldots, q$. Recall that the common inter-cluster coupling condition is imposed, on M, one hence has

$$\dot{\hat{x}} = \left(\hat{A} - \hat{B}\left(\mathsf{L} \otimes I_{p+m}\right)\bar{C}\right)\hat{x}, \tag{8}$$

where $\bar{C} = \mathrm{diag}\{\bar{C}_1, \ldots, \bar{C}_q\}$. Following the analysis in [13], assume, at no loss of generality, that M 1) contains no exponentially stable modes, 2) contains only modes that are observable at the output, and 3) is non-trivial with dimension m. In view of such assumption, there exists a matrix $S \in \mathbb{R}^{m\times m}$ such that[1]

$$\left(\hat{A} - \hat{B}\left(\mathsf{L} \otimes I_{p+m}\right)\bar{C}\right)\Phi = \Phi S,$$

Partition Φ into

$$\Phi = [\Pi_1^{\mathrm{T}}, \Sigma_1^{\mathrm{T}}, \ldots, \Pi_N^{\mathrm{T}}, \Sigma_N^{\mathrm{T}}]^{\mathrm{T}}.$$

One then has

$$\hat{A}_i[\Pi_i^{\mathrm{T}}, \Sigma_i^{\mathrm{T}}]^{\mathrm{T}} - \hat{B}_i \sum_{j=1}^{N} l_{ij}\left(C_j\Pi_j + \Sigma_j\right) = [\Pi_i^{\mathrm{T}}, \Sigma_i^{\mathrm{T}}]^{\mathrm{T}}S,$$

this completing the first part of the proof with $\Gamma_i = K_{1i}\Pi_i + K_{2i}\Sigma_i$, and $\Lambda_j = -G_{2i}(C_j\Pi_j + \Sigma_j)$. Next, we will prove that $C_i\Pi_i = R_\ell$, $\forall i \in \mathsf{V}_\ell$. Since $y_i = y_j$, one has $C_i\Pi_i = C_j\Pi_j, \forall i, j \in \mathsf{V}_\ell$. Then, there exists some matrix R_ℓ such that $C_i\Pi_i = R_\ell$, $\forall i \in \mathsf{V}_\ell$, for $\ell = 1, \ldots, q$. By the fact that modes in M are observable at the output, one has (S, R_ℓ) is observable for $\ell = 1, \ldots, q$. ∎

[1] It is always possible to find a transformation matrix $T = [\Phi, \Sigma]$ such that $T^{-1}\hat{A}T = \begin{bmatrix} S & 0 \\ 0 & H \end{bmatrix}$, where $\sigma(S) \in \bar{\mathbb{C}}_+$ and H is Hurwitz.

Remark 1 *The coupled term* $\sum_{j=1}^{N} l_{ij}\Lambda_j$ *in* (6a) *arises from the employment of* e_ζ^i *and* e_y^i *in* (4a). *If the latter two terms* e_ζ^i *and* e_y^i *are removed in controller design, then* (6a) *reduces to the form obtained in [13].*

4 Output Group Consensus Using Relative Controller State

In this section, we will solve the output group consensus problem by designing appropriate control laws u_i, $i = 1, \ldots, N$. To resolve this group consensus problem, we first introduce an assumption, which is required such that the problem is feasible, based on the necessary condition proposed in the preceding section. Similar conditions can also be found in [5, 11, 13].

Assumption 1 *For each* $i \in \mathsf{V}_\ell$, $\ell = 1, \ldots, q$, *there exist compatible matrices* Γ_i, Π_i, *and* Ψ_i, *such that*

$$\begin{cases} A_i\Pi_i + B_i\Gamma_i = \Pi_iS, \\ C_i\Pi_i = R_\ell, \\ B_i\Psi_i = \Pi_i, \end{cases} \tag{9}$$

where $S \in \mathrm{R}^{m\times m}$, $R_\ell \in \mathrm{R}^{p\times m}$, $\sigma(S) \in j\mathbb{R}$.

