Time-Inconsistent Mean-Field Stochastic Linear-Quadratic Optimal Control

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Abstract: This paper is concerned with a kind of time-consistent solution of the time-inconsistent mean-field stochastic linear-quadratic optimal control. Different from standard stochastic linear-quadratic problems, both the system matrices and the weighting matrices are dependent on initial time, and the conditional expectations of the control and state enter quadratically into the cost functional. Such features will ruin the Bellman optimality principle and result in the time-inconsistency of the optimal control. Due to the dynamical nature of control problems, a kind of open-loop time-consistent equilibrium control is thoroughly investigated in this paper. It is shown that the existence of an open-loop time-consistent equilibrium control for a fixed initial pair is equivalent to the solvability of a set of forward-backward stochastic difference equations with stationary conditions and convexity conditions. By decoupling the forward-backward stochastic difference equations, the existence of the open-loop equilibrium control for all the initial pairs is shown to be equivalent to the solvability of a set of coupled constraint generalized difference Riccati equations and a set of coupled constraint linear difference equations.

Key Words: Time-inconsistency, mean-field control, stochastic linear-quadratic optimal control

1 Introduction

The Bellman optimality principle is an essential property in optimal control theory, which provides the theoretical foundation of the dynamic programming approach. From the Bellman optimality principle, an optimal control for an initial pair is also optimal along the optimal trajectory. Such a phenomenon is referred as the time-consistency of the optimal control, which ensures that one needs only to solve an optimal control problem for a given initial pair, and the obtained optimal control is also optimal along the optimal trajectory.

However, in reality, the time-consistency fails quite often. For instance, when the initial time or initial state enters into the system dynamics or cost functional explicitly, or even more, the conditional expectations of the state or control enters nonlinearly into the cost functional, the corresponding problems are time-inconsistent. See examples in [13] and [5] about the hyperbolic discounting and quasi-geometric discounting. The problem with nonlinear term of the conditional expectation in cost functional is called as mean-field stochastic optimal control. In this case, the smoothing property of the conditional expectation will not be in effect to ensure the time-consistency of the optimal control. A particular example of this case is the known mean-variance utility [3] and [5].

Due to the dynamical nature of the control problems, it is reasonable to handle the time-inconsistency in a dynamic manner. Instead of seeking an "optimal control", some kind of equilibrium controls are concerned with. This is mainly motivated by practical applications such as in mathematical finance and economics, and has recently attracted considerable interest and efforts. The mathematical formulation of the time-inconsistency was first reported by [22], and a qualitative analysis was given in [21]. Following [22], the works [10], [13], [14] and [19] are for discrete dynamic systems or simple ordinary differential equations (ODEs). Subsequently, [6] and [7] studied the non-exponential discounting problems both for simple ODEs and stochastic differential equations and introduced the notion of time-consistent control. [5] discussed the problems of general Markovian time-inconsistent stochastic optimal control. [24] and [25] addressed the deterministic continuous-time linear-quadratic (LQ) optimal control by an essentially cooperative game approach. [27] considered the stochastic LQ problem of mean-field case, which is called the closed-loop formulation there. Different from [24] and [25], [12] studied another kind of time-consistent equilibrium control, which is an infinitesimally open-loop optimal control. Roughly speaking, the controller given by [12] can commit to the equilibrium control in an infinitesimal manner. [27] thoroughly investigated both the open-loop and the closed-loop time-consistent solutions for general mean-field stochastic LQ problems. It is shown ([27]) that the existence of the open-loop equilibrium control and the closed-loop equilibrium strategy is ensured via the solvability of certain systems of Riccati-type differential equations. However, all these existing results focus on the definite LQ problems.

