Robust Internal Models for Nonlinear Output Regulation with Uncertain Exosystems

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Abstract: This paper presents a nonlinear internal model construction method by introducing certain coined Luenberger observer-like trial samples and further using Lyapunov's auxiliary theorem for confirmation. Specifically, it comes up with several novel types of parameterized nonlinear internal models with certain output in a constructive fashion, distinguishing themselves from existing ones proposed in the literature. In particular, they are most effective in tackling non-adaptive output regulation with uncertain exosystems, *i.e.*, stabilizing control for the resulting augmented system can be treated independent of any adaptive control law. For an exemplary application, an output consensus control problem is proven solvable for multi-agent systems with uncertain leaders/exosystems under general directed communication topologies. As a major consequence, the hurdles can be circumvented arising in the same problem if the conventional canonical internal model were used.

Key Words: Nonlinear control; regulation theory; internal model; robust stabilization.

1 Introduction

Output regulation, or called generalized servomechanism, aims at addressing tracking control and/or disturbance rejection for a composite system, to say, a basic single-input single-output scenario, described by a set of equations

$$\begin{cases} \dot{x} = f(x, u, v, w),\ \dot{w} = 0, \\ e = h(x, u, v, w),\ \dot{v} = S(\sigma)v,\ \dot{\sigma} = 0 \end{cases} \tag{1}$$

with the plant state $x \in \mathbb{R}^n$, the exosystem state $v \in \mathbb{R}^{n_v}$, the control input $u \in \mathbb{R}$, the regulated output (or tracking error) $e \in \mathbb{R}$, uncertain parameters (or real parametric uncertainties) $w \in \mathbb{W} \subset \mathbb{R}^{n_w}$ of the plant and $\sigma \in \mathbb{S} \subset \mathbb{R}^{n_\sigma}$ of the exosystem. For more background materials, we shall refer to [9] for a landmark paper on the topic and just to name but a few, [6, 10] and references therein for a brief overview of up-to-date studies on nonlinear output regulation.

It is known that for robust error output feedback control, the internal model is indispensable and plays a central role in the problem; see [7] for the general framework and characterization of steady-state generators and internal models. The internal model is essentially recognized as a dynamic compensator. It quite affects effective problem conversions from output regulation to stabilization of the so-called augmented system, composed of the plant dynamics and the internal model. Therefore, internal model design is a most crucial issue for tackling nonlinear output regulation. In the literature, the underlying idea of designing internal model is to seek a steady-state generator shaping an internal model candidate. For example, when the input feedforward function is a finite number of harmonics, it can be well established thanks to the equivalence to its characteristic polynomial as addressed in [4]. In this way, it builds up the canonical internal model; see [20]. However, when the frequencies in question are unknown, it merely gives a steady-state generator with *uncertain* output and the internal model as well (see Definition 2.1 of this paper). As a result, stabilization of the augmented system should be done by means of incorporating suitable adaptive control techniques. Such a design refers to adaptive internal model approach or adaptive output regulation in literature.

Considerable interest in adaptive output regulation has extensively developed over the past few years for a number of distinguished scenarios using such canonical internal models, pioneered in [20] and thereafter in [2, 16, 23, 28] for a broad range of normal-form nonlinear systems, in [19] for linear systems, and in [24] for a bi-directed multi-agent network. However, it would bring formidable obstacles in at least three situations: *(i)* it would not be applicable to local robust output regulation design and thus prevents its application to more systems; *(ii)* it remarkably complicates the auxiliary recursive as well as distributed stabilization design; see, e.g., [2, 25] for general lower triangular systems control and [26] for multi-agent systems control under general communication topologies; *(iii)* the involved adaptive control itself may increase difficulties for robust output regulation design, including the parameter convergence question; see [16, 23]. These factors motivate us to develop new techniques on constructive internal model design giving rise to much more flexible or tractable augmented systems.

One main objective of this paper is to explore robust internal models with the idea that: *Some trial parameterized servo compensators may indicate steady-state generators with certain output to shape internal models with certain output.* The investigation is substantially inspired by an observer design technique developed in [12] using Lyapunov's auxiliary theorem. For different control objectives and control goals, we aim to discover useful compensators, modified from [13], to establish potential steady-state generators and eventually construct useful internal models with certain output.

Contribution. The main contribution of the present study is three-fold. Firstly, a systematic approach is proposed toward designing the internal model with certain output. Secondly, a couple of novel nonlinear internal models are proposed serving the non-adaptive output regulation design. The internal model at issue is actually made to operate the parameter adaptation as an inner process, preventing adaptive stabilization for the augmented systems. Thirdly,

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as an application, the proposed internal model enables us to confirm solvability of an interesting consensus control problem with uncertain leaders and under general directed communication topologies. It is a distinguished scenario in contrast to distributed control problems such as [3, 15, 24, 26] and a decentralized one addressed in [28]. To the best of our knowledge, such problems have rarely been addressed.

