Theory and Design of PID Controller

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Outline

1. Overview of PID Control
2. Mathematical Formulation
3. Theory and Design of PID
4. Concluding Remarks

A brief history of PID

- Proportional feedback in the form of a centrifugal governor was used to regulate the speed of windmills around 1750.
- In 1788 James Watt used a similar system for speed control of steam engines.
- The first mathematical analysis of a steam engine with a governor was made by Maxwell in 1868.
- One of the earliest examples of a PID-type controller was intuitively developed by Elmer Sperry in 1911.
- It was not until 1922 that PID controllers were analytically developed by N.Minorsky for automatic ship steering.

The Impact of PID

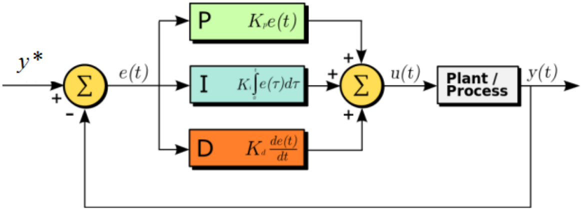
- Despite of the remarkable progresses of modern control theory over the past half a century, the classical PID controller is still the most widely used ones in engineering systems today.

  As an example, 95% control loops are of PID type in process control, and most loops are actually PI control(Åström and Hägglund,1995).

- In 2016, IFAC publicized a survey conducted by a "Pilot" Industry Committee launched by IFAC and chaired by Tariq Samad. The survey shows that the PID control has much higher impact rating than other 12 advanced control technologies, and "we still have nothing compares with PID", see,

  http://blog.ifac-control.org/

The structure of PID



- Linear feedback structure of the form "present-past-future":

$$u(t) = k_p e(t) + k_i \int_0^t e(s)ds + k_d \frac{de(t)}{dt}$$

where $e(t) = y^* - y(t)$.

Why the PID so effective?

- It is simple, model-free and easy-to-use.
- It can eliminate steady state offsets via the integral action.
- It can anticipate the tendency through the derivative action.
- It has strong robustness w.r.t both system uncertainties and controller parameters.
- The well-known Newton's law plays a fundamental role in modeling physical systems.
- ... ...

Fundamental Theoretical Problems

- How to properly design the PID parameters?
- How to guarantee the desired control performance?
- What is the maximum capability of PID feedback?

Classical Design Methods

Ziegler-Nichols method

Two classical methods for determining the parameters of PID controllers were presented by Ziegler and Nichols in 1942. These methods are still widely used, either in their original form or in some modification.

- It is based on some features of the process dynamics extracted from experiments, conducted by either the step response method or the frequency response method, for linear time-invariant systems.
- The PID controller $u(t) = K\left(e(t) + \frac{1}{T_i}\int_0^t e(\tau)d\tau + T_d\frac{de(t)}{dt}\right)$.

See: Ziegler J G, Nichols N B,1942.

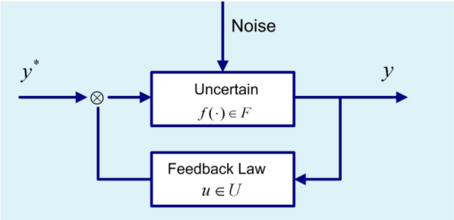
Other Methods

Many other methods including tuning and adaptation for the design of the PID parameters have also been proposed but mainly for linear systems.

References

- Åström K J, Hägglund T. (1995,2006)
- Blanchini F, Lepschy A, Miani S, et al. (2004)
- Hara S, Iwasaki T, Shiokata D. (2006)
- Ho M T, Lin C Y. (2003)
- Keel LH, Bhattacharyya S P. (2008)
- Killingworth N J, Kristic M.(2006)
- O'Dwyer. (2006)
- Silva G J, Datta A, Bhattacharyya S P.(2005)
- Söylemez M T, Munro N, Baki H.(2003)
- ......

Understanding PID: Uncertainty, nonlinearity and feedback



- To understand PID, we have to face with uncertainties and nonlinearities, because they always exist in practical systems.
- Basic questions: why it is so powerful? how much uncertainty can it deal with?
- As pointed out by Åström and Hägglund(1995,2006), better understanding of PID control may improve its widespread practice, and so contribute to better product quality.

