Pinning Control and Controllability of Complex Networks

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Dedicated to the Memory of

Rudolf E Kalman
(1930-5-19 — 2016-7-3)
Motivational Examples
Example:

C. elegans

In its Neural Network:

Neurons: 300~500   Synapses: 2500~7000
The worm Caenorhabditis elegans has 297 nerve cells. The neurons switch one another on or off, and, making 2345 connections among themselves. They form a network that stretches through the nematode’s millimeter-long body.

“How many neurons would you have to commandeer to control the network with complete precision?”

The answer is, on average: 49


Here, control = stimuli
Another Example

“... very few individuals (approximately 5%) within honeybee swarms can guide the group to a new nest site.”


These 5% of bees can be considered as “controlling” or “controlled” agents

**Leader-Followers network**
Given a network of identical dynamical systems (e.g., ODEs)

Given a specific control objective (e.g., synchronization)

Assume: a certain class of controllers (e.g., local linear state-feedback controllers) have been chosen to use
Questions:

Objective: To achieve the control goal with good performance

- How many controllers to use?
- Where to put them? (which nodes to “pin”)

--- “Pinning Control”

\[
\frac{dx_i}{dt} = f(x_i), \quad x_i \in \mathbb{R}^n
\]

\[
u_i = -H_i x_i
\]
Network Model

Linearly coupled network:

\[
\frac{dx_i}{dt} = f(x_i) + c \sum_{j=1}^{N} a_{ij} H x_j \quad x_i \in \mathbb{R}^n \quad i = 1,2,\ldots,N
\]

- a general assumption is that \(f(.)\) is Lipschitz
- coupling strength \(c > 0\) and \(H\) - input matrix
- coupling matrices (undirected):

\[
A = [a_{ij}]_{N \times N}
\]

\(A\): If node \(i\) connects to node \(j\) \((j \neq i)\), then \(a_{ij} = a_{ji} = 1\); else, \(a_{ij} = a_{ji} = 0\);
and \(a_{ii} = d_i\) where \(d_i\) - degree of node \(i\)

Note: For undirected networks, \(A\) is symmetrical; for directed networks, it is not so
What kind of controllers? How many? Where?

\[
\frac{dx_i}{dt} = f(x_i) + c \sum_{j=1}^{N} a_{ij} Hx_j \quad \Leftarrow \quad + u_i \quad i = 1,2,\ldots, N
\]

\[
(u_i = -\Gamma x_i)
\]

\[
\frac{dx_i}{dt} = f(x_i) + c \sum_{j=1}^{N} a_{ij} Hx_j - \delta_i \Gamma x_i \quad i = 1,2,\ldots, N
\]

\[
\delta_i = \begin{cases} 
1 & \text{if to - control} \\
0 & \text{if not - control} 
\end{cases}
\]

Q: How many \( \delta_i = 1 \)? Which \( i \)?
Pinning Control: Our Research Progress


Controllability Theory
MATHEMATICAL DESCRIPTION OF LINEAR DYNAMICAL SYSTEMS*

R. E. KALMAN†

Abstract. There are two different ways of describing dynamical systems: (i) by means of state variables and (ii) by input/output relations. The first method may be regarded as an axiomatization of Newton’s laws of mechanics and is taken to be the basic definition of a system.

It is then shown (in the linear case) that the input/output relations determine only one part of a system, that which is completely observable and completely controllable. Using the theory of controllability and observability, methods are given for calculating irreducible realizations of a given impulse-response matrix. In particular, an explicit procedure is given to determine the minimal number of state variables necessary to realize a given transfer-function matrix. Difficulties arising from the use of reducible realizations are discussed briefly.
State Controllability

Linear Time-Invariant (LTI) system

\[ \dot{x}(t) = Ax(t) + Bu(t) \]

\( x \in \mathbb{R}^n \) : state vector

\( u \in \mathbb{R}^p \) : control input

\( A \in \mathbb{R}^{n \times n} \) : state matrix

\( B \in \mathbb{R}^{n \times p} \) : control input matrix

**[Concept] State Controllable:**

The system orbit can be driven by an input from any initial state to the origin in finite time

State Controllability Theorems

\[ \dot{x}(t) = Ax(t) + Bu(t) \]

(i) Kalman Rank Criterion

The controllability matrix \( Q \) has full row rank:

\[ Q = [B\ AB\ \cdots\ A^{n-1}B] \]

