Input-to-State Stabilization of Nonlinear Discrete-Time Systems with Event-Triggered Control

Pengpeng Zhang[1], Tengfei Liu[1]*, Zhong-Ping Jiang[1,2]

1. State Key Laboratory of Synthetical Automation for Process Industries, Northeastern University, Shenyang, 110004, China
E-mail: ppzhang@hotmail.com; tfliu@mail.neu.edu.cn

2. Tandon School of Engineering, New York University, 5 MetroTech Center, Brooklyn, NY 11201, USA
E-mail: zjiang@nyu.edu

Abstract: This paper studies the event-triggered control problem of nonlinear discrete-time systems in the presence of external disturbances. In particular, we focus on the case in which whether or not the system state is sampled at step k is determined by the feedback information at step $k-1$. This problem is motivated by practical applications, and the proposed solution drastically differs from the existing event-based control methods. One of the fundamental difficulties is caused by the existence of external disturbances in predicting $x(k)$ by using $x(k-1)$. In this paper, refined tools of input-to-state stability (ISS) and the nonlinear small-gain theorem are developed to estimate the influence of external disturbances, and an input-to-state stabilizing design is proposed to solve the event-triggered control problem.

Key Words: Event-Triggered Control, Nonlinear Discrete-Time Systems, Input-to-State Stabilization.

1 Introduction

Nowadays, practical control systems increasingly rely on the use of computers. The computer-controlled systems can be considered as sampled-data systems [1]. In order to improve the performance of the controlled system and reduce the waste of computing and communication resources, state-dependent event-triggered sampling has been proposed [2]. Compared with the traditional time-triggered control, the sampling time instants of event-triggered control are determined by state-dependent events. Early papers [2, 3] have shown the advantages of event-triggered control over traditional time-triggered control. See also [4, 5] for a literature review and tutorial.

In the past ten years, tremendous effort has been paid to the study of event-triggered control. The focus of this research area has moved from first-order systems to multivariable, high-order systems; see, e.g., [6–13] and the references therein. Most of these published papers deal with linear systems or simplified nonlinear models. In event-triggered control, threshold signals are usually employed to generate the events of data sampling. Constant threshold signals are used in earlier results, e.g., [14, 15], while in the recent results, e.g., [16–18], the threshold signals are designed to depend on the previously sampled data or the real-time state. Standard control design methods have been refined for event-triggered control. For instance, the concept of input-to-state stability (ISS) is used to characterize the robustness of the control system to sampling error, and the threshold signal is designed as a function of the real-time state of the controlled system [9, 19]. References [18, 20] present a Lyapunov approach to event-triggered control. Event-triggered control with disturbances has also been studied. For instance, reference [21] analyzes the robustness with respect to disturbances in a self-triggered implementation. An extended result is obtained in the reference [22] in which the proposed control strategy can guarantee exponential input-to-state stability of the closed-loop system with respect to disturbances. The event-triggered control problem of continuous-time nonlinear systems in the presence of external disturbances is discussed in [23]. The theoretical results have also been extended to the setting of distributed control and output-feedback control [24, 25]. For more recent work, see [26–28] and the references therein.

The main purpose of this paper is to study the event-triggered control problem for discrete-time nonlinear systems subject to external disturbances. In [29], event-triggered condition and self-triggered formulation are proposed for discrete-time systems without external disturbances by using the notion of ISS. Reference [8] proposes sensor-controller and controller-actuator event-triggering mechanisms, and also model-based periodic event-triggered control strategies to achieve $\mathcal{L}_2$ property gain with respect to external disturbances. In [30, 31], the event-triggered consensus problem of discrete-time multi-agent systems is investigated, and the triggering condition is designed based on the measurement error to ensure multi-agent consensus.

In this paper, we consider discrete-time systems with nonlinear uncertain dynamics and external disturbances. In view of practical applications, there is always a gap between event-triggering and actuator realization. To address this issue, this paper focuses on the case in which we only use the current state to determine the logics of data-sampling event at the next step. To the best of our knowledge, this problem has not been systematically studied in the past literature. The major contribution of this paper lies in a new event-triggering mechanism based on the refined tools of ISS [32, 33] and the nonlinear small-gain theorem [34–36] for discrete-time systems. With the proposed design, the closed-loop event-triggered system can be rendered ISS with the external disturbance as the input.

The rest of the paper is organized as follows. Section 2 gives the problem formulation of event-triggered control for discrete-time nonlinear systems. In Section 3, we present our main results on the event-triggered robust stabilization.

This work was supported in part by NSF grant ECCS-1230040, in part by NSFC grants 61374042, 61522305 and 61533007, in part by the Fundamental Research Funds for the Central Universities under Grants N130108001 and N140805001 in China, and in part by State Key Laboratory of Intelligent Control and Decision of Complex Systems.