Remark 2 *The first two conditions in Assumption 1 are broadly used in the existing literature concerning consensus of heterogeneous linear systems [11]. Intuitively, these two conditions require that all the system matrices contain a common eigen-space, which is reflected by the matrix S. The states of the agents in each cluster are finally controlled into the common eigen-space. The third condition is made to facilitate the controller design such that the influence from common inter-cluster couplings can be compensated.*

Inspired by Remark 2, we consider the following reference generator for each agent i, $i \in \mathsf{V}_\ell$, $\ell = 1, \ldots, q$,

$$\dot{\zeta}_i(t) = S\zeta_i(t), \tag{10}$$

where $\zeta_i \in \mathbb{R}^m$. These types of generators produce trajectories for agents in each cluster to track. To synchronize the above reference generators in each cluster, we consider the following distributed control protocol for agent $i \in \mathsf{V}_\ell$, $\ell = 1, \ldots, q$,

$$\dot{\zeta}_i(t) = S\zeta_i(t) + \sum_{j=1}^{N} H_{\bar{i}}c_{ij}a_{ij}^{\sigma(t)}\left(\zeta_j(t) - \zeta_i(t)\right), \tag{11}$$

where $c_{ij} = c_\ell$, $\bar{i} = \bar{j} = \ell$, otherwise, $c_{ij} = 1$; $H_{\bar{i}}$ is to be designed later. In the sequel, we call c_ℓ the intra-cluster coupling strength, which is used to reflect strong versus weak couplings.

To track the trajectories of the reference generators for each agent, the following feedback controller is exploited

$$\begin{cases} \dot{\tilde{x}}_i(t) = A_i\tilde{x}_i(t) + B_iu_i(t) + H_i\left(\tilde{y}_i(t) - y_i(t)\right), & \text{(12a)} \\ u_i(t) = K_{1i}\tilde{x}_i(t) + K_{2i}\zeta_i(t) + \Psi_i \sum_{j=1}^{N} c_{ij}l_{ij}^{\sigma(t)}\zeta_j, & \text{(12b)} \end{cases}$$

for $\forall i \in \mathsf{V}_\ell$, $\ell = 1, \ldots, q$, where (12a) is a Luenberger observer. The feedback controller (12b) uses the information from the Luenberger observer and the reference generators.

The controller of the above form can be seen as extension of those proposed in [13]. In this paper, we choose K_{1i} as K_i, and $K_{2i} = -K_i\Pi_i + \Gamma_i$, where K_i is to be determined.

Next, we will prove that output group consensus for heterogeneous system (1) can be achieved via (11), (12a), and (12b). In what follows, we first prove that the reference generators coupled as in (11) can realize group consensus exponentially fast. Then, by dynamic controller (12a), (12b), we will show that the output of each agent tracks that of its corresponding reference generator, thereby leading to output group consensus.

4.1 Group Consensus for Coupled Reference Generators

In this subsection, we will show that the coupled reference generators (11) achieve group consensus in an exponential manner. To this purpose, we transform the group consensus problem into a stability problem by introducing proper error variables. Then, the stability analysis is performed with respect to the error system dynamics.

First, we introduce some notations. Let $\tilde{\mathsf{L}}_{\ell k}^{\sigma(t)}$, $\ell, k = 1, \ldots, q, \ell \neq k$, be the sub-matrix of the following matrix

$$\Delta_\ell \mathsf{L}_{\ell k}^{\sigma(t)} \Delta_k = \begin{bmatrix} -d_{\ell k} & * \\ \mathbf{0}_{N_\ell - 1} & \tilde{\mathsf{L}}_{\ell k}^{\sigma(t)} \end{bmatrix},$$