Prior Work. Recently, a few notable internal model design techniques have been proposed serving such a non-adaptive output regulation design; see [6] for an overview and also refer to references therein for more results on robust output regulation, especially the internal model design techniques. In particular, [21] addressed a robust internal model for a linear output regulation problem without using any adaptive control law. [10] revealed that any finite number of harmonics can be generated by a uniformly observable steady-state generator with certain output and it was further applied to solve a semi-global output regulation problem thereof. [25, 27] proposed some nonlinear internal model design techniques based on translated steady-state generators for global robust output regulation problems. In contrast to those results, the finding of new internal models is from a different perspective of Lyapunov's auxiliary theorem.

Outline. After this section, Section 2 formulates the problem of robust output regulation, gives a definition of the concerned internal model, and shows motivating materials of the present study. Section 3 addresses a new systematic approach on constructing nonlinear internal models with certain output. Section 4 shows solvability of a consensus control problem by applying the proposed internal model. Section 5 closes this paper.

Terminology. $\|\cdot\|$ is the Euclidean norm. I_n is the n-dimensional identity matrix for a positive integer n. For a square matrix $A \in \mathbb{R}^{n\times n}$, $[\cdot]^{\textbf{vec}}$ stands for the column stacking operator and $[\cdot]^{\textbf{mat}}$ is its inverse, *i.e.*, $[A]^{\textbf{vec}} = [A_1^\intercal, \cdots, A_n^\intercal]^\intercal$ and $[[A]^{\textbf{vec}}]^{\textbf{mat}} = A$, where A_i is the ith column vector of A for $1 \le i \le n$ and the symbol $^\intercal$ is matrix transpose. The symbol $\otimes$ denotes the Kronecker product.

Due to page limit, all the relevant proofs and simulation results are omitted in this paper.

2 Formulation & Background

Consider the composite system (1). Suppose the exosystem therein is capable of generating trigonometric functions of the sum form

$$\sum\nolimits_{i=1}^{\ell} \Omega_i \sin(\sigma_i t + \phi_i) \tag{2}$$

with uncertain parameters $\Omega_i, \sigma_i, \phi_i \in \mathbb{R}$ for $1 \le i \le \ell$ and an integer $\ell > 0$, relying on initial conditions and uncertainties of (1). The sum function (2) is general enough to model or approximate most periodic references and disturbances. It relates to a fundamental question in systems and control to cope with trigonometric functions with unknown amplitudes, frequencies and phases. Thus, it is of great interest in tracking or consensus control. From this aspect, we use the following standing assumption.

Assumption 1 For the exosystem, $S(\sigma)$ has distinct eigenvalues lying on the imaginary axis for each $\sigma \in \mathbb{S}$. The initial condition $v(0) \in \mathbb{V} \subset \mathbb{R}^{n_v}$ where $\mathbb{V}$ is a given invariant compact set for the exosystem.

2.1 Problem of Robust Output Regulation

For the composite system (1), *the global robust output regulation is to seek a smooth error feedback controller*

$$\dot{x}_c = f_c(x_c, e),\ u = h_c(x_c, e) \tag{3}$$

such that, for each $(v(0), \sigma, w) \in \mathbb{D} := \mathbb{V} \times \mathbb{S} \times \mathbb{W}$ *and for each of the other initial conditions, (i) the trajectory of the closed-loop system, composed of (1) and (3), exists for all* $t \ge 0$ *and is bounded over the time interval* $[0, +\infty)$*; (ii) the regulated output satisfies* $\lim_{t\to+\infty} e(t) = 0$.

Recall that to handle the robust output regulation, a d-dimensional internal model

$$\dot{\eta} = \gamma(\eta, u),\ \eta \in \mathbb{R}^d \tag{4}$$

acting as a dynamic compensator embedded in the controller (3), is basically indispensable, *cf.* [5, Lemma 1.21]. On the other hand, stabilization of the augmented system composed of (1) and (4) would be much complicated due to system augmentation. In view of these aspects, our chief objective is to develop nonlinear internal models enabling us to pursue non-adaptive output regulation. Also, the latter result will be demonstrated by a consensus control problem as a significant extension of that addressed in [26].

2.2 Definition & Background

To show the background of the present study, let us begin with a standing assumption on solvability of the so-called regulator equations and a popular internal model design condition; see, e.g., [4, 9] for more details.

Assumption 2 For the system (1), there are smooth functions $\mathbf{x} : \mathbb{D} \to \mathbb{R}^n$ and $\mathbf{u} : \mathbb{D} \to \mathbb{R}$ such that, for all $\mu \in \mathbb{D}$,

$$\begin{aligned}&[\partial \mathbf{x}(\mu)/\partial v] S(\sigma) v = f(\mathbf{x}(\mu), \mathbf{u}(\mu), v, w),\\ &0 = h(\mathbf{x}(\mu), \mathbf{u}(\mu), v, w),\ \mu(t) := [v^\intercal(t), \sigma^\intercal, w^\intercal]^\intercal\end{aligned} \tag{5}$$

Moreover, $\mathbf{u}(\mu) = \mathbf{u}(v, \sigma, w)$ is polynomial in v.