Corresponding author: tfliu@mail.neu.edu.cn (Tengfei Liu)

We first consider the case of state-dependent disturbance, in which the disturbance magnitude depends on the state magnitude. A nonlinear small-gain design is proposed such that the closed-loop event-triggered system is robustly stable. Then, the case of state-independent disturbance is studied. In this second case, the event-triggered sampling mechanism is designed to be able to predict the influence of the external disturbance, and input-to-state stabilization is achieved. In Section 4, we employ an illustrative example to show the effectiveness of the theoretical results.

For completeness of the paper, some notations and definitions are given here. Recall that a function $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ is a $\mathcal{K}$-function if it is continuous, strictly increasing and $\alpha(0) = 0$; it is a $\mathcal{K}_\infty$-function if it is a $\mathcal{K}$-function and is unbounded. In this paper, $\gamma \circ \rho$ represents the composition function of γ and ρ. For $\gamma, \rho \in \mathcal{K}$, $\gamma \circ \rho < \mathrm{Id}$ means $\gamma(\rho(s)) < s$ for all $s > 0$. We use $\mathbb{Z}_+$ to denote the set of all nonnegative integers. For any vector $x \in \mathbb{R}^n$, $|x|$ represents its Euclidean norm.

2 Problem Formulation

Consider a class of discrete-time nonlinear systems with external disturbances:

$$x(k+1) - x(k) = f(x(k), u(k), d(k)), \quad k \in \mathbb{Z}_+ \tag{1}$$

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^m$ is the control input, $d \in \mathbb{R}^{n_d}$ represents external disturbances, and $f : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{n_d} \to \mathbb{R}^n$ is a continuous function representing the rate of change of state x.

We consider the state-feedback control law in the form of

$$u(k) = \kappa(x(k_i)), \quad k_i \le k < k_{i+1}, \quad i \in \mathbb{S} \tag{2}$$

where $\kappa : \mathbb{R}^n \to \mathbb{R}^m$ is a continuous function, $\{k_i\}_{i\in\mathbb{S}}$ is the sequence of sampling times with $\mathbb{S} = \{0, 1, 2, \cdots\} \subseteq \mathbb{Z}_+$ and $k_0 = 0$. In the problem setting of event-triggered control, $\mathbb{S}$ is determined by a state-dependent function corresponding to the events.

In practice, some response time is often required for data sampling after the triggering of an event. In this paper, we consider the case in which whether the sampling event is triggered at step k is determined by state $x(k-1)$. In particular, the event-triggering mechanism proposed in this paper is defined in the form of

$$k_{i+1} = \min\{k \ge k_i : \varphi(x(k), x(k_i)) > 0\} + 1 \tag{3}$$

where $\varphi : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is the triggering function, to be designed later.

Set

$$w(k) = x(k_i) - x(k), \quad k_i \le k < k_{i+1}, \quad i \in \mathbb{S} \tag{4}$$

as the error caused by event-triggered sampling.

By substituting (2) and (4) into (1), we obtain the closed-loop system

$$\begin{aligned} x(k+1) - x(k) &= f(x(k), \kappa(x(k) + w(k)), d(k)) \\ &=: \tilde{f}(x(k), w(k), d(k)), \end{aligned} \tag{5}$$

for which, the sampling error w and the external disturbance d are considered as the inputs. Without loss of generality, assume $\tilde{f}(0, 0, 0) = 0$.

This paper focuses on the design of a desired event-triggering mechanism. Like the work of others on event-based nonlinear control, we assume that system (1) already has a stabilizing control law (2), and that the closed-loop system (5) is ISS with respect to w and d.

Assumption 1. *System* (5) *is ISS with w and d as the inputs, and admits an ISS-Lyapunov function $V : \mathbb{R}^n \to \mathbb{R}_+$ satisfying the following conditions:*

1) There exist $\underline{\alpha}, \overline{\alpha} \in \mathcal{K}_\infty$ such that

$$\underline{\alpha}(|x|) \le V(x) \le \overline{\alpha}(|x|), \quad \forall x; \tag{6}$$

2) There exist functions $\alpha \in \mathcal{K}_\infty$ and $\gamma_w, \gamma_d \in \mathcal{K}$, such that

$$\begin{aligned} &V(\bar{f}(x, w, d)) - V(x) \\ &\le -\alpha(|x|) + \max\{\gamma_w(|w|), \gamma_d(|d|)\}, \quad \forall x, w, d \end{aligned} \tag{7}$$

where $\bar{f}(x, w, d) = \tilde{f}(x, w, d) + x$.

Without loss of generality, the following standing assumption is made on $\tilde{f}$.

Assumption 2. *There exist functions $\psi^x_{\tilde{f}}$, $\psi^w_{\tilde{f}}$, $\psi^d_{\tilde{f}} \in \mathcal{K}_\infty$ such that*

$$|\tilde{f}(x, w, d)| \le \psi^x_{\tilde{f}}(|x|) + \psi^w_{\tilde{f}}(|w|) + \psi^d_{\tilde{f}}(|d|) \tag{8}$$

for all $x \in \mathbb{R}^n$, $w \in \mathbb{R}^n$ and $d \in \mathbb{R}^{n_d}$.

According to the definition of $w(k)$ in (4), the closed-loop event-triggered system can be represented in the feedback form shown in Fig.1.