where $\Delta = \operatorname{diag}\{\Delta_1, \cdots, \Delta_q\}$, $\Delta_\ell = \begin{bmatrix} 1 & 0_{1\times(N_\ell-1)} \\ \mathbf{1}_{N_\ell-1} & -I_{N_\ell-1} \end{bmatrix}$. Suppose that the underlying topology of each cluster contains a directed spanning tree during each interval $[t_k, t_{k+1})$, $k = 0, 1, 2, \ldots$, a positive definite matrix, say $\Xi_\ell^{\sigma(t_k)}$ corresponding to $-\tilde{\mathsf{L}}_{\ell\ell}^{\sigma(t_k)}$, exists such that $-\tilde{\mathsf{L}}_{\ell\ell}^{\sigma(t_k)\mathrm{T}}\Xi_\ell^{\sigma(t_k)} - \Xi_\ell^{\sigma(t_k)}\tilde{\mathsf{L}}_{\ell\ell}^{\sigma(t_k)} < 0$. Let $N^{\sigma(t_k)} = \operatorname{diag}\{c_1(\Xi_1^{\sigma(t_k)}\tilde{\mathsf{L}}_{11}^{\sigma(t_k)} + \tilde{\mathsf{L}}_{11}^{\sigma(t_k)\mathrm{T}}\Xi_1^{\sigma(t_k)}), \cdots, c_q(\Xi_q^{\sigma(t_k)}\tilde{\mathsf{L}}_{qq}^{\sigma(t_k)} + \tilde{\mathsf{L}}_{qq}^{\sigma(t_k)\mathrm{T}}\Xi_q^{\sigma(t_k)})\}$, $\mathcal{N}^{\sigma(t_k)} = \tilde{\mathsf{L}}^{\sigma(t_k)\mathrm{T}}\Xi^{\sigma(t_k)} + \Xi^{\sigma(t_k)}\tilde{\mathsf{L}}^{\sigma(t_k)} - N^{\sigma(t_k)}$, where $\Xi^{\sigma(t_k)} = \operatorname{diag}\{\Xi_1^{\sigma(t_k)}, \ldots, \Xi_q^{\sigma(t_k)}\}$. Now, we are ready to present our result.

Lemma 1 *Suppose that the common inter-cluster coupling condition* (3) *is satisfied. If for each $t_k = 0, 1, 2\ldots$, the underlying graph $\mathsf{G}_\ell^{\sigma(t_k)}$ of each cluster contains a directed spanning tree, and intra-cluster coupling strength c_ℓ, $\ell = 1, \ldots, q$, satisfies*

$$c_\ell > \frac{\gamma\lambda_{\max}(P) + \lambda_{\max}\left(PS + S^{\mathrm{T}}P\right) - \phi\lambda_{\min}^2(P)}{\eta\lambda_{\min}^2(P)}, \tag{13}$$

then group consensus for the coupled generators (11) *can be achieved exponentially fast with $\gamma > 0$ satisfying the following inequalities*

$$\begin{cases} PS + S^{\mathrm{T}}P - (c_\ell\eta + \phi)PP \leq -\gamma P, \\ \ln(\kappa) - \gamma\tau_0 < 0, \end{cases}$$

where P is symmetric positive definite, κ satisfies $\lambda_{\max}(\Xi_\ell^{\sigma(t_{k+1})} \otimes P) < \kappa\lambda_{\min}(\Xi_\ell^{\sigma(t_k)} \otimes P)$ for $t_k = 0, 1, \ldots$, ϕ and $\eta > 0$ are defined such that

$$\lambda_{\min}\left(\mathcal{N}^{\sigma(t_k)}\right) \geq \phi\lambda_{\max}\left(\Xi^{\sigma(t_k)}\right),$$

$$\Xi_\ell^{\sigma(t_k)}\tilde{\mathsf{L}}_{\ell\ell}^{\sigma(t_k)} + \tilde{\mathsf{L}}_{\ell\ell}^{\sigma(t_k)\mathrm{T}}\Xi_\ell^{\sigma(t_k)} \geq \eta\Xi_\ell^{\sigma(t_k)},\ \ell = 1, \ldots, q.$$

Proof. See Appendix for the detailed proof. ∎

Remark 3 *In the statement of Lemma 1, the variables ϕ, κ and η are well defined in light of that there are finite different topologies for the communication network to take.*

4.2 Output Group Consensus via Dynamic Controller

Based on the result established in the preceding subsection, in what follows, we shall prove that the output group consensus is asymptotically achieved for heterogeneous systems (1). Especially, we will show that each heterogeneous linear system tracks the trajectory of the corresponding reference generator.