(ii) Popov-Belevitch-Hautus (PBH) Test

The following relationship holds:

\[ v^T A = \lambda v^T, \quad v^T B \neq 0 \]

\( \lambda \) : eigenvalue of \( A \)

\( v \) : nonzero left eigenvector with \( \lambda \)
What about networks? -- Some earlier attempts

- **Leader-follower multi-agent systems**
  H.G. Tanner, *CDC*, 2004

- **Pinning state-controllability of complex networks**

- **Structural controllability of complex networks**
Structural Controllability

A network of single-input/single-output (SISO) node systems, where the node systems can be of higher-dimensional
In the controllability matrix $Q$:

$$Q = [B \ AB \ \cdots \ A^{n-1} B]$$

All 0 are fixed

There is a realization of independent nonzero parameters such that $Q$ has full row-rank

Example 1:

$$Q = \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}$$

Realization: All admissible parameters

$a \neq 0, \ d \neq 0$

Example 2: Frobenius Canonical Form

$$Q = \begin{bmatrix} -a_1 & -a_2 & -a_3 & \cdots & -a_{n-1} & -a_n \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{bmatrix}$$
Examples: Structure matters

\[ C = [B, A \cdot B, A^2 \cdot B] \]

\[
\begin{bmatrix}
1 & 0 & 0 \\
0 & a_{21} & 0 \\
0 & 0 & a_{32}a_{21}
\end{bmatrix}, \quad \begin{bmatrix}
1 & 0 & 0 \\
0 & a_{21} & 0 \\
0 & a_{31} & 0
\end{bmatrix}, \quad \begin{bmatrix}
1 & 0 & 0 \\
0 & a_{21} & 0 \\
0 & a_{31} & a_{33}a_{31}
\end{bmatrix}, \quad \begin{bmatrix}
1 & 0 & 0 \\
0 & a_{21} & a_{23}a_{31} \\
0 & a_{31} & a_{32}a_{21}
\end{bmatrix}
\]

\[
\text{rank } C = 3 = n \quad \text{rank } C = 2 < n = 3 \quad \text{rank } C = 3 = n \quad \text{rank } C = ?
\]

controllable \quad \text{uncontrollable} \quad \text{controllable} \quad \text{controllable?}

Partially controllable \quad \text{Structurally controllable}

In retrospect: large-scale systems theory

Structural Controllability (and Structural Observability)

Cactus is the minimum structure which contains no inaccessible nodes and no dilations.
Structural Controllability Theorem

The following two criteria are equivalent:

1. Algebraic:
   The LTI control system $(A,B)$ is structurally controllable

2. Geometric:
   The digraph $G(A,B)$ is spanned by a cactus

Matching in Directed Networks

- **Matching**: a set of directed edges without common heads and tails
- **Unmatched node**: the tail node of a matching edge

Maximum matching:
Cannot be extended

Perfect matching:
All nodes are matched nodes

← Maximum but not perfect matching
Minimum Inputs Theorem

Q: How many?
A: The minimum number of inputs $N_D$ needed is:

Case 1: If there is a perfect matching, then
$N_D = 1$

Case 2: If there is no perfect matching, then
$N_D =$ number of unmatched nodes

Q: Where to put them?
A: Case 1: Anywhere
Case 2: At unmatched nodes

State Controllability

A network of multi-input/multi-output (MIMO) node systems, where the node systems are of higher-dimensional
Some Earlier Progress

Consider a network of $N$ identical discrete-time LTI node-systems, with the $i$ th ($i = 1, 2, \ldots, N$) sub-system

$$
\begin{bmatrix}
  x(t+1,i) \\
  z(t,i) \\
  y(t,i)
\end{bmatrix} = 
\begin{bmatrix}
  A_{TT}(i) & A_{TS}(i) & B_T(i) & 0 \\
  A_{ST}(i) & A_{SS}(i) & B_S(i) & 0 \\
  C_T(i) & C_S(i) & D_d(i) & D_w(i)
\end{bmatrix}
\begin{bmatrix}
  x(t,i) \\
  v(t,i) \\
  d(t,i) \\
  w(t,i)
\end{bmatrix}
$$

where $x(t)$ – state; $y(t)$ – observation; $d(t)$ – disturbance; $w(t)$ – noise;