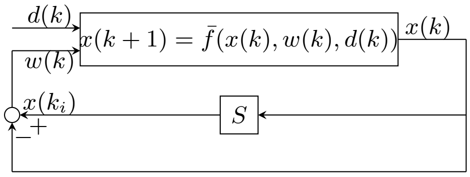

Fig. 1: The block diagram of the closed-loop system (5), where S represents the event-triggered sampler defined by (3).

3 Event-Triggered Robust Stabilization

Following the standard line of robustness analysis, this section studies the event-triggered robust stabilization problem for discrete-time nonlinear systems by considering the cases of state-dependent disturbance and state-independent disturbance, respectively. Input-to-state stabilization with respect to external disturbances is achieved.

3.1 State-Dependent Disturbance

In this subsection, we suppose that the external disturbance d is upper bounded by a function of system state x. In particular, there exists a $\gamma \in \mathcal{K}_\infty$ such that

$$|d(k)| \le \gamma(|x(k)|) \tag{9}$$

for all $k \in \mathbb{Z}_+$.

Motivated by previously existing results of event-triggered control, we propose an event-triggering mechanism such that the error w satisfies

$$|w(k)| \leq \rho(|x(k)|) \tag{10}$$

for all $k \in \mathbb{Z}_+$, where ρ is an appropriately chosen $\mathcal{K}_\infty$ function.

Then, the closed-loop system (5) with interconnections satisfying (9) and (10) can be described as a network of three subsystems as shown in Fig.2.

Considering the network structure and the gain interconnection, we employ the cyclic-small-gain theorem to analyze the robust stability of the closed-loop system. Loosely speaking, according to cyclic-small-gain theorems [37, 38], the closed-loop system is stable if the following cyclic-small-gain conditions are satisfied:

$$\alpha^{-1} \circ (\mathrm{Id} - \delta)^{-1} \circ \gamma_d \circ \gamma < \mathrm{Id}, \tag{11}$$

$$\alpha^{-1} \circ (\mathrm{Id} - \delta)^{-1} \circ \gamma_w \circ \rho < \mathrm{Id}, \tag{12}$$

where α is defined in Assumption 1, and δ is a continuous and positive definite function satisfying $\mathrm{Id} - \delta \in \mathcal{K}_\infty$.

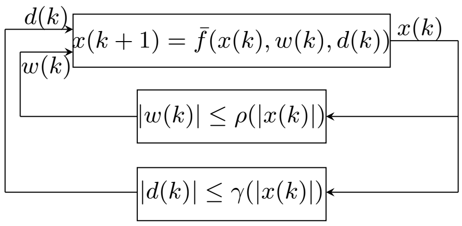

Fig. 2: The block diagram of the closed-loop event-triggered system (5) with state-dependent disturbance.

Since w is determined by the event-triggered sampling mechanism, we find an appropriate event trigger to satisfy both conditions (10) and (12).

Specifically, the function φ of the event-triggering mechanism (3) is defined as follows:

$$\varphi(r_1, r_2) = -\chi_x(|r_1|) + \chi_w(|r_1 - r_2|) \tag{13}$$

for all $r_1, r_2 \in \mathbb{R}^n$, with functions χ_x and χ_w to be chosen later.

From (13), it holds that

$$\begin{aligned}\varphi(x(k), x(k_i)) &= -\chi_x(|x(k)|) + \chi_w(|x(k) - x(k_i)|) \\ &= -\chi_x(|x(k)|) + \chi_w(|w(k)|),\end{aligned} \tag{14}$$

which, together with (3), determines the sequence of sampling times k_i.

Compared with the triggering function normally used in the previously published results, the triggering function φ is in a more general form. The advantage of such φ is discussed in Remark 1. Our first main result on event-triggered stabilization for the case of state-dependent disturbance is given by Theorem 1.

Theorem 1. *Consider system* (5). *Under Assumptions 1 and 2, if the external disturbance d satisfies* (9) *and* (11), *then global asymptotic stabilization can be achieved with the event-triggering mechanism* (3) *and* (13), *where*

$$\chi_x = \rho \circ (\mathrm{Id} + \rho)^{-1} - (\psi_{\bar{f}}^x + \psi_{\bar{f}}^d \circ \gamma) \tag{15}$$

$$\chi_w = \psi_{\bar{f}}^w + \mathrm{Id} \tag{16}$$

with $\rho \in \mathcal{K}_\infty$ satisfying (12).

Proof. The event-triggering mechanism (3) uses $x(k)$ and $w(k)$ to determine whether $x(k+1)$ should be sampled for feedback.

Note that $|w(0)| = 0 \leq \rho(|x(0)|)$. With φ defined in (13), we prove that for any given $k \in \mathbb{Z}_+$ and any $x(k)$ and $w(k)$,

$$|w(k+1)| \leq \rho(|x(k+1)|). \tag{17}$$

We study two cases: (a) $\varphi(x(k), x(k_i)) > 0$; (b) $\varphi(x(k), x(k_i)) \leq 0$.