Definition 2.1 For the system (1), a dynamic compensator of the form (4) is called *an internal model with (linearly parameterized) uncertain output* u if, there exist a triple of well-defined smooth functions $\theta : \mathbb{D} \to \mathbb{R}^s$, $\Gamma : \mathbb{D} \to \mathbb{R}^{\bar{s}}$, $\hbar : \mathbb{S} \to \mathbb{R}^{1\times \bar{s}}$ for integers $s, \bar{s} > 0$, such that, for all $\mu \in \mathbb{D}$,

$$[\partial \theta(\mu)/\partial v] S(\sigma) v = \gamma(\theta(\mu), \mathbf{u}(\mu)),\ \mathbf{u}(\mu) = \hbar(\sigma)\Gamma(\theta(\mu)). \tag{6}$$

Moreover, it is called *an internal model with certain output* u if, instead of (6), there exist a pair of well-defined smooth functions $\theta : \mathbb{D} \to \mathbb{R}^s$, $\bar{\Gamma} : \mathbb{D} \to \mathbb{R}$ such that, for all $\mu \in \mathbb{D}$,

$$[\partial \theta(\mu)/\partial v] S(\sigma) v = \gamma(\theta(\mu), \mathbf{u}(\mu)),\ \mathbf{u}(\mu) = \bar{\Gamma}(\theta(\mu)). \tag{7}$$

The system (6) is called a *steady-state generator with uncertain output* u. In contrast, the system (7) is called a *steady-state generator with certain output* u.

Remark 2.1 The preceding definition is given in the same spirit of [5, Definition 6.2] to characterize the internal model candidates of interest in the present study. The output of the system (6) or (7) formulates the *necessary* input information to assure asymptotic convergence of tracking error $e(t)$ in

(1). The mapping $\hbar(\sigma)\Gamma(\theta)$ or $\bar{\Gamma}(\theta)$ offers design output of the internal model. Because the required steady-state input $\mathbf{u}(\mu)$ is not available, it will be eventually duplicated by the output of an effective internal model (4). In fact, one can always split up the control input u of (1) by the equation[1]

$$u(t) = \hat{u}(t) + \bar{u}(t) \text{ with } \hat{u}(t) = \hat{\Gamma}(\eta(t)) \tag{8}$$

where $\hat{\Gamma}(\cdot)$ *coincides* with the output mapping of (7) in the sense that $\hat{\Gamma}(\theta(\mu)) = \bar{\Gamma}(\theta(\mu))$ for all $\mu \in \mathbb{D}$. In the equation (8), following $\hat{u}(t)$ by the internal model design, the stabilization part $\bar{u}(t)$ is designed by the second-step stabilizing control for the augmented system.

Hence, what qualifies (4) for a well-defined internal model at issue is that the corresponding condition (6) or (7) is verifiable. On the other hand, characterization of the system like (6) or (7) guides construction of the internal model candidates. Compared with (6), the output mapping $\bar{\Gamma}(\cdot)$ in (7) is independent of any uncertainties. To be explained at the end of this section, this may quite facilitate and simplify the output regulation design. The focus of the present study is on the latter one as a basic criterion to seek new internal models.

In the rest of this section, we shall review the canonical linear internal model design of (11) in the literature. Then we explain why the internal model with certain output is much more advantageous than those with uncertain outputs.

Firstly, it is known that, if $\mathbf{u}(\mu)$ satisfying (5) is a nonlinear polynomial in v, its *minimal* zeroing polynomial (see [5]) can be derived leading to a steady-state generator (that is essentially nonlinear when σ is unknown)

$$[\partial\tau(\mu)/\partial v]S(\sigma)v = \Phi(a)\tau(\mu),\ \dot{a} = 0,\ \mathbf{u}(\mu) = \Psi\tau(\mu) \tag{9}$$

with output u, where $\tau : \mathbb{D} \to \mathbb{R}^s$, $a : \mathbb{S} \to \mathbb{R}^s$ for an integer $s > 0$, and

$$\Phi(a) = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ -a_1(\sigma) & -a_2(\sigma) & \cdots & -a_s(\sigma) \end{bmatrix},\ \Psi = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}^{\mathsf{T}}.$$

Note that $\Phi(a)$ has distinct eigenvalues lying on the imaginary axis and $(\Psi, \Phi(a))$ is observable. For the generator (9) with a fixed region of interest, we denote $\mathbb{T}$ as the image of the set $\mathbb{D}$ under the map $\tau : \mathbb{D} \to \mathbb{R}^s$ and $\mathbb{A}$ as the image of $\mathbb{S}$ under the map $a : \mathbb{S} \to \mathbb{R}^s$.