Theorem 2 *Under Assumption 1, suppose that the conditions stated in Lemma 1 hold. Then the output group consensus for multi-agent system* (1) *can be achieved through dynamic controller* (11), (12a), *and* (12b) *where K_i and H_i are designed such that $A_i + B_iK_i$ and $A_i + H_iC_i$ are Hurwitz for $i = 1, \ldots, N$.*

Proof. Define error variables $\epsilon_i = x_i - \Pi_i\zeta_i$, and $\nu_i = x_i - \tilde{x}_i$. One has

$$\begin{aligned} \dot{\epsilon}_i &= (A_i + B_iK_i)\epsilon_i - B_iK_i\nu_i \\ \dot{\nu}_i &= (A_i + H_iC_i)\nu_i, \end{aligned}$$

for $i = 1, \ldots, N$. Since $A_i + H_i$ is Hurwitz, ν_i tends to zero as time approaches infinity at an exponential rate. Recall that K_i is designed such that $A + B_iK_i$ is Hurwitz, it is therefore concluded that ϵ_i reaches zero exponentially fast for $i = 1, \ldots, N$.

Next, we will verify output consensus. Bearing the above conclusion in mind, it can be obtained that $x_i \to \Pi_i\zeta_i$ and $x_i \to \tilde{x}_i$ exponentially fast as time tends to infinity. Recalling the fact that $y_i = C_ix_i$, by Assumption 1, one has

$$y_i \to C_i\Pi_i\zeta_i = R_\ell\zeta_i,$$

exponentially fast. Since $\zeta_i \to \zeta_j$, $\forall \bar{i} = \bar{j} = \ell$, according to Lemma 1, $y_i \to y_j$, $\forall \bar{i} = \bar{j} = \ell$, the result is valid. ∎

5 An Illustrative Example

In this section, we present an example to illustrate our theoretical findings.

Example 1: Consider the five linear systems moving on the plane with two possible network topologies shown in Fig. 1. The communication graph switches from graph a to graph b periodically with a period $T = 1s$. Two clusters are considered in this example such that $\mathsf{V}_1 = \{1, 2\}$, $\mathsf{V}_2 = \{3, 4, 5\}$. The systems dynamics are chosen such that

$$A_1 = \begin{bmatrix} 1 & -3 \\ 1 & -1 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad C_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} 0.3 & -2 \\ 0.1 & -0.2 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 1.5 & 0 \\ 0 & 0.9 \end{bmatrix}, \quad C_2 = \begin{bmatrix} 2 & -2 \\ -2 & 3 \end{bmatrix},$$

$$A_3 = \begin{bmatrix} 1 & -7 \\ 3 & -3 \end{bmatrix}, \quad B_3 = \begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix}, \quad C_3 = \begin{bmatrix} -0.2 & 1.4 \\ 0.5 & 1.5 \end{bmatrix},$$

$$A_4 = \begin{bmatrix} 1 & -6 \\ 2 & -1 \end{bmatrix}, \quad B_4 = \begin{bmatrix} 2 & 1 \\ 1 & -2 \end{bmatrix}, \quad C_4 = \begin{bmatrix} -0.2 & 1.2 \\ 0.5 & 2.1 \end{bmatrix},$$

$$A_5 = \begin{bmatrix} 3 & -5 \\ 4 & -1 \end{bmatrix}, \quad B_5 = \begin{bmatrix} 0 & 1 \\ 2 & +3 \end{bmatrix}, \quad C_5 = \begin{bmatrix} 2 & 3 \\ -2 & 3 \end{bmatrix}.$$