Case (a): In this case, by directly using (3), we have $k_{i+1} = k+1$, and thus

$$|w(k+1)| = 0, \tag{18}$$

which guarantees (17).

Case (b): In this case, $x(k+1)$ is not sampled, which means $k_{i+1} > k+1$. Thus, we have

$$|w(k+1)| = |x(k+1) - x(k_i)|. \tag{19}$$

Recall that $x(k+1)$ is determined by $x(k)$, $x(k_i)$ and $d(k)$, and also $d(k)$ satisfies (9). We estimate an upper bound of $|w(k+1)|$ by using $x(k)$ and $x(k_i)$.

From (19), we have

$$\begin{aligned}|w(k+1)| &= |x(k+1) - x(k) + x(k) - x(k_i)| \\ &\leq |x(k+1) - x(k)| + |x(k) - x(k_i)|.\end{aligned} \tag{20}$$

By substituting (8) and (9) into the right-hand side of (20), we get an upper bound of $w(k+1)$ as

$$\begin{aligned}|w(k+1)| &\leq |x(k+1) - x(k)| + |x(k) - x(k_i)| \\ &\leq \psi_{\bar{f}}^x(|x(k)|) + \psi_{\bar{f}}^w(|x(k) - x(k_i)|) \\ &\quad + \psi_{\bar{f}}^d(|d(k)|) + |x(k) - x(k_i)| \\ &\leq \psi_{\bar{f}}^x(|x(k)|) + \psi_{\bar{f}}^d \circ \gamma(|x(k)|) \\ &\quad + (\psi_{\bar{f}}^w + \mathrm{Id})(|x(k) - x(k_i)|) \\ &\leq (\psi_{\bar{f}}^x + \psi_{\bar{f}}^d \circ \gamma)(|x(k)|) \\ &\quad + (\psi_{\bar{f}}^w + \mathrm{Id})(|x(k_i) - x(k)|).\end{aligned} \tag{21}$$

Using $\varphi(x(k), x(k_i)) \leq 0$, we have $(\psi_{\bar{f}}^x + \psi_{\bar{f}}^d \circ \gamma)(|x(k)|) - \rho \circ (\mathrm{Id} + \rho)^{-1}(|x(k)|) + (\psi_{\bar{f}}^w + \mathrm{Id})(|x(k_i) - x(k)|) \leq 0$, and thus

$$\begin{aligned}(\psi_{\bar{f}}^x + \psi_{\bar{f}}^d \circ \gamma)(|x(k)|) + (\psi_{\bar{f}}^w + \mathrm{Id})(|x(k_i) - x(k)|) \leq \\ \rho \circ (\mathrm{Id} + \rho)^{-1}(|x(k)|).\end{aligned} \tag{22}$$

From the first and the last inequalities of (21), we also have

$$|x(k+1) - x(k)| \leq \rho \circ (\mathrm{Id} + \rho)^{-1}(|x(k)|). \tag{23}$$

By using [23, Lemma A.1], it follows that

$$\begin{aligned}|x(k+1)-x(k)| &\leq \rho\circ(\mathrm{Id}+\rho)^{-1}(|x(k)|)\\ &\leq \rho(|x(k+1)|).\end{aligned} \tag{24}$$

Then, (17) is proved for Case (b) by combining (21), (22), and (24).

From the discussions above, we have

$$|w(k)| \leq \rho(|x(k)|) \tag{25}$$

for all $k \in \mathbb{Z}_+$.

Then, with the satisfaction of (11) and (12), property (7) implies

$$\begin{aligned}&V(\bar{f}(x,w,d)) - V(x)\\ &\leq -\alpha(|x|) + \max\{\gamma_w\circ\rho(|x|), \gamma_d\circ\gamma(|x|)\}\\ &\leq -\alpha(|x|) + \max\{(\mathrm{Id}-\delta)\circ\alpha(|x|), (\mathrm{Id}-\delta)\circ\alpha(|x|)\}\\ &\leq -\delta\circ\alpha(|x|)\\ &=: -\hat{\alpha}(|x|)\end{aligned} \tag{26}$$

for all x, w, d. This ends the proof of Theorem 1. □

Remark 1. *It can be observed that* $|w(k+1)| \leq \rho(|x(k+1)|)$ *cannot be directly guaranteed by* $|w(k)| \leq \rho_0(|x(k)|)$ *for some* $\rho_0 \in \mathcal{K}_\infty$*, since the amplitude of* $x(k+1)$ *depends on the uncertain* $\tilde{f}(x(k), w(k), d(k))$*. The problem is solved by introducing a new event-triggering function* φ*, for which* χ_x *is not necessarily positive definite, and* χ_w *is defined to be of class* $\mathcal{K}_\infty$*. In the case of* $\chi_x(|x(k)|) < 0$*, it can be directly concluded that*

$$k_{i+1} = k+1. \tag{27}$$

If, moreover, χ_x *is negative definite, then* (27) *holds for all* $k \in \mathbb{Z}_+, i \in \mathbb{Z}_+$*. In this case, each* $x(k)$ *for* $k \in \mathbb{Z}_+$ *should be sampled for feedback.*