Keep the above materials in mind. We next pick an s-dimensional controllable pair (M, N) and M is Hurwitz. Then the parameterized algebraic Sylvester equation $T(a)\Phi(a) = MT(a) + N\Psi$ has an invertible solution $T(a)$ by [17, Theorem 2], because (M, N) is controllable, $(\Psi, \Phi(a))$ is observable, and M and $\Phi(a)$ have no related eigenvalues in the sense that, for each $a \in \mathbb{R}^s$, the eigenvalues $\{\lambda_i\}_{i=1}^s$ of M and $\{\lambda'_i\}_{i=1}^s$ of $\Phi(a)$ are so that $\sum_{i=1}^s c_i\lambda_i \neq \lambda'_j$, $\forall c_1, \cdots, c_s > 0$, $\forall j : 1 \leq j \leq s$. It follows that $\theta(\tau, a) = T(a)\tau$ defines a similarity transformation for the system (9). Then we have another steady-state generator written by

$$[\partial\theta(\tau, a)/\partial\tau]\Phi(\sigma)\tau = M\theta(\tau, a) + N\Psi\tau,$$
$$\Psi\tau = \Psi T^{-1}(a)\theta(\tau, a) \tag{10}$$

with output u. As a result, the translated generator (10) immediately shapes the so-called canonical internal model

$$\dot{\eta} = M\eta + Nu,\ \eta \in \mathbb{R}^d \text{ with } d = s \tag{11}$$

that is with uncertain output in the sense of Definition 2.1.

If the design of (11) is adopted for output regulation design, then an adaptive design has to be done for the augmented system composed of (1) and (11) because of the uncertain function $\Psi T^{-1}(a)$ appearing in the output of (10) and therefore appearing in the control input (8); see, e.g., [16, 19, 20, 23] and references therein. Moreover, in distributed control problems such as [26], this adaptation will unfortunately bring serious obstacles, though it can be done in some special cases such as bi-directed multiple agent networks [24] and decentralized large-scale systems [28]. Thus, it is viable and of greater interest to develop internal models with *certain* output, leading to a relatively straightforward non-adaptive stabilization.

Now, we show in detail the technical superiority of internal models with certain output in tackling nonlinear output regulation. Following an internal model design of (4) for the composite system (1), we may write the augmented system by the equations

$$\begin{cases} \dot{x} = f(x, \hat{\Gamma}(\eta) + \bar{u}, v, w),\ \dot{\eta} = \gamma(\eta, \hat{\Gamma}(\eta) + \bar{u}), \\ e = h(x, \hat{\Gamma}(\eta) + \bar{u}, v, w),\ \dot{v} = S(\sigma)v \end{cases} \tag{12}$$

that is established by attaching (4) to (1), bringing the control input (8), and ignoring the trivial dynamics. The system (12) is autonomous. Set $\bar{u} \equiv 0$. It is observed that the unforced system

$$\dot{x} = f(x, \hat{\Gamma}(\eta), v, w),\ \dot{\eta} = \gamma(\eta, \hat{\Gamma}(\eta))$$

enjoys a zero-error manifold (see [8])

$$\big\{(x, \eta, \mu) \mid x = \mathbf{x}(\mu), \eta = \theta(\mu), \mu \in \mathbb{D}\big\}. \tag{13}$$

Moreover, within (13), it assures $h(\mathbf{x}(\mu), \hat{\Gamma}(\theta(\mu)), v, w) = 0$ for all $\mu(t) \in \mathbb{D}$. In consequence, to achieve the output regulation design, we can perform the second-step stabilizing control $\bar{u}(t)$ to make the zero-error manifold (13) for (12) attractive in some sense; see [8] for rigorous arguments.

Boosted in [7], the preceding stabilizing can be managed by making a suitably *translated* system of (12) robustly asymptotically stable at an equilibrium point with a domain of attraction covering a desired initial region. For example, the translated system may be described with certain error states or translated coordinates like $(\bar{x}, \bar{\eta}) = (x - \mathbf{x}, \eta - \theta)$ (see [5] for concrete problems). Then, roughly speaking, convergence of $(\bar{x}(t), \bar{\eta}(t)) \to (0, 0)$ as $t \to \infty$ by stabilizing control implies attractiveness of (13). If this stabilization can be done after the internal model design, then it solves the original output regulation problem and consequently validate the compensator in (12) to be an effective internal model. The above design framework offers to a problem conversion

[1] For any effective regulator (3), the total control input $u(t)$ can be split in two parts: the *compensation* one $\hat{u}(t)$ and the *stabilization* one $\bar{u}(t)$ in the sense that $\hat{u}(t)$ may not vanish while $\bar{u}(t)$ vanishes at the infinity.