Description of Uncertainty

- Uncertainty is mathematically described by a set F, either parametric or functional.
- The control of uncertain systems is by definition the control of all possible systems related to this set.

The Maximum Capability of Feedback

Consider the following control system:

$$y_{t+1} = f(y_t) + u_t + w_{t+1}, \quad y_0 \in \mathbb{R}$$

with $f \in F_L$ where

$$F_L = \{f : \mathbb{R} \to \mathbb{R} \Big| sup_{x \neq y} \frac{|f(x) - f(y)|}{|x - y|} \leq L\}$$

L: Serves as a measure of uncertainty

Theorem(Xie-Guo,2000). The above class of uncertain nonlinear dynamical systems described by F_L is globally stabilizable by the feedback mechanism if and only if

$$L < \frac{3}{2} + \sqrt{2}$$

Framework of PID Theory

Following a similar theoretical framework as the investigation of the maximum capability of the feedback mechanism:

- The maximum capability of feedback is defined by the largest possible class of nonlinear functions that can be dealt with by the feedback mechanism.
- The size of the uncertain functional class is characterized by the corresponding Lipschitz constant.

See:

- Xie L L, Guo L. IEEE Trans Automat Control, 2000.
- Guo L. Plenary Lecture at the 19th IFAC World Congress, Cape Town, 2014.

Mathematical Formulation

Mathematical Formulation

Background

- The Newton's second law plays a fundamental role in modeling dynamical systems of the physical world, which is actually a second order ordinary differential equation of the position of a moving body.

- PID control is sufficient for processes where the dominant dynamics are of the second order. For such processes there are no benefits gained by using a more complex controller. (Åström K J, Hägglund T. 1995)

Mathematical formulation

- Consider a moving body of unit mass in $\mathbb{R}$ which is regraded as a controlled system.
- $x(t)$, $v(t)$, $a(t)$ are its position, velocity and acceleration at the time instant t
- Assume that the external forces acting on the body consist of f and u.
- $f = f(x, v, t)$ is a nonlinear function of the position x, velocity v and time t and u is the control force.

The equation of motion

$$ma(t) = f(x(t), v(t), t) + u(t)$$

Objective:

To understand when and how the PID controller can guarantee that the position converges to a given constant reference value y^* for any initial position and any initial velocity.

State space equation

Denote $x_1(t) = x(t)$ and $x_2(t) = \frac{dx(t)}{dt} = \dot{x}(t)$, then the state space equation of this basic mechanic system under PID control is

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = f(x_1, x_2, t) + u(t) \\ \\ u(t) = k_p e(t) + k_i \int_0^t e(s)ds + k_d \frac{de(t)}{dt} \end{cases} \tag{1}$$

where $x_1(0), x_2(0) \in \mathbb{R}$ and $e(t) = y^* - x_1(t)$.

Theory and Design of PID

The Class of Uncertain Functions

Define a functional class:

$$\mathscr{F}_{L_1,L_2} = \left\{ f \in C^1(\mathbb{R}^2 \times \mathbb{R}^+) \middle| \frac{\partial f}{\partial x_1} \leq L_1,\ |\frac{\partial f}{\partial x_2}| \leq L_2, \forall x_1, x_2 \in \mathbb{R},\ \forall t \in \mathbb{R}^+ \right\}$$

where L_1 and L_2 are positive constants, and $C^1(\mathbb{R}^2 \times \mathbb{R}^+)$ denotes the space of all functions from $\mathbb{R}^2 \times \mathbb{R}^+$ to $\mathbb{R}$ which are locally Lipschitz in (x_1, x_2) uniformly in t and piecewise continuous in t, with continuous partial derivatives with respect to (x_1, x_2).