If χ_x *is positive definite, then for each* $i \in \mathbb{S}$*, we have* $\varphi(x(k_i), x(k_i)) = -\chi_x(|x(k_i)|) \leq 0$*, and* $min\{k \geq k_i : \varphi(x(k), x(k_i)) > 0\} \geq k_i + 1$*, which implies*

$$k_{i+1} \geq k_i + 2. \tag{28}$$

As a result, if $x(k)$ *is sampled for feedback, then* $x(k+1)$ *will not be sampled.*

Remark 2. *Suppose that system* (1) *is an Euler approximation of a continuous-time system*

$$\dot{x}(t) = g(x(t), u(t), d(t)). \tag{29}$$

Then, we have

$$f(x(k), u(k), d(k)) = Tg(x(k), u(k), d(k)) \tag{30}$$

where constant $T > 0$ *is the sampling period for the continuous-time system* (29)*.*

Clearly, by choosing T *small enough, one can make* $\psi^x_{\tilde{f}}, \psi^w_{\tilde{f}}, \psi^d_{\tilde{f}}$ *small enough, and thus* χ_x *positive definite. Intuitively, this means that if* T *is small, then some* $x(k)$ *is not necessarily sampled; otherwise, each* $x(k)$ *should be sampled for feedback.*

There is a trade-off between the sampling period for continuous-time system and the number of sampling events for the discrete-time system.

3.2 State-Independent Disturbance

In this subsection, condition (9) used in Subsection 3.1 is not assumed, and we aim to show the effectiveness of the proposed event-triggering mechanism for input-to-state stabilization of the closed-loop event-triggered system. The analysis in Subsection 3.1 is not valid for this case. Specifically, if $|d(k)| > \gamma(|x(k)|)$ for some k, then the event trigger (3) with function φ defined in (13) cannot directly guarantee (17), and thus the stability of the closed-loop event-triggered system cannot be directly proved. This leads to one major difference between the event-triggered control system studied in this paper and the conventional discrete-time nonlinear systems.

We prove that the closed-loop event-triggered system is ISS, though the resulted ISS gain does not have the standard relation with the γ satisfying (11) or the γ_d defined in Assumption 1.

Theorem 2. *Consider system* (5)*. Under Assumptions 1 and 2, if the event-triggering mechanism* (3) *with function* φ *defined in* (13) *satisfies* (15) *and* (16)*, we have*

$$\begin{aligned}&V(\bar{f}(x,w,d)) - V(x)\\ &\leq -\breve{\alpha}(|x|) + \max\{\breve{\gamma}_d(|d(k-1)|), \gamma_d(|d(k)|)\}, \quad \forall x, w, d\end{aligned}$$

where

$$\breve{\alpha}(s) = \min\{\hat{\alpha}(s), \alpha(s)\} \tag{31}$$

for all $s \in \mathbb{R}_+$*, and*

$$\breve{\gamma}_d = \gamma_w \circ \lambda_d \tag{32}$$

with $\lambda_d = (\psi^x_{\tilde{f}} + \rho\circ(\psi^w_{\tilde{f}} + \mathrm{Id}))\circ\gamma^{-1} + \psi^d_{\tilde{f}}$*.*

Proof. The basic idea of the proof is to reveal that the closed-loop system is ISS by finding the true ISS gain, which is different from the proof of Theorem 1. For any specific $k^* \in \mathbb{Z}_+$, we consider the cases of $|w(k^*)| \leq \rho(|x(k^*)|)$ and $|w(k^*)| > \rho(|x(k^*)|)$, respectively.

Case (a): $|w(k^*)| \leq \rho(|x(k^*)|)$. Under Assumption 1, with condition (12) satisfied, we have

$$\begin{aligned}V(x(k^*+1)) - V(x(k^*)) &\leq -\alpha(|x(k^*)|)\\ &+ \max\{(\mathrm{Id}-\delta)\circ\alpha(|x(k^*)|), \gamma_d(|d(k^*)|)\}.\end{aligned} \tag{33}$$

If, moreover, (9) is satisfied, then property (33) implies

$$V(x(k^*+1)) - V(x(k^*)) \leq -\hat{\alpha}(|x(k^*)|) \tag{34}$$

where $\hat{\alpha}$ is defined as for (26).

If, on the other hand, condition (9) is not satisfied, then property (33) leads to

$$V(x(k^*+1)) - V(x(k^*)) \leq -\alpha(|x(k^*)|) + \gamma_d(|d(k^*)|). \tag{35}$$

By combining (34) and (35), we have

$$V(x(k^*+1)) - V(x(k^*)) \leq -\hat{\alpha}(|x(k^*)|) + \gamma_d(|d(k^*)|). \tag{36}$$

Case (b): $|w(k^*)| > \rho(|x(k^*)|)$. In this case, we first prove

$$|w(k^*-1)| \leq \rho(|x(k^*-1)|), \tag{37}$$

$$|d(k^*-1)| > \gamma(|x(k^*-1)|). \tag{38}$$

By contradiction, assume that (37) does not hold, that is, $|w(k^*-1)| > \rho(|x(k^*-1)|)$. Then, $\varphi(x(k^*-1), x(k_{i-1})) > 0$, and thus $x(k^*)$ is sampled, i.e., $x(k_i) = x(k^*)$. From the definition of w in (4), we have $|w(k^*)| = 0 \le \rho(|x(k^*)|)$. This contradicts with $|w(k^*)| > \rho(|x(k^*)|)$, and thus (37) is proved.