Table 1: Exemplary Trial Servo Compensators

Symbol	Design Parameter	Form
$\mathcal{C}_0$	$\{M, N\}$	$\dot{\eta} = M\eta + Nu$ with $\eta \in \mathbb{R}^d$ (Canonical form)
$\mathcal{C}_1$	$\{M, N\}$	$\dot{\eta}_{\mathbf{a}} = M\eta_{\mathbf{a}} + Nu$, $\dot{\eta}_{\mathbf{b}} = -\eta_{\mathbf{a}}(\eta_{\mathbf{a}}^{\top}\eta_{\mathbf{b}} - u)$, with $\eta = [\eta_{\mathbf{a}}^{\top}, \eta_{\mathbf{b}}^{\top}]^{\top}$, $\eta_{\mathbf{a}} \in \mathbb{R}^s$, $\eta_{\mathbf{b}} \in \mathbb{R}^s$
$\mathcal{C}_2$	$\{M, N, A, B, C\}$	$\dot{\eta}_{\mathbf{a}} = M\eta_{\mathbf{a}} + NC\eta_{\mathbf{c}}$, $\dot{\eta}_{\mathbf{b}} = -\eta_{\mathbf{a}}(\eta_{\mathbf{a}}^{\top}\eta_{\mathbf{b}} - C\eta_{\mathbf{c}})$, $\dot{\eta}_{\mathbf{c}} = A\eta_{\mathbf{c}} + Bu$, with $\eta = [\eta_{\mathbf{a}}^{\top}, \eta_{\mathbf{b}}^{\top}, \eta_{\mathbf{c}}^{\top}]^{\top}$, $\eta_{\mathbf{a}}, \eta_{\mathbf{b}} \in \mathbb{R}^s$, $\eta_{\mathbf{c}} \in \mathbb{R}^{n_f}$
$\mathcal{C}_3$	$\{M, N, A, B, C\}$	$\dot{\eta}_{\mathbf{a}} = M\eta_{\mathbf{a}} + NC\eta_{\mathbf{e}}$, $\dot{\eta}_{\mathbf{b}} = -\eta_{\mathbf{b}} + [\eta_{\mathbf{a}}\eta_{\mathbf{a}}^{\top}]^{\mathbf{vec}}$, $\dot{\eta}_{\mathbf{c}} = -\eta_{\mathbf{c}} + \eta_{\mathbf{a}}\eta_{\mathbf{e}}$, $\dot{\eta}_{\mathbf{d}} = -[\eta_{\mathbf{b}}]^{\mathbf{mat}}\eta_{\mathbf{d}} + \eta_{\mathbf{c}}$, $\dot{\eta}_{\mathbf{e}} = A\eta_{\mathbf{e}} + Bu$, with $\eta = [\eta_{\mathbf{a}}^{\top}, \eta_{\mathbf{b}}^{\top}, \eta_{\mathbf{c}}^{\top}, \eta_{\mathbf{d}}^{\top}, \eta_{\mathbf{e}}^{\top}]^{\top}$, $\eta_{\mathbf{a}}, \eta_{\mathbf{c}}, \eta_{\mathbf{d}} \in \mathbb{R}^s$, $\eta_{\mathbf{b}} \in \mathbb{R}^{s^2}$, $\eta_{\mathbf{e}} \in \mathbb{R}^{n_f}$

from output regulation to more tractable stabilization control. Hence, by an internal model with certain output, *i.e.*, the above $\hat{\Gamma}(\cdot)$ is provided independent of any uncertainty, the latter stabilization can be more simplified than those using internal models with uncertain output like (11).

3 Design of Internal Models with Certain Output

We address a new internal model construction method by introducing a set of coined trial samples and further using Lyapunov's auxiliary theorem (see [12]) to do the confirmation according to Definition 2.1. In this way, it may come up with well-defined nonlinear internal models with certain output serving the non-adaptive output regulation.

3.1 Selecting/Coining Trial Servo Compensators

In the present study, we shall limit ourselves to some coined Luenberger observer-like servo compensators as trial samples, listed in Table 1, toward nonlinear internal model design in a constructive fashion. Specifically, the class $\mathcal{C}_0$ was originally introduced in [18] for linear systems and further addressed in [14] for nonlinear systems. $\mathcal{C}_1$ was adopted from [13]. Both $\mathcal{C}_2$ and $\mathcal{C}_3$ are introduced based on specific modifications of Kreisselmeier's parameterized observers proposed in [13].

Known from [14], note that the case $\mathcal{C}_0$ can be done generically by enhancing the system dimension. However, the calculation of $\bar{\Gamma}(\cdot)$ would be difficult and unfortunately, the appealing smoothness or even Lipschitz continuity would not be assured in general. This case is lack of interest in smooth stabilization of the augmented system and is beyond the scope of the present study.

In Table 1, viewed as design parameters, (A, B, C) is any n_f-dimensional controllable and observable triple and (M, N) is any s-dimensional controllable pair. In the present study and for simplicity, we may set $(A, B, C) = (-1, 1, 1)$,

$$M = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ -m_1 & -m_2 & \cdots & -m_s \end{bmatrix},\ N = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

with $m = [m_1, \cdots, m_s]^{\top}$ given such that M is Hurwitz.

3.2 Internal Model Construction

Let us now carry out the internal model design. Loosely speaking, our main task is to examine the compensators in

Table 2: Confirming Internal Models with Certain Output

STEP 1	Select a sample	$\dot{\eta} = \gamma(\eta, u),\ \eta \in \mathbb{R}^d$
STEP 2	Solve the equations:	$[\partial\theta(\tau, a)/\partial\tau]\Phi(a)\tau = \gamma(\theta(\tau, a), \Psi\tau)$, $\Psi\tau = \bar{\Gamma}(\theta(\tau, a))$ with $(\tau, a) \in \mathbb{T} \times \mathbb{A}$
	– If $\{\theta, \bar{\Gamma}\}$ are well-defined, – Otherwise,	the internal model is confirmed. END. turn to STEP 1.