In this paper, we shall investigate a time-inconsistent mean-field stochastic LQ optimal control problem, whose system dynamics and cost functional are also dependent on the initial time. It is worth noting that the definiteness constraints are not posed on the state and control weighting matrices. Here, we adopt a dynamic manner to attack the time-inconsistency and intend seeking a kind of open-loop equilibrium control of Problem (LQ). After giving the definition of open-loop equilibrium pair, its existence for a fixed initial pair is shown to be equivalent to the solvability of a set of forward-backward stochastic difference equations (FBSΔEs) with stationary conditions and convexity conditions. If for a fixed initial pair Problem (LQ) admits an open-loop equilibrium pair, then a set of constrained linear difference equations (LDEs) is solvable, and the open-loop equilibrium control admits a closed-loop representation. Here, the closed-loop representation is a linear feedback of the cur-

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rent value of equilibrium state, whose gains are computed via the solutions of the constraint LDEs (11) and a set of generalized difference Riccati equations (GDREs) (14). For any initial pair, Problem (LQ) admitting an open-loop equilibrium pair is shown to be equivalent to that the constraint LDEs (15) and a set of constraint GDREs (16) are solvable. Different from (14), (16) is of constrained GDREs. Interestingly, if solvable, the set of GDREs (16) does not have symmetric structure, i.e., its solution is not symmetric.

In [16], a simplified version of Problem (LQ) is considered, where the system dynamics and the cost functional do not contain mean-field terms. Hence, this paper is a continuation of [16]. If the system dynamics and cost functional are independent of the initial time, the corresponding LQ problem is a dynamic version of that considered in [17], where a static version is studied with the conditional expectation operator being replaced by the expectation operator. For details on mean-field stochastic optimal control and related mean-field games, we refer to, for example, [8] [17] [18] [26] [4] [11] [15] and the references therein.

The rest of the paper is organized as follows. Section 2 presents the results mentioned above, and Section 3 gives some concluding remarks.

2 Open-loop time-consistent equilibrium control

Consider the following controlled stochastic difference equation (SΔE)

$$\begin{cases} X_{k+1}^t = \big(A_{t,k}X_k^t + \bar{A}_{t,k}\mathbb{E}_tX_k^t \\ \qquad\quad + B_{t,k}u_k + \bar{B}_{t,k}\mathbb{E}_tu_k\big) \\ \qquad\quad + \big(C_{t,k}X_k^t + \bar{C}_{t,k}\mathbb{E}_tX_k^t \\ \qquad\quad + D_{t,k}u_k + \bar{D}_{t,k}\mathbb{E}_tu_k\big)w_k, \\ X_t^t = x, \;\; k \in \mathbb{T}_t, \;\; t \in \mathbb{T}, \end{cases} \tag{1}$$

where $\mathbb{T} = \{0, \dots, N-1\}$, $\mathbb{T}_t = \{t, \cdots, N-1\}$, and $A_{t,k}, \bar{A}_{t,k}, C_{t,k}, \bar{C}_{t,k} \in \mathbb{R}^{n\times n}$, $B_{t,k}, \bar{B}_{t,k}, D_{t,k}, \bar{D}_{t,k} \in \mathbb{R}^{n\times m}$ are deterministic matrices; $\{X_k^t, k \in \widetilde{\mathbb{T}}_t\} \triangleq X^t$ and $\{u_k, k \in \mathbb{T}_t\} \triangleq u$ with $\widetilde{\mathbb{T}}_t$ being $\{0, \dots, N\}$ are the state process and control process, respectively. Here, the initial time t is parameterized in the matrices and state to emphasize that the matrices and state may change according to the initial time t. Similar notations will be used throughout the paper. In (1), $\mathbb{E}_t$ is the conditional mathematical expectation $\mathbb{E}[\,\cdot\,|\mathcal{F}_t]$ with respect to $\mathcal{F}_t = \{x_0, w_l, l = 0, 1, \cdots, t-1\}$ and $\mathcal{F}_{-1}$ is understood as $\{\emptyset, \Omega\}$. The noise $\{w_k, k \in \mathbb{T}\}$ is assumed to be a martingale difference sequence defined on a probability space $(\Omega, \mathcal{F}, P)$ with