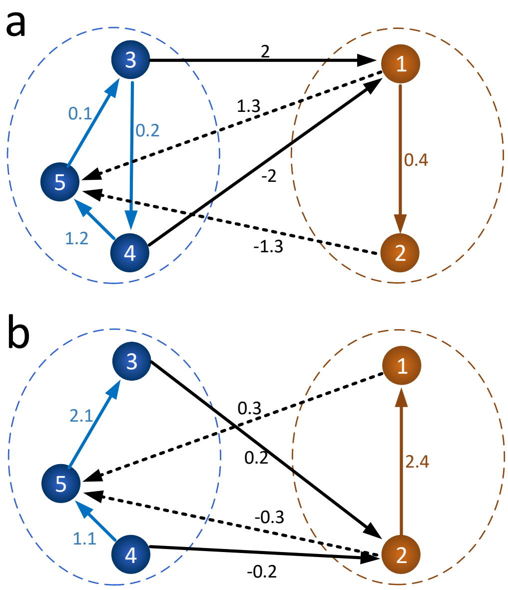


Fig. 1: Two different interaction topologies among five heterogeneous agents. Underlying topology of each cluster contains a directed spanning tree. The inter-cluster couplings satisfy in-degree balanced condition with $d_1 = d_2 = 0$.

By computation, one has

$$S_1 = \begin{bmatrix} 1 & -3 \\ 1 & -1 \end{bmatrix},\ R_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},\ S_2 = \begin{bmatrix} 3 & -5 \\ 3 & -3 \end{bmatrix},\ R_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

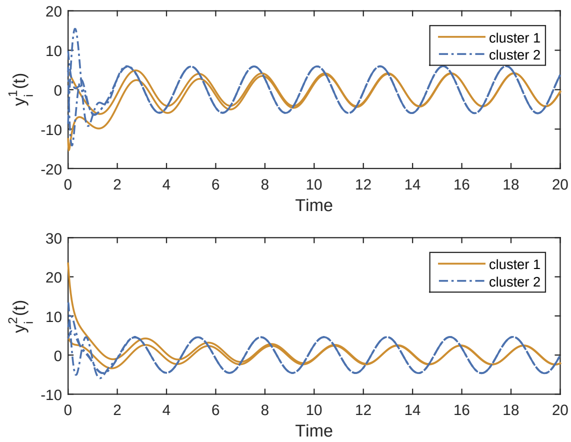
The initial values of the agents are randomly chosen from interval $[-10, 10] \times [-10, 10] \in \mathbb{R}^2$.

The output trajectories of the five agents are shown in Fig. 2. While the output tracking error is depicted in Fig. 3. It is observed obviously that group consensus is achieved asymptotically.

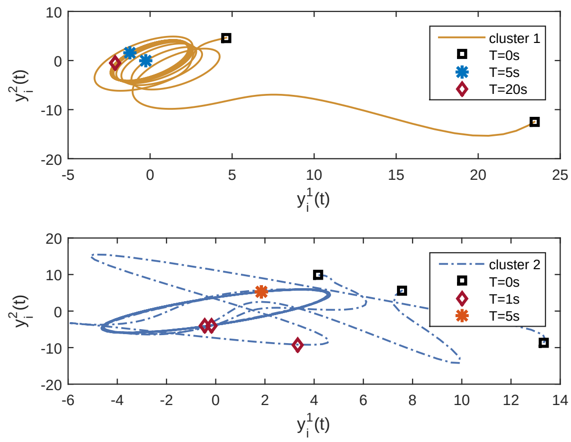
6 Conclusion

In this paper, we have investigated the output group consensus control for a network of heterogeneous linear systems communicating over switching topology. A necessary condition has been derived in terms of the system dynamics, with which a dynamic controller is then designed to solve the output group consensus problem in two separative steps: 1) consensus of reference generators; and 2) output tracking for each agent. An appropriate controller is then proposed in order to track the output of the reference generators to realize group consensus. The simulation has validated the effectiveness of our theoretical findings.

Future works involve further discussions on how inter-cluster couplings influence the evolution of the reference generators and relaxation of the assumption imposed on system dynamics.



(a) 1D plot of the output trajectories of the five agents.



(b) 2D plot of the output trajectories of the five agents.

Fig. 2: The output trajectories of $y_i^{\mathrm{T}} = [y_i^1 \quad y_i^2]$, $i = 1, \ldots, 5$. It is observed that group consensus is asymptotically achieved.