Table 1 and to verify the respective conditions in Table 2 by adjusting the design parameters. It determines the internal model of interest according to Definition 2.1 by a two-step procedure as a method of trial and error.

Note that one feasible design for the case $\mathcal{C}_0$ of Table 1 can be referred to Section 2.2 for an internal model with uncertain output by letting $d = s$. For the design of an internal model with certain output, $\mathcal{C}_0$ may be confirmed, not shown here in detail, by [14, Theorem 2] for a large integer $d > s$. Associated with $\mathcal{C}_3$, it relates to the set of equations

$$\begin{aligned}
[\partial\theta_{\mathbf{a}}(\tau, a)/\partial\tau]\Phi(a)\tau &= M\theta_{\mathbf{a}}(\tau, a) + N\theta_{\mathbf{e}}(\tau, a), \qquad (14)\\
[\partial\theta_{\mathbf{b}}(\tau, a)/\partial\tau]\Phi(a)\tau &= -\theta_{\mathbf{b}}(\tau, a) + [\theta_{\mathbf{a}}(\tau, a)\theta_{\mathbf{a}}^{\top}(\tau, a)]^{\mathbf{vec}},\\
[\partial\theta_{\mathbf{c}}(\tau, a)/\partial\tau]\Phi(a)\tau &= -\theta_{\mathbf{c}}(\tau, a) + \theta_{\mathbf{a}}(\tau, a)\theta_{\mathbf{e}}(\tau, a),\\
0 &= -[\theta_{\mathbf{b}}(\tau, a)]^{\mathbf{mat}}\theta_{\mathbf{d}}(a) + \theta_{\mathbf{c}}(\tau, a),\\
[\partial\theta_{\mathbf{e}}(\tau, a)/\partial\tau]\Phi(a)\tau &= -\theta_{\mathbf{e}}(\tau, a) + \Psi\tau,\ \Psi\tau = \bar{\Gamma}(\theta(\tau, a))
\end{aligned}$$

with $\theta(\tau, a) := [\theta_{\mathbf{a}}^{\top}, \cdots, \theta_{\mathbf{e}}^{\top}]^{\top}$ for smooth maps $(\tau, a) \mapsto \theta_{\mathbf{a}}(\tau, a), \theta_{\mathbf{b}}(\tau, a), \theta_{\mathbf{c}}(\tau, a), \theta_{\mathbf{e}}(\tau, a)$, $a \mapsto \theta_{\mathbf{d}}(a)$, and $\theta \mapsto \bar{\Gamma}(\theta)$. It is worth noting that a *closed-form solution* of (14) can be derived by translating the PDEs into algebraic equations. As a major consequence, constructiveness of the internal model design is assured for all the cases $\mathcal{C}_1$ to $\mathcal{C}_3$. The main conclusion of this section is stated.

Theorem 3.1 *For the composite system (1) under Assumptions 1 & 2, each of $\mathcal{C}_0$ to $\mathcal{C}_3$ in Table 1 can be made an internal model with certain output u per Definition 2.1.*

Remark 3.1 The design method as shown in Tables 1 & 2 is tractable for a two-fold reason. Firstly, it is practically viable to establish effective trial candidates by taking into account of the necessary conditions of Lyapunov's auxiliary theorem and carefully adapting popular forms in system observation and identification, owning to their common characteristics. Secondly, the set of PDEs may reduce to certain algebraic equations. It indeed provides a tractable path and much flexibility for internal model construction.

Remark 3.2 The trial servo compensators in Table 1 have been coined inspired by the observation research of [13]. From the system observation viewpoint, the proposed internal models are conceptually closer to "output observers" because only the output function $\bar{\Gamma}(\cdot)$ is required in the internal model design problem. On the other hand, by a slight modification of equations in Table 2, we can impose a "stronger" condition, for suitable mappings $\{\theta, \bar{\Gamma}\}$,

$$[\partial\theta(\tau, a)/\partial\tau]\Phi(a)\tau = \gamma(\theta(\tau, a), \mathbf{u}(\mu)),\ \begin{bmatrix} \tau \\ a \end{bmatrix} = \bar{\Gamma}(\theta(\tau, a)). \qquad (15)$$

The condition (15) is exactly an observer design condition for (9) in the coordinate (τ, a); see [12, Definition 1] using Lyapunov's auxiliary theorem and refer to [14, Theorem 1] for a general result on solvability of (15). By contrast, the condition in Table 2 is never stronger than that of (15). Also note that instead of convergence analysis for error dynamics in observer design problems, a subsequent stabilization design should be performed in output regulation design problems or specifically for the resulting augmented systems. This is a crucial question to accomplish output regulation control goal. At this point, it is stressed that $\mathcal{C}_2$ and $\mathcal{C}_3$ in Table 1 are strictly extended from the standing observer forms proposed in [14]. With a consideration of rendering the aforementioned stabilization problem tractable, both $\mathcal{C}_2$ and $\mathcal{C}_3$ are coined to adapt output regulation design. This will be further explained by Remark 4.2.