With (37) proved, if $|d(k^*-1)| \le \gamma(|x(k^*-1)|)$, then following the proof of Theorem 1, we have $|w(k^*)| \le \rho(|x(k^*)|)$. This contradicts with $|w(k^*)| > \rho(|x(k^*)|)$, and thus (38) is proved.

Then, the following relation between $|w(k^*)|$ and $|d(k^*-1)|$ can be observed:

$$
\begin{aligned}
&|w(k^*)| = |x(k^*) - x(k_i)| \\
&\le |x(k^*) - x(k^*-1)| + |w(k^*-1)| \\
&\le \psi_{\tilde{f}}^x(|x(k^*-1)|) + \psi_{\tilde{f}}^w(|w(k^*-1)|) \\
&\quad + \psi_{\tilde{f}}^d(|d(k^*-1)|) + |w(k^*-1)| \\
&\le (\psi_{\tilde{f}}^x + \rho \circ (\psi_{\tilde{f}}^w + \mathrm{Id}))(|x(k^*-1)|) + \psi_{\tilde{f}}^d(|d(k^*-1)|) \\
&\le \lambda_d(|d(k^*-1)|) \qquad (39)
\end{aligned}
$$

where $\lambda_d = (\psi_{\tilde{f}}^x + \rho \circ (\psi_{\tilde{f}}^w + \mathrm{Id})) \circ \gamma^{-1} + \psi_{\tilde{f}}^d$.

From (7), we have

$$
\begin{aligned}
&V(x(k^*+1)) - V(x(k^*)) \\
&\le -\alpha(|x(k^*)|) + \max\{\gamma_w \circ \lambda_d(|d(k^*-1)|), \gamma_d(|d(k^*)|)\} \\
&= -\alpha(|x(k^*)|) + \max\{\breve{\gamma}_d(|d(k^*-1)|)\gamma_d(|d(k^*)|)\}
\end{aligned} \tag{40}
$$

where $\breve{\gamma}_d = \gamma_w \circ \lambda_d$.

Then, (36) and (40) together imply

$$
\begin{aligned}
&V(x(k^*+1)) - V(x(k^*)) \\
&\le -\breve{\alpha}(|x(k^*)|) + \max\{\breve{\gamma}_d(|d(k^*-1)|), \gamma_d(|d(k^*)|)\}
\end{aligned}
$$

where $\breve{\alpha}(s) = \min\{\hat{\alpha}(s), \alpha(s)\}$ for all $s \in \mathbb{R}_+$.

By applying this reasoning repeatedly, we have, for any $k \in \mathbb{Z}_+$,

$$
\begin{aligned}
&V(x(k+1)) - V(x(k)) \\
&\le -\breve{\alpha}(|x(k)|) + \max\{\breve{\gamma}_d(|d(k-1)|), \gamma_d(|d(k)|)\},
\end{aligned} \tag{41}
$$

which means ISS of the closed-loop event-triggered system. This ends the proof of Theorem 2. □

Remark 3. *It can be observed that the right-hand side of* (39) *depends on* $d(k^*-1)$. *Intuitively, this is caused by the "+1" term in* (3). *More specifically, for the case of* $|w(k^*)| > \rho(|x(k^*)|)$ *of the proof of Theorem 2, some upper bound of* $|w(k^*)|$ *is needed to guarantee the ISS-like relation between* $|x|$ *and* $|d|$. *This is achieved by using event-triggering mechanism* (3).

4 An Illustrative Example

In this section, we employ an example to show the effectiveness of the obtained results in this paper. We consider event-triggered state-feedback control system in the following form:

$$
\begin{aligned}
&x(k+1) = x(k) + 0.05|x(k)| + 0.01d(k) + 0.1u(k) \\
&u(k) = -x(k_i),\ k_i \le k < k_{i+1},\ i \in \mathbb{S}
\end{aligned} \tag{42}
$$

where x is the state, u is the control input, and d is the external disturbance. With w defined in (4), we obtain the closed-loop system

$$
x(k+1) = 0.9x(k) + 0.05|x(k)| + 0.01d(k) - 0.1w(k). \tag{43}
$$

By transforming system (43) into the form of (5), it can be directly proved that

$$
|\tilde{f}(x, w, d)| \le 0.15|x| + 0.1|w| + 0.01|d| \tag{44}
$$

holds for all x, w, d.