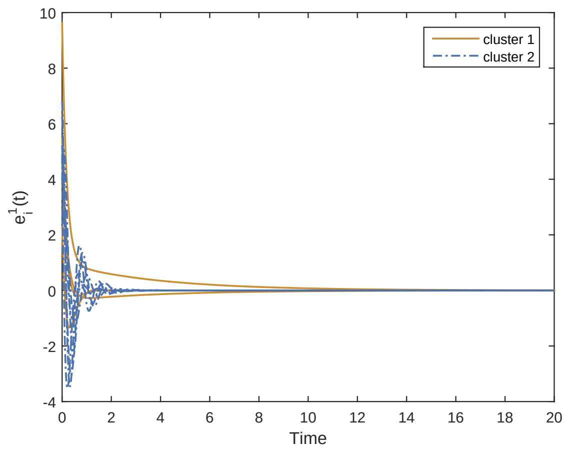


Fig. 3: The tracking error trajectories of $e_i = y_i - R_{\bar{i}}\zeta_i$, $i = 1, \ldots, 5$. It is clear that the tracking error asymptotically vanishes.

7 Appendix: Proof of Lemma 1

In this proof, we consider a general system dynamics for every $i \in \mathsf{V}_\ell$,

$$\dot{\zeta}_i(t) = S_\ell \zeta_i(t) + \sum_{j=1}^{N} H_{\bar{i}} c_{ij} a_{ij}^{\sigma(t)} \left(\zeta_j(t) - \zeta_i(t)\right). \qquad (15)$$

Pre-multiplying the above system with $\Delta \otimes I_m$ and noting that $\Delta_\ell \Delta_\ell = I_{N_\ell}$, for $\ell = 1, \ldots, q$, one then arrives at

$$\begin{aligned}\dot{\delta} =& \mathrm{diag}\left\{I_{N_1-1} \otimes S_1, \ldots, I_{N_q-1} \otimes S_q\right\} \delta \\ & - \left[\tilde{\mathsf{L}}^{\sigma(t)} \otimes I_m\right] \mathrm{diag}\left\{I_{N_1-1} \otimes H_1, \ldots, I_{N_q-1} \otimes H_q\right\} \delta, \quad (16)\end{aligned}$$

where $\delta_\ell = [(\zeta_{\kappa_{\ell-1}+1} - \zeta_{\kappa_{\ell-1}+2})^{\mathrm{T}}, \ldots, (\zeta_{\kappa_{\ell-1}+1} - \zeta_{\kappa_\ell})^{\mathrm{T}}]^{\mathrm{T}}$, $\delta = [\tilde{\delta}_1^{\mathrm{T}}, \ldots, \tilde{\delta}_q^{\mathrm{T}}]^{\mathrm{T}}$, $\tilde{\mathsf{L}}^{\sigma(t)}$ takes the following form

$$\tilde{\mathsf{L}}^{\sigma(t)} = \begin{bmatrix} c_1 \tilde{\mathsf{L}}_{11}^{\sigma(t)} + \tilde{D}_1^{\sigma(t)} & \cdots & \tilde{\mathsf{L}}_{1q}^{\sigma(t)} \\ \vdots & \ddots & \vdots \\ \tilde{\mathsf{L}}_{q1}^{\sigma(t)} & \cdots & c_q \tilde{\mathsf{L}}_{qq}^{\sigma(t)} + \tilde{D}_q^{\sigma(t)} \end{bmatrix}. \qquad (17)$$

Evidently, group consensus is asymptotically achieved if system (16) is asymptotically stable. Now, consider the following Lyapunov function candidate for the error system dynamics (16): $V(t) = \sum_{\ell=1}^{q} \delta_\ell^{\mathrm{T}} \left(\Xi_\ell^{\sigma(t_k)} \otimes P_\ell\right) \delta_\ell$, $t \in [t_k, t_{k+1})$. Choosing $H_{\bar{i}} = P_\ell$, for $i \in \mathsf{V}_\ell$, and taking the derivative of $V(t)$ along the trajectory of error system dynamics (16) gives