4 Application to Output Consensus

This section is devoted to an exemplary application of the case $\mathcal{C}_3$ for an extended study of a global (distributed) robust output regulation problem addressed in [26] when the exosystems or leaders are unknown.

4.1 Problem Description

For a concrete investigation of the general setting (1), let us turn to a composite system consisting of p agents

$$\begin{cases} \dot{z}_i = F(w)z_i + \Delta_0(y_i, w), \\ \dot{y}_i = G(w)z_i + \Delta_1(y_i, w) + u_i, \\ e_i = y_i - q(v),\ 1 \le i \le p,\ \dot{v} = S(\sigma)v \end{cases} \tag{16}$$

where each agent or subsystem takes an output feedback normal form (see, e.g., [3, 16]). In (16), it is assumed that $F(w)$ is Hurwitz for each unknown parameter $w \in \mathbb{W}$ and functions Δ_0, Δ_1, q are *polynomials* in their argument(s).

The system (16) may be regarded as a leader-follower multi-agent nonlinear system. In contrast to that posed in (1), the exosystem here is named a leader to characterize the synchronous output $q(v)$ for a consensus control goal. It is now the situation that each local agent measurement $\chi_i := \chi_i(e)$ for $1 \le i \le p$ is given by

$$\chi = [\chi_1, \cdots, \chi_p]^\intercal = He \ \text{ with } \ e = [e_1, \cdots, e_p]^\intercal \tag{17}$$

for an *invertible* matrix $H \in \mathbb{R}^{p\times p}$ relating to general directed communication topologies. Indeed, H is stemmed from output interactions in the sense of [26, H1 & Remark 3.3] (*cf.* [24, Assumption 5], [3, Assumption 4], and [15, Assumption 3.2] and references thereof). Also note that there is a positive definite matrix $R = \mathrm{diag}(r_1, \cdots, r_p)$ such that $RH + H^\intercal R \ge I_p$.

For the above described system (16) with the agent-wise measurement (17), *the global robust distributed output consensus problem (GROC) is to seek a smooth controller*

$$\dot{\eta} = \gamma(\eta, u_\jmath),\ u_i = -\rho(\chi_i) + \Gamma_{\mathbf{cs}}(\eta),\ 1 \le i \le p \tag{18}$$

for an index $\jmath$ satisfying $0 \le \jmath \le p$ (namely, the host agent[2] *in line with [26]), that solves the relevant distributed non-adaptive output regulation without a prior of σ in (16).*

4.2 Output Consensus Design Using $\mathcal{C}_3$

To solve the GROC, it is easy to show that Assumption 2 is verifiable for each agent or subsystem of (16). Its regulator equations are solvable with a common solution. That is, they has the same regulator equations

$$\begin{aligned} [\partial \mathbf{z}(\mu)/\partial v]S(\sigma)v &= F(w)\mathbf{z}(\mu) + \Delta_0(q(v), w), \\ [\partial q(v)/\partial v]S(\sigma)v &= G(w)\mathbf{z}(\mu) + \Delta_1(q(v), w) + \mathbf{u}(\mu) \end{aligned} \tag{19}$$

with a solution $\{\mathbf{z}(\mu), \mathbf{u}(\mu)\}$. In particular, each steady-state input can be given by $\mathbf{u}(\mu) = [\partial q(v)/\partial v]S(\sigma)v - \Delta_1(\mathbf{z}(\mu), q(v), w)$ that is polynomial in v by assumption. Moreover, known from [4], for the function $\mathbf{u}(\mu)$, there is an integer $s^\star > 0$ such that $\mathbf{u}(\mu)$ can be expressed by $\mathbf{u}(\mu(t)) = \sum_{i=1}^{s^\star} C_i(v(0), \sigma, w)e^{\imath\hat{\omega}_i t}$ where $\imath$ is the imaginary unit and $\hat{\omega}_i$ for $1 \le i \le s^\star$ are distinct real numbers.

Assumption 3 The condition $(v(0), \sigma, w) \in \mathbb{D}$ is generic so that $C_i(v(0), \sigma, w) \ne 0$ for each $1 \le i \le s^\star$.

Remark 4.1 Assumption 3 is standing and the trivial case is excluded from the present problem. It is noted that Assumption 3 also assures a certain stabilizability property of the augmented system to be given by (22). A similar condition is often used to assure convergence in observation problems such as [13], understood as a persistency of excitation (PE) condition (see [22, pp. 265]).