$$\mathbb{E}_{k+1}[(w_{k+1})^2] = 1, \; k \in \mathbb{T}. \tag{2}$$

The cost functional associated with system (1) is

$$\begin{aligned} J(t,x;u) = \sum_{k=t}^{N-1} \mathbb{E}_t\Big[&(X_k^t)^TQ_{t,k}X_k^t + (\mathbb{E}_tX_k^t)^T\bar{Q}_{t,k}\mathbb{E}_tX_k^t \\ &+ u_k^TR_{t,k}u_k + (\mathbb{E}_tu_k)^T\bar{R}_{t,k}\mathbb{E}_tu_k \\ &+ \mathbb{E}_t\big[(X_N^t)^TG_tX_N^t\Big] + (\mathbb{E}_tX_N^t)^T\bar{G}_t\mathbb{E}_tX_N^t, \end{aligned} \tag{3}$$

where $Q_{t,k}, \bar{Q}_{t,k}, R_{t,k}, \bar{R}_{t,k}, k \in \mathbb{T}_t, G_t, \bar{G}_t$ are deterministic symmetric matrices of appropriate dimensions. In (1), x is in $L^2_{\mathcal{F}}(t;\mathbb{R}^n)$, which is a set of random variables such that any $\xi \in L^2_{\mathcal{F}}(t;\mathbb{R}^n)$ is $\mathcal{F}_t$-measurable and $\mathbb{E}|\xi|^2 < \infty$. Let $L^2_{\mathcal{F}}(\mathbb{T}_t;\mathcal{H})$ be a set of $\mathcal{H}$-valued processes such that for any $\nu = \{\nu_k, k \in \mathbb{T}_t\} \in L^2_{\mathcal{F}}(\mathbb{T}_t;\mathcal{H})$, ν_k is $\mathcal{F}_k$-measurable and $\sum_{k=t}^{N-1}\mathbb{E}|\nu_k|^2 < \infty$. Then, we pose the following optimal control problem.

Problem (LQ). *Concerned with (1)(3) and the initial pair (t,x), find a $u^* \in L^2_{\mathcal{F}}(\mathbb{T}_t;\mathbb{R}^m)$, such that*

$$J(t,x;u^*) = \inf_{u\in L^2_{\mathcal{F}}(\mathbb{T}_t;\mathbb{R}^m)} J(t,x;u). \tag{4}$$

Instead of solving Problem (LQ) for the static precommitted optimal control, we adopt the concept of dynamic equilibrium control, which is optimal in an infinitesimal sense and consistent with the dynamical nature of Problem (LQ).

Definition 2.1 *Given $t \in \mathbb{T}$ and $x \in L^2_{\mathcal{F}}(t;\mathbb{R}^n)$, a state-control pair $(X^{t,x,*}, u^{t,x,*})$ with $u^{t,x,*} \in L^2_{\mathcal{F}}(\mathbb{T}_t;\mathbb{R}^m)$ is called an open-loop equilibrium pair of Problem (LQ) for the initial pair (t,x) if $X_t^{t,x,*} = x$, and*

$$J(k, X_k^{t,x,*}; u^{t,x,*}|_{\mathbb{T}_k}) \leq J(k, X_k^{t,x,*}; (u_k, u^{t,x,*}|_{\mathbb{T}_{k+1}})) \tag{5}$$

holds for any $u_k \in L^2_{\mathcal{F}}(k;\mathbb{R}^m)$ and $k \in \mathbb{T}_t$. Here, $u^{t,x,*}|_{\mathbb{T}_k}$ and $u^{t,x,*}|_{\mathbb{T}_{k+1}}$ are the restrictions of $u^{t,x,*}$ on $\mathbb{T}_k$ and $\mathbb{T}_{k+1}$, respectively. Furthermore, such a $u^{t,x,*}$ is called an open-loop equilibrium control for the initial pair (t,x).