The Parameter Manifold

Denote

$$(\overline{k}_p, \overline{k}_i, \overline{k}_d) = (k_p - L_1, k_i, k_d - L_2)$$

and introduce

$$\Omega_{pid} = \left\{ \begin{bmatrix} k_p \\ k_i \\ k_d \end{bmatrix} \middle| \overline{k}_p > 0, \overline{k}_i > 0, \overline{k}_p \overline{k}_d > \overline{k}_i + L_2 \sqrt{\overline{k}_i (\overline{k}_d + 2L_2)} \right\} \quad (2)$$

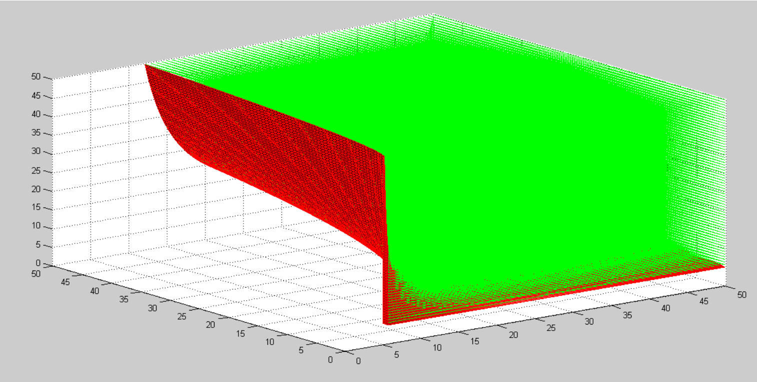
Theorem 1: Assume that $f \in \mathscr{F}_{L_1,L_2}$ and that $f(y,0,t) = f(y,0,0)$ for all $t \in \mathbb{R}^+$ and $y \in \mathbb{R}$. Then, whenever $(k_p, k_i, k_d) \in \Omega_{pid}$, the PID controlled system (1) will satisfy

$$\lim_{t \to \infty} x_1(t) = y^*, \qquad \lim_{t \to \infty} x_2(t) = 0$$

for any $(x_1(0), x_2(0)) \in R^2$ and any setpoint $y^* \in \mathbb{R}$.

An Illustration: $L_1 = 5$ and $L_2 = 5$

The set Ω_{pid} when restricted to the domain $0 \leq k_p, k_i, k_d \leq 50$.



Remarks

- The selection of the PID parameters has wide flexibility (Ω_{pid} is open and unbounded).
- Theorem 1 gives a global convergence result.
- The selection of the PID parameters does not depend on the initial states and the setpoint y^*.
- PID controller has strong robustness with respect to uncertain nonlinear functions and to the selection of parameters.

Remarks

- L_1 and L_2 represent the "anti-stiffness" coefficient and the "anti-damping" coefficient of the nonlinear system, respectively.
- For any $k_p > L_1$ and $k_d > L_2$, we have $(k_p, k_i, k_d) \in \Omega_{pid}$ for all sufficiently small $k_i > 0$.
- The results can be generalized by replacing the conditions on the partial derivatives with Lipschitz-like properties.

Necessity of Parameter Manifold

If we have more constrains on the unknown function $f(x_1, x_2, t)$, such as f is independent of t and $\frac{\partial^2 f}{\partial x_2^2} = 0$, then we can find a larger and necessary parameter manifold to stabilize the system.

Examples:

- f is of the form $f(x_1, x_2, t) = a(x_1) + b(x_1)x_2$.
- f is merely a function of the variable x_1, i.e., the open-loop system is conservative.

Definitions

Let us introduce the following functional class,

$$\mathscr{G}_{L_1,L_2} = \left\{ f \in C^2(\mathbb{R}^2) \middle| \frac{\partial f}{\partial x_1} \leq L_1,\ \frac{\partial f}{\partial x_2} \leq L_2,\ \frac{\partial^2 f}{\partial x_2^2} = 0,\ \forall x_1, x_2 \in \mathbb{R} \right\},$$

where $L_1 > 0$, $L_2 > 0$ are constants and $C^2(\mathbb{R}^2)$ is the space of twice continuously differentiable functions from $\mathbb{R}^2$ to $\mathbb{R}$.