In the following, we fix the host agent $\jmath = 1$ and employ the internal model $\mathcal{C}_3$. Further define a smooth and compactly supported[3] function $\bar{\Gamma}_{\mathbf{cs}}(\eta)$ satisfying

$$\bar{\Gamma}_{\mathbf{cs}}(\eta) = \bar{\Gamma}(\eta),\ \forall \eta \in \big\{\eta = \theta(\tau(\mu), a(\sigma)) \mid \mu \in \mathbb{D}\big\}. \tag{20}$$

In other words, $\bar{\Gamma}_{\mathbf{cs}}(\eta)$ should coincide with $\bar{\Gamma}(\eta)$. We are ready to establish a tractable augmented system by letting

$$\begin{aligned} &\bar{z} = [\bar{z}_1^\intercal, \cdots, \bar{z}_p^\intercal]^\intercal,\ \bar{z}_i = z_i - \mathbf{z}(\mu),\ 1 \le i \le p, \\ &\xi = \begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \\ \xi_4 \end{bmatrix} = \begin{bmatrix} \eta_{\mathbf{a}} - \theta_{\mathbf{a}}(\tau(\mu), a(\sigma)) \\ \eta_{\mathbf{b}} - \theta_{\mathbf{b}}(\tau(\mu), a(\sigma)) \\ \eta_{\mathbf{c}} - \theta_{\mathbf{c}}(\tau(\mu), a(\sigma)) \\ \eta_{\mathbf{e}} - \theta_{\mathbf{e}}(\tau(\mu), a(\sigma)) - e_1 \end{bmatrix}, \\ &\zeta = \eta_{\mathbf{d}} - \theta_{\mathbf{d}}(a(\sigma)),\ \chi = He, \\ &\bar{u} = [\bar{u}_1, \cdots, \bar{u}_p]^\intercal,\ \bar{u}_i = u_i - \bar{\Gamma}_{\mathbf{cs}}(\eta),\ 1 \le i \le p \end{aligned} \tag{21}$$

where $\{\theta_{\mathbf{a}}, \theta_{\mathbf{b}}, \theta_{\mathbf{c}}, \theta_{\mathbf{d}}, \theta_{\mathbf{e}}\}$ satisfy (14), $\mathbf{z}(\mu)$ satisfies (19), and χ is given by (17). For the augmented system composed of (16) and $\mathcal{C}_3$, using the new coordinates and input (21) we obtain a translated system described by

$$\begin{cases} \dot{\bar{z}} = \big[I_p \otimes F(w)\big]\bar{z} + f^a(\chi, \mu), \\ \dot{\xi} = \begin{bmatrix} M & 0 \\ 0 & -I_{s^2+s+1} \end{bmatrix} \xi + f^b(\bar{z}, \xi, \chi, \mu), \\ \dot{\zeta} = -[\theta_{\mathbf{b}}(\tau, a)]^{\mathbf{mat}}\zeta + f^c(\xi, \zeta, \mu), \\ \dot{\chi} = H\bar{u} + Hg(\bar{z}, \xi, \zeta, \chi, \mu) \end{cases} \tag{22}$$

for some smooth functions f^a, f^b, f^c, and g. The augmented system (22) enjoys an equilibrium $(\bar{z}, \xi, \zeta, \chi) = (0, 0, 0, 0)$ and moreover, for any $(\bar{z}, \xi, \chi, \mu)$, $g(\bar{z}, \xi, \zeta, \chi, \mu)$ is bounded in ζ due to the specific design of $\bar{\Gamma}_{\mathbf{cs}}(\cdot)$ by (20).

As a result, the GROC can be solved as long as the equilibrium $(\bar{z}, \xi, \zeta, \chi) = (0, 0, 0, 0)$ can be made globally robustly

[2]The case $\jmath = 0$ may be understood as the non-host agent design.

[3]A function is *compactly supported* if it is zero outside a compact set.

asymptotically stable by a *decentralized mode* controller of the form

$$\bar{u} = -\bar{\rho}(\chi) := -[\rho(\chi_1), \cdots, \rho(\chi_p)]^{\mathsf{T}},\ \rho : \mathbb{R} \to \mathbb{R}. \quad (23)$$

This problem conversion is effective and exactly achieved in the spirit of [7]. The main conclusion of this section is stated.

Theorem 4.1 *For the system (16) under Assumptions 1 & 3, the GROC is solvable by a controller of the form (18).*

Remark 4.2 Regarding the coordinate transformation (21), we shall note that, what makes the candidate $\mathcal{C}_1$ difficult to be performed for output regulation design is that derivation of an augmented system of the normal form like (22) having well-defined uniform relative degrees would be very difficult and not clear in most situations. Nevertheless, the modified candidates $\mathcal{C}_2$ and $\mathcal{C}_3$ can get through this obstacle in virtue of their interconnection structures.

On the other hand, the resulting augmented system (22) has both static and dynamic uncertainties (see [11]), $\mu(t)$ and $(\bar{z}(t), \xi(t), \zeta(t))$, respectively. It is should be noted that some of the dynamic uncertainties may merely satisfy certain integral input-to-state stability (ISS) (see [1]). The encountered globally stabilizing control for (22) by (23) is a distinguished scenario and much more challenging than the ISS case addressed in [26]. Nonetheless, solvability of this stabilization can still be assured due to a stronger integral ISS property of the proposed internal models, that is also why they are named robust internal models in this study. Hence, the stabilizing design (23) for (22) is the main body of the proof of Theorem 4.1.

5 Conclusion

Several novel types of robust nonlinear internal models with certain output have been proposed based on an approach of introducing certain coined trial samples and using Lyapunov's auxiliary theorem for confirmation. For an exemplary application, a distributed output regulation problem has been shown solvable.

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