$A_{*,\#} = \text{col}\left\{A_{*,\#}(i) \mid i = 1, \ldots, N\right\}$ \hspace{1cm} $B_{*} = \text{diag}\left\{B_{*}(i) \mid i = 1, \ldots, N\right\}$ \hspace{1cm} $C_{*} = \text{diag}\left\{C_{*}(i) \mid i = 1, \ldots, N\right\}$

in which $*, \# = T$ or $S$ \hspace{1cm} (Note: All nodes are subject to control input)
**Result:** Assume that all the transfer function matrices $\overline{G}_i^{[1]} |_{\lambda = 1}$ of the network have full column normal rank. Then, the network is controllable if and only if for every $k \in \{1, 2, \ldots, m\}$, where $m$ is the number of distinctive transmission zeros of $G^{[1]}(\lambda)$, and for every $\bar{y}^{[k]} \in \bar{Y}^{[k]}$, one has $\Phi^T \overline{G}^{[2]}(\lambda_0^{[k]}) \bar{y}^{[k]} \neq \bar{y}^{[k]}$.

Here, $\Phi$ is the transfer matrix in $z(t) = \Phi v(t)$ and, for $i = 1, 2, \ldots, N$,

$G_i^{[1]}(\lambda) = C_S(i) + C_T(i) [\lambda I - A_{JT}(i)]^{-1} A_{TS}(i), \quad G_i^{[2]}(\lambda) = A_{ss}(i) + A_{st}(i) [\lambda I - A_{TT}(i)]^{-1} A_{TS}(i)$

$\bar{Y}^{[k]} = \left\{ \begin{array}{l}
\bar{y} = \text{col} \left\{ \begin{array}{c}
0_{mz(\tilde{k}(i)+1)}, \ldots, 0_{mz(\tilde{k}(i+1)-1)} , \\
\bar{y}^{[k]}_{i+1,0}, \bar{y}^{[k]}_{i+1,0} \vdots , 0_{mz(\tilde{k}(3[k])+1)}, \ldots, 0_{mzN} \end{array} \right. \\
\bar{y}^{[k]}_{i,0} \in \bar{Y}^{[k]}_i, \ i = 1, 2, \ldots, s^{[k]}; \ y \neq 0
\end{array} \right\}$
A Network of Multi-Input/Multi-Output LTI Systems

Node system
\[ \dot{x}_i = Ax_i + Bu_i, \quad y_i = Cx_i, \quad x_i \in R^n, \quad y_i \in R^m, \quad u_i \in R^p \]

Networked system
\[ \dot{x}_i = Ax_i + \sum_{j=1}^{N} \beta_{ij} Hy_j, \quad y_i = Cx_i, \quad i = 1, 2, \ldots, N \]

Networked system with external control
\[ \dot{x}_i = Ax_i + \sum_{j=1}^{N} \beta_{ij} HCx_j + \delta_i Bu_i, \quad i = 1, 2, \ldots, N \]

\[ \delta_i = 1: \text{with external control} \quad \delta_i = 0: \text{without external control} \]

Some notations

Node system \((A,B,C)\)  
Network structure \(L = [\beta_{ij}] \in R^{N \times N}\)  
Coupling matrix \(H\)  
External control inputs \(\Delta = \text{diag}(\delta_1, \ldots, \delta_N)\)

Some counter-intuitive examples

Network structure

![Network structure diagram]

Node system

### Example

\[ L = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \]

\[ A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \]

\[ B = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \]

\[ C = \begin{bmatrix} 0 & 1 \end{bmatrix} \]

\[ H = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \]

- \(L\) is structurally controllable
- \((A,B)\) is controllable
- \((A,C)\) is observable
- The state is uncontrollable

Networked MIMO system

![Networked MIMO system diagram]
Some counter-intuitive examples

Network structure

Node system

Networked MIMO system

\[ \begin{align*}
L &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \\
A &= \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \\
B &= \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\
C &= \begin{bmatrix} 0 & 1 \end{bmatrix}
\end{align*} \]

\[ H = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \]