We define an ISS-Lyapunov function $V(x) = |x|$ for system (43). Then, it holds that

$$
\begin{aligned}
&V(\bar{f}(x, w, d)) - V(x) \\
&\le -0.05|x| + \max\{0.2|w|, 0.02|d|\}, \quad \forall x, w, d.
\end{aligned} \tag{45}
$$

In the simulation, the disturbance $d(k)$ is selected as

$$
d(k) = 0.01\sin(0.03k) + 0.01\cos(0.09k) + 0.03. \tag{46}
$$

By choosing $\delta = 0.1, \rho = 0.22, \gamma = 0.2$, we calculate $\lambda_d = 1.97$, and thus $\breve{\gamma}_d = 0.394$. The simulation result for the gain of system (43) with initial states $x(0) = -1$ is shown in Fig.3. Clearly, in case of state-independent disturbance, the γ_d in (7) is not the true ISS gain.

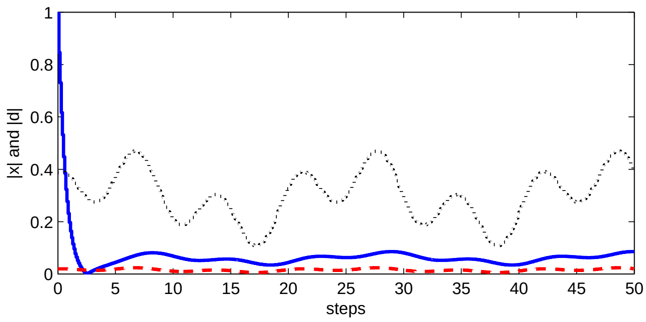

Fig. 3: The state trajectory $|x(k)|$ ("—") and related signals $\max\{\tilde{\alpha} \circ \breve{\gamma}_d(|d(k-1)|), \tilde{\alpha} \circ \gamma_d(|d(k)|)\}$ ("$\cdots$") and $\tilde{\alpha} \circ \gamma_d(|d(k)|)$ ("- - -"), where $\tilde{\alpha}(s) = \underline{\alpha}^{-1} \circ (\alpha \circ \overline{\alpha}^{-1})^{-1}(s)$ with $\underline{\alpha}(s) = 0.9s, \overline{\alpha}(s) = 1.1s$.

5 Conclusions

This paper proposes a new event-triggering mechanism for event-triggered control of nonlinear discrete-time systems in the presence of external disturbances. Compared with the huge literature of event-based control on continuous-time systems, the study in this paper reveals new phenomena appearing in event-triggered controlled discrete-time nonlinear systems. In particular, we consider the case in which whether or not the sampling event is triggered at step k is determined by state $x(k-1)$, which together with the external disturbance causes one of the major difficulties. Moreover, in case of state-independent disturbance, the standard tool of input-to-state stability cannot be directly used to guarantee the input-to-state stabilization of the closed-loop event-triggered system. Following this line of research, further directions may include the study of event-triggered control of the systems with partial-state feedback, for which not only external disturbances but also dynamic uncertainties should be taken into consideration.

References

[1] K. Åström and B. Wittenmark. *Computer Controlled Systems*. Englewood Cliffs: Prentice-Hall, 1996.

[2] K. Årzén. A simple event-based PID controller. In *Proceedings the IFAC World Congress*, pages 423–428, 1999.

[3] K. Åström and B. Bernhardsson. Comparison of periodic and event based sampling for first order stochastic systems. In *Proceedings of the IFAC World Congress*, pages 301–306, Beijing, 1999.

[4] K. Åström. *Event Based Control, Analysis and Design of Nonlinear Control Systems*. Springer, 2008.

[5] W.P.M.H. Heemels, K. Johansson, and P. Tabuada. An introduction to event-triggered and self-triggered control. In *Procceeding of the 51st IEEE Conference on Decision and Control*, pages 3270–3285, 2012.

[6] J. Lunze and D. Lehmann. A state-feedback approach to event-based control. *Automatica*, 46:211–215, 2010.

[7] W.P.M.H. Heemels, M. Donkers, and A. Teel. Periodic event-triggered control for linear systems. *IEEE Transactions on Automatic Control*, 58:847–861, 2013.

[8] W.P.M.H. Heemels and M. Donkers. Model-based periodic event-triggered control for linear systems. *Automatica*, 49:698–711, 2013.

[9] P. Tabuada. Event-triggered real-time scheduling of stabilizing control tasks. *IEEE Transactions on Automatic Control*, 52:1680–1685, 2007.

[10] A. Anta and P. Tabuada. To sample or not to sample: Self-triggered control for nonlinear systems. *IEEE Transactions on Automatic Control*, 55:2030–2042, 2010.

[11] P. Gawthrop and L. Wang. Event-driven intermittent control. *International Journal of Control*, 82:2235–2248, 2009.

[12] M. Donkers and W.P.M.H. Heemels. Output-based event-triggered control with guaranteed $\mathcal{L}_\infty$-gain and improved and decentralized event-triggering. *IEEE Transactions on Automatic Control*, 57:1362–1376, 2012.

[13] N. Marchand, S. Durand, and J. Castellanos. A general formula for event-based stabilization of nonlinear systems. *IEEE Transactions on Automatic Control*, 58:1332–1337, 2013.