Proposition 1

Assume that $f \in \mathscr{G}_{L_1,L_2}$ does not depend on time t. Then for any $f \in \mathscr{G}_{L_1,L_2}$ and any setpoint $y^* \in \mathbb{R}$, the closed-loop system (1) satisfies

$$\lim_{t\to\infty} x_1(t) = y^* \qquad \lim_{t\to\infty} x_2(t) = 0$$

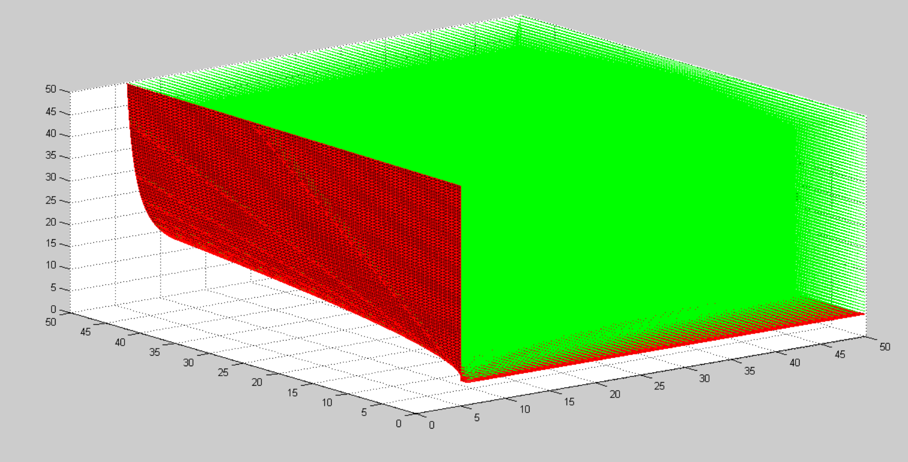
if and only if the PID parameters (k_p, k_i, k_d) lie in the following 3-dimensional manifold:

$$\Omega'_{pid} = \left\{ \begin{bmatrix} k_p \\ k_i \\ k_d \end{bmatrix} \middle| \overline{k}_p > 0, \overline{k}_i > 0, \overline{k}_p \overline{k}_d > \overline{k}_i \right\} \tag{3}$$

where $(\overline{k}_p, \overline{k}_i, \overline{k}_d) = (k_p - L_1, k_i, k_d - L_2)$.

An Illustration: $L_1 = 5$ and $L_2 = 5$:

The set Ω'_{pid} when restricted to the domain $0 \leq k_p, k_i, k_d \leq 50$.



PD control

When $(y^*, 0)$ is an equilibrium point of the open-loop systems, i.e. $f(y^*, 0) = 0$, the I-term is not necessary for regulation.
Define a functional class $\mathscr{F}_{L_1, L_2, y^*}$ as follows,

$$\left\{ f \in C^1(\mathbb{R}^2) \middle| \frac{\partial f}{\partial x_1} \leq L_1, \frac{\partial f}{\partial x_2} \leq L_2, \forall x_1, x_2, f(y^*, 0) = 0 \right\}$$

Theroem 2: Consider the PD controlled system

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = f(x_1, x_2) + u(t) \\ u(t) = k_p e(t) + k_d \frac{de(t)}{dt} \end{cases} \tag{4}$$

where the unknown $f \in \mathscr{F}_{L_1,L_2,y^*}$. Then for any $f \in \mathscr{F}_{L_1,L_2,y^*}$, we have

$$\lim_{t\to\infty} x_1(t) = y^*, \qquad \lim_{t\to\infty} x_2(t) = 0$$

if and only if the PD parameters (k_p, k_d) lie in the following 2-dimensional manifold:

$$\Omega_{pd} = \left\{ (k_p, k_d) \middle| k_p > L_1, k_d > L_2 \right\}. \tag{5}$$

A Generalization

The next theorem is a generalization of **Theorem 2**, where the second state variable is not the derivative of the first in general.