$$\begin{aligned}\dot{V}(t) =& 2 \sum_{\ell=1}^{q} \delta_\ell^{\mathrm{T}} \left(\Xi_\ell^{\sigma(t_k)} \otimes P_\ell S_\ell\right) \delta_\ell \\ & -2 \sum_{\ell=1}^{q} \delta_\ell^{\mathrm{T}} \left[\left(c_\ell \Xi_\ell^{\sigma(t_k)} \tilde{\mathsf{L}}_{\ell\ell}^{\sigma(t_k)} + \Xi_\ell^{\sigma(t_k)} \tilde{D}_\ell^{\sigma(t_k)}\right) \otimes P_\ell P_\ell\right] \delta_\ell \\ & -2 \sum_{\ell=1}^{q} \sum_{j=1, j\neq\ell}^{q} \delta_\ell^{\mathrm{T}} \left(\Xi_\ell^{\sigma(t_k)} \tilde{\mathsf{L}}_{\ell j}^{\sigma(t_k)} \otimes P_\ell P_j\right) \delta_j.\end{aligned}$$

Let us first consider the last two terms in the derivative of $V(t)$. Define $y_\ell(t) = (I_{N_\ell} \otimes P_\ell) \delta_\ell$, then one has

$$\begin{aligned}& 2 \sum_{\ell=1}^{q} \sum_{j=1, j\neq i}^{q} \delta_\ell^{\mathrm{T}} \left(\Xi_\ell^{\sigma(t_k)} \tilde{\mathsf{L}}_{\ell j}^{\sigma(t_k)} \otimes P_\ell P_j\right) \delta_j \\ & + 2 \sum_{\ell=1}^{q} \delta_\ell^{\mathrm{T}} \left(\Xi_\ell^{\sigma(t_k)} \tilde{D}_\ell^{\sigma(t_k)} \otimes P_\ell P_\ell\right) \delta_\ell \\ =& y^{\mathrm{T}} \left[\mathcal{N}^{\sigma(t_k)} \otimes I_m\right] y \\ \geq& \phi y^{\mathrm{T}} \left(\Xi^{\sigma(t_k)} \otimes I_m\right) y = \phi \sum_{\ell=1}^{q} \delta_\ell^{\mathrm{T}} \left(\Xi_\ell^{\sigma(t_k)} \otimes P_\ell P_\ell\right) \delta_\ell. \qquad (18)\end{aligned}$$

Therefore, one obtains the following inequality

$$\begin{aligned}\dot{V} \leq& \sum_{\ell=1}^{q} \delta_\ell^{\mathrm{T}} \left(\Xi_\ell^{\sigma(t_k)} \otimes \left[P_\ell S_\ell + S_\ell^{\mathrm{T}} P_\ell - (c_\ell \eta + \phi) P_\ell^2\right]\right) \delta_\ell \\ \leq& -\gamma \sum_{\ell=1}^{q} \delta_\ell^{\mathrm{T}} \left(\Xi_\ell^{\sigma(t_k)} \otimes P_\ell\right) \delta_\ell = -\gamma V, \qquad (19)\end{aligned}$$

where it is assumed that

$$P_\ell S_\ell + S_\ell^{\mathrm{T}} P_\ell - (c_\ell \eta + \phi) P_\ell^2 < \gamma P_\ell.$$

Hence, $V(t) \leq e^{-\gamma(t-t_k)} V(t_k)$ for $t \in [t_k, t_{k+1})$.

For any $t > 0$, it is always possible to find an integer s such that $t_s \leq t < t_{s+1}$. Therefore, one has

$$\begin{aligned}V(t) \leq& \exp^{-\gamma(t-t_s)} V(t_s) \leq \exp^{-\gamma(t-t_s)} \exp^{\ln(\kappa)-\gamma(t_s-t_{s-1})} V(t_{s-1}) \\ \leq& \cdots \leq \exp\left\{-\gamma(t-t_s) + \sum_{j=1}^{s} \left(\ln(\kappa) - \gamma(t_j - t_{j-1})\right)\right\} V(t_0).\end{aligned}$$

If $\ln(\kappa) - \gamma\tau_0 < 0$ holds, then one has $\delta(t) \to 0$ as $t \to +\infty$ exponentially fast. This completes the proof.

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