\( (A,B) \) is uncontrollable

\( (A,C) \) is observable

state controllable

coupling matrix \( H \)
is important

Hou B Y, Li X, Chen G (2016)
A Network of Multi-Input/Multi-Output LTI Systems

A necessary and sufficient condition

\[
\begin{align*}
\dot{x}_i &= Ax_i + \sum_{j=1}^{N} \beta_{ij} HCx_j + \sum_{k=1}^{s} \delta_{ik} Bu_k, & x_i &\in \mathbb{R}^n, \quad i = 1, \ldots N \\
y_l &= \sum_{j=1}^{N} m_{ij} Dx_j, & u_k &\in \mathbb{R}^p, \quad k = 1, \ldots s \\
L &= [\beta_{ij}] \in \mathbb{R}^{N \times N}, \quad \Delta = [\delta_{ij}] \in \mathbb{R}^{N \times s}
\end{align*}
\]

State Controllable

If and only if

\[
\Delta^T XB = 0, \quad L^T XHC = X(\lambda \mathbf{I} - A)
\]

has a unique solution \(X = 0\)

A network with SISO nodes is **controllable** if and only if

(A,H) is controllable,

(A,C) is observable,

for any \( s \in \sigma(A) \) and \( \alpha \in \Gamma(s), \alpha L \neq 0 \) if \( \alpha \neq 0 \),

for any \( s \notin \sigma(A) \), \( \text{rank}(I - L\gamma, \Delta \eta) = N \), with \( \gamma = C(sI - A)^{-1}H, \eta = C(sI - A)^{-1}B. \)
Some most recent progress
Temporally Switching Networks

Adjacency matrix:

\[ [A_k]_{ji} = a_{ji}(k) \begin{cases} 
\neq 0, & \text{edge}(i, j, [t_{k-1}, t_k]) \neq \emptyset \\
= 0, & \text{otherwise} 
\end{cases} \]

\(a_{ji}\) are constants, but appear and disappear in a temporal manner.

Network topology is temporally switching.
State Controllability of Temporally Switching Systems

Temporally Switching Systems

\[ \dot{x}(t) = A(t)x(t) + Bu(t), \quad x(t_0) = x_0 \]
\[ x(t) \in \mathbb{R}^n, \quad u(t) \in \mathbb{R}^r, \quad B \in \mathbb{R}^{n \times r} \]
\[ A(t) \in \mathbb{R}^{n \times n} \text{ is piecewise constant} \]
\[ (A(t), B) \text{ can be described by matrix pair } (A_i, B) \]
when \( t \) belongs to \([t_{i-1}, t_i)\)

State Controllability:

any given initial state \( x(t_0) = x_0 \)

input signal \( u(\cdot) \)
defined on \([t_0, t_1]\)

final state \( x(t_m) = 0 \)

Necessary and Sufficient Condition

State Controllable \( \iff \)

Controllability matrix

\[ C = (e^{A_m(t-t_{m-1})} \ldots e^{A_2(t_2-t_1)}C_1, \ldots, e^{A_m(t-t_{m-1})}C_{m-1}, C_m) \]

has full rank, where \( C_i = (A_i^{n-1}B, \ldots, A_iB, B) \)
Structural Controllability of Temporally Switching Networks

Temporally Switching Systems
\[ \dot{x}(t) = A(t)x(t) + Bu(t), \quad x(t_0) = x_0 \]
\[ x(t) \in \mathbb{R}^n, \quad u(t) \in \mathbb{R}^r, \quad B \in \mathbb{R}^{n \times r} \]
\[ A(t) \in \mathbb{R}^{n \times n} \text{ is piecewise constant} \]
\[ (A(t), B) \text{ can be described by matrix pair } (A_i, B) \]
\[ \text{when } t \text{ belongs to } [t_{i-1}, t_i) \]

Structural Controllability: There exist a set of parameter values such that

any given initial state \( x(t_0) = x_0 \)

input signal \( u(\cdot) \)

defined on \([t_0, t_1]\)

final state \( x(t_m) = 0 \)

---

Necessary and Sufficient Condition

Structural Controllability if and only if

Controllability matrix
\[ C = (e^{A_m(t-t_{m-1})} \ldots e^{A_2(t_2-t_1)} C_1, \ldots, e^{A_m(t-t_{m-1})} C_{m-1}, C_m) \]

has full rank for some set of parameter values
Research Outlook

General networks of linear time-varying (LTV) node-systems

General networks of non-identical node-systems

General temporal networks of LTI or LTV node-systems

Some special types of networks of nonlinear node-systems

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There are more, of course
Thanks
• Motter, A E, Networkcontrology, Chaos, 25: 097621, 2015