Consider the following uncertain nonlinear system with unknown $f = (f_1, f_2) \in C^1(\mathbb{R}^2 \to \mathbb{R}^2)$,

$$\begin{cases} \dot{x}_1 = f_1(x_1, x_2) \\ \dot{x}_2 = f_2(x_1, x_2) + u(t) \\ u(t) = k_p e(t) + k_d \frac{de(t)}{dt} \end{cases} \tag{6}$$

Definition

Define a functional class $\mathscr{G}_{L_1,L_2,y^*} \subset C^1(\mathbb{R}^2 \to \mathbb{R}^2)$ as follows,

$$\left\{ f = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \,\middle|\, \frac{\partial f_1}{\partial x_2} > 0, -(\frac{\partial f_1}{\partial x_2})^{-1} \det(Df) \le L_1, (\frac{\partial f_1}{\partial x_2})^{-1} \mathrm{tr}(Df) \le L_2, f(y^*, 0) = 0 \right\},$$

where $\det(Df)$ is the determinant of the Jacobian matrix of f defined by

$$Df = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{pmatrix}$$

and $\mathrm{tr}(Df)$ is the trace of Df defined by $\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2}$.

Theorem

Let the unknown $f \in \mathscr{G}_{L_1,L_2,y^*}$*, and* $u(t)$ *is the PD control:*

$$u(t) = k_p e(t) + k_d \dot{e}(t), \qquad e(t) = y^* - x_1(t).$$

Then for any $f \in \mathscr{G}_{L_1,L_2,y^*}$*, the closed-loop system satisfies*

$$\lim_{t\to\infty} x_1(t) = y^* \qquad \lim_{t\to\infty} x_2(t) = 0$$

for any initial value $(x_1(0), x_2(0)) \in \mathbb{R}^2$ *if and only if the PD parameters*

$$(k_p, k_d) \in \Omega_{pd} = \left\{(k_p, k_d)\middle| k_p > L_1, k_d > L_2\right\}$$

.

Remark. If $f_1(x_1, x_2) = x_2$, then the functional class $\mathscr{G}_{L_1,L_2,y^*}$ reduces to $\mathscr{F}_{L_1,L_2,y^*}$.

Markus-Yamabe Conjecture(or Jacobian Conjecture)

Let $f \in C^1(\mathbb{R}^n, \mathbb{R}^n)$, $f(0) = 0$. Consider the following n-dimensional autonomous differential equation,

$$\dot{x} = f(x)$$

If for any $x \in \mathbb{R}^n$, the eigenvalues of the Jacobian matrix $\frac{\partial f(x)}{\partial x}$ of f at x have negative real parts, then it is conjectured that the zero solution of the differential equation is globally asymptotically stable.

Markus-Yamabe Theorem: The above conjecture is true for $n = 2$.

References

- Markus L, Yamabe H. Osaka Math J, 1960.
- Feßler R, Ann Polon Math, 1995.
- Chen P N, He J X, Qin H S. Acta Math Sin, 2001.

First order systems

Finally, it is worth mentioning that for first order systems, PI control is sufficient. The next proposition gives a rigorous description. Define

$$\mathscr{F}_L = \{f \in H(\mathbb{R}\times\mathbb{R}^+) : |f(x,t)-f(y,t)| \leq L|x-y|, \forall x, y \in \mathbb{R},\ \forall t \in \mathbb{R}^+\},$$

where $L > 0$ is a constant and $H(\mathbb{R} \times \mathbb{R}^+)$ is the space of functions from $\mathbb{R} \times \mathbb{R}^+$ to $\mathbb{R}$, which are piecewise continuous in the second variable t.

Proposition 2

Consider the following first order nonlinear system

$$\dot{x} = f(x, t) + u$$

where the unknown $f \in \mathscr{F}_L$ and the PI control is defined by:

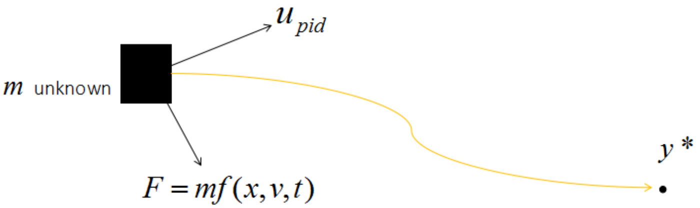
$$u(t) = k_p e(t) + k_i \int_0^t e(s) ds.$$