[14] W.P.M.H. Heemels, J. Sandee, and P. Van Den Bosch. Analysis of event-driven controllers for linear systems. *International Journal of Control*, 81:571–590, 2008.

[15] T. Henningsson, E. Johannesson, and A. Cervin. Sporadic event-based control of first-order linear stochastic systems. *Automatica*, 44:2890–2895, 2008.

[16] D. Dimarogonas and K. Johansson. Event-triggered control for multi-agent systems. In *Proceedings of the 48th IEEE Conference on Decision and Control and 28th Chinese Control Conference*, pages 7131–7136, 2009.

[17] X. Wang and M. Lemmon. *Event-Triggering in Distributed Networked Systems with Data Dropouts and Delays, Hybrid Systems: Computation and Control*. Heidelberg: Springer Berlin, 2009.

[18] X. Wang and M. Lemmon. Event design in event-triggered feedback control systems. In *Proceedings of the 47th IEEE Conference on Decision and Control*, pages 2105–2110, 2008.

[19] A. Girard. Dynamic triggering mechanisms for event-triggered control. *IEEE Transactions on Automatic Control*, 60:1992–1997, 2015.

[20] P. Tallapragada and N. Chopra. On event triggered tracking for nonlinear systems. *IEEE Transactions on Automatic Control*, 58:2343–2348, 2013.

[21] M. Mazo Jr and P. Tabuada. Input-to-state stability of self-triggered control systems. In *Proceedings of the 48th IEEE Conference on Decision and Control and 28th Chinese Control Conference*, pages 928–933, 2009.

[22] M. Mazo Jr, A. Anta, and P. Tabuada. An ISS self-triggered implementation of linear controllers. *Automatica*, 46:1310–1314, 2010.

[23] T. Liu and Z.P. Jiang. A small-gain approach to robust event-triggered control of nonlinear systems. *IEEE Transactions on Automatic Control*, 60:2072–2085, 2015.

[24] E. Kofman and J. Braslavsky. Level crossing sampling in feedback stabilization under data-rate constraints. In *Proceedings of the 45th IEEE Conference on Decision and Control*, pages 4423–4428, 2006.

[25] D. Lehmann and J. Lunze. Event-based output-feedback control. In *Proceedings of the 19th Mediterranean Conference on Control and Automation*, pages 982–987, 2011.

[26] F. Brunnery, T. Gommansz, W.P.M.H. Heemels, and F. Allgöwer. Resource-aware set-valued estimation for discrete-time linear systems. In *Proceedings of the 54th IEEE Conference on Decision and Control*, pages 5480–5486, 2015.

[27] B. Khashooei, D. Antunes, and W.P.M.H. Heemels. An event-triggered policy for remote sensing and control with performance guarantees. In *Proceedings of the 54th IEEE Conference on Decision and Control*, pages 5480–5486, 2015.

[28] A. Molin, C. Ramesh, H. Esen, and K. Johansson. Innovations-based priority assignment for control over CAN-like networks. In *Proceedings of the 54th IEEE Conference on Decision and Control*, pages 4163–4169, 2015.

[29] A. Eqtami, D. Dimarogonas, and K. Kyriakopoulos. Event-triggered control for discrete-time systems. In *Proceedings the 2010 American Control Conference*, pages 4719–4724, 2010.

[30] X. Chen and F. Hao. Event-triggered average consensus control for discrete-time multi-agent systems. *IET Control Theory and Applications*, 6:2493–2498, 2013.

[31] L. Li, D. Ho, and S. Xu. A distributed event-triggered scheme for discrete-time multi-agent consensus with communication delays. *IET Control Theory and Applications*, 8:830–837, 2014.

[32] M. Lazar, W.P.M.H. Heemels, and A. Teel. Further input-to-state stability subtleties for discrete-time systems. *IEEE Transactions on Automatic Control*, 58:1609–1613, 2013.

[33] L. Grüne and C. Kellett. ISS-Lyapunov functions for discontinuous discrete-time systems. *IEEE Transactions on Automatic Control*, 59:3098–3103, 2014.

[34] Z.P. Jiang and Y. Wang. Input-to-state stability for discrete-time nonlinear systems. *Automatica*, 37:857–869, 2001.

[35] C. De Persis, R. Sailer, and F. Wirth. Parsimonious event-triggered distributed control: A Zeno free approach. *Automatica*, 49:2116–2124, 2013.

[36] T. Liu, Z.P. Jiang, and D. Hill. *Nonlinear Control of Dynamic Networks*. Boca Raton: CRC Press, 2014.

[37] Z.P. Jiang and Y. Wang. A generalization of the nonlinear small-gain theorem for large-scale complex systems. In *Proceedings the 7th World Congress on Intelligent Control and Automation*, pages 1188–1193, 2008.

[38] T. Liu, D. Hill, and Z.P. Jiang. Lyapunov formulation of ISS cyclic-small-gain in continuous-time dynamical networks. *Automatica*, 47:2088–2093, 2011.