Then for any $f \in \mathscr{F}_L$ and any setpoint y^* satisfying $f(y^*, t) = f(y^*, 0)$ for all $t \in \mathbb{R}^+$, the closed-loop system is globally stable and satisfies $\lim_{t\to\infty} x(t) = y^*$ if and only if the PI parameters lie in the following 2-dimensional manifold:

$$\Omega_{pi} = \{(k_p, k_i) \in \mathbb{R}^2 \big| k_p > L, k_i > 0\}.$$

Uncertainty in control channel

If we only know the upper bound M of the mass of the moving body, then the control channel would contain an unknown parameter, say b, where $b = \frac{1}{m}$ is an unknown positive constant with a known lower bound $\underline{b} = \frac{1}{M} > 0$. We assume the unknown disturbance F is proportional to the mass m.



Proposition 3

In this case, the state space equation in Theorem 1 under PID control is

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = f(x_1, x_2, t) + bu(t) \\ u(t) = k_p e(t) + k_i \int_0^t e(s)ds + k_d \frac{de(t)}{dt} \end{cases} \tag{7}$$

where unknown $f \in \mathscr{F}_{L_1,L_2}$ and unknown $b \geq \underline{b} > 0$. Then for any $L_1, L_2 > 0$, the closed-loop system will satisfy

$$\lim_{t\to\infty} x_1(t) = y^*, \qquad \lim_{t\to\infty} x_2(t) = 0$$

for any initial value $(x_1(0), x_2(0))$ and any constant setpoint $y^* \in \mathbb{R}$ if the parameters $(\underline{b}k_p, \underline{b}k_i, \underline{b}k_d) \in \Omega_{pid}$.

Remark All the above results remain to be true as long as $(\underline{b}k_p, \underline{b}k_i, \underline{b}k_d)$ are chosen from the corresponding manifolds.

Concluding Remarks

What we have done

- We have presented a mathematical theory together with a design method for the well-known PID controller of a basic class of second order nonlinear uncertain dynamical systems.
- We have investigated several related issues including global stabilization and asymptotic regulation.
- The PID design rules given in this paper is quite simple and is almost necessary for global stabilization.
- Both our theory and design methods demonstrate that the PID controller is indeed quite robust with respect to both the design parameters and the nonlinear uncertainties.

The limitations of PID

The above theoretical results may not be true in the following cases:

- The nonlinearity has a growth rate "faster" than linear growth. For example, $f(x_1, x_2) = (x_1^2 + x_2^2)^\delta$, where $\delta > \frac{1}{2}$.

- Systems described by differential equations of order ≥ 3.

$$\begin{cases} \dot{x}_1 &= x_2 \\ \cdots & \\ \dot{x}_n &= f(x_1, \cdots, x_n) + k_p e(t) + k_i \int_0^t e(s)ds + k_d \dot{e}(t) \end{cases} \tag{8}$$

even if $f(x_1, \cdots, x_n)$ is linear, known.

Reference

Zhao C, Guo L, 2017. To appear in 2017 IFAC World Congress.

Some generalizations

- Similar results can also be established for PID controlled nonlinear uncertain stochastic systems.
- Any $n-$dimensional nonlinear uncertain system of the form

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = f(x_1, x_2) + u \\ u = k_p e(t) + k_i \int_0^t e(s)ds + k_d \dot{e}(t) \end{cases}$$

can be stabilized globally by PID, as long as the nonlinearity satisfies a global Lipschitz condition.

References

- Cong X R, Guo L, 2017. Submitted to 56th IEEE-CDC, 2017.
- Zhao C, Guo L, 2017. To appear in 2017 IFAC World Congress.

Some future problems

- To extend the results and methods on PID to more general nonlinear uncertain systems.

- To improve the existing results on control of uncertain nonlinear systems in the literature, by either improving the structure of PID or using the analytical methods developed here.

- To investigate under what additional conditions, the Jacobian Conjecture is also true for high-dimensional systems.

- To consider more complicated situations such as time-delayed inputs and sampled-data PID controllers under a prescribed sampling rate, and to connect the related boundaries established for the maximum capability of the general feedback mechanism.

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THANK YOU!

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