For a $u \in L^2_{\mathcal{F}}(\mathbb{T}_t;\mathbb{R}^m)$, the requirement that u_k is $\mathcal{F}_{k-1}$-measurable is parallel to the standard statement on the admissible controls of continuous-time stochastic optimal control; see [9], [23] for details. In other words, u_k is determined from $\{k, x_0, w_0, ..., w_{k-1}\}$ only, irrespective of how the state process develops. From this and the standard arguments about the open-loop control [2], $u \in L^2_{\mathcal{F}}(\mathbb{T}_t;\mathbb{R}^m)$ can be viewed as an open-loop control. Hence, we call $u^{t,x,*}$ an open-loop equilibrium control. Furthermore, noting that $u^{t,x,*}|_{\mathbb{T}_k} = (u_k^{t,x,*}, u^{t,x,*}|_{\mathbb{T}_{k+1}})$, the control $(u_k, u^{t,x,*}|_{\mathbb{T}_{k+1}})$ on the right hand of the inequality of (5) differs from $u^{t,x,*}|_{\mathbb{T}_k}$ only at stage k. Therefore, (5) is viewed as a local optimality condition. Similarly to [16], we can show that $\{u_t^{t,x,*}, ..., u_{N-1}^{t,x,*}\}$ is the Nash equilibrium of a multi-person game with hierarchical structure. By its definition, an open-loop equilibrium control $u^{t,x,*}$ is time-consistent in the sense that for any $k \in \mathbb{T}_t$, $u^{t,x,*}|_{\mathbb{T}_k}$ is an open-loop equilibrium control for the initial pair $(k, X_k^{t,x,*})$. In other words, $u^{t,x,*}$ is time-consistent along the trajectory of the equilibrium state $X^{t,x,*}$.

By a formula of the difference of cost functionals, we can derive the following necessary and sufficient condition to the existence of the open-loop equilibrium pair for a given initial pair, whose proof is omitted due to space limitations.

Theorem 2.1 *Given $t \in \mathbb{T}$ and $x \in L^2_{\mathcal{F}}(t;\mathbb{R}^n)$, the following statements are equivalent.*

(i) There exists an open-loop equilibrium pair of Problem (LQ) for the initial pair (t,x).

(ii) There exists a $u^{t,x,*} \in L^2_{\mathcal{F}}(\mathbb{T}_t;\mathbb{R}^m)$ such that for any $k \in \mathbb{T}_t$, the following FBSΔE admits a solution

$(X^{k,t,x}, Z^{k,t,x})$

$$\left\{\begin{array}{l} X^{k,t,x}_{\ell+1} = \big(A_{k,\ell}X^{k,t,x}_\ell + \bar{A}_{k,\ell}\mathbb{E}_k X^{k,t,x}_\ell \\ \qquad + B_{k,\ell}u^{t,x,*}_\ell + \bar{B}_{k,\ell}\mathbb{E}_k u^{t,x,*}_\ell\big) \\ \qquad + \big(C_{k,\ell}X^{k,t,x}_\ell + \bar{C}_{k,\ell}\mathbb{E}_k X^{k,t,x}_\ell \\ \qquad + D_{k,\ell}u^{t,x,*}_\ell + \bar{D}_{k,\ell}\mathbb{E}_k u^{t,x,*}_\ell\big)w_\ell, \\ Z^{k,t,x}_\ell = A^T_{k,\ell}\mathbb{E}_\ell Z^{k,t,x}_{\ell+1} + \bar{A}^T_{k,\ell}\mathbb{E}_k Z^{k,t,x}_{\ell+1} \\ \qquad + C^T_{k,\ell}\mathbb{E}_\ell(Z^{k,t,x}_{\ell+1}w_\ell) + \bar{C}^T_{k,\ell}\mathbb{E}_k(Z^{k,t,x}_{\ell+1}w_\ell) \\ \qquad + Q_{k,\ell}X^{k,t,x}_\ell + \bar{Q}_{k,\ell}\mathbb{E}_k X^{k,t,x}_\ell, \\ X^{k,t,x}_k = X^{t,x,*}_k, \\ Z^{k,t,x}_N = G_k X^{k,t,x}_N + \bar{G}_k\mathbb{E}_k X^{k,t,x}_N, \\ \ell \in \mathbb{T}_k \end{array}\right. \tag{6}$$

with the stationary condition

$$0 = (R_{k,k} + \bar{R}_{k,k})u^{t,x,*}_k + (B_{k,k} + \bar{B}_{k,k})^T\mathbb{E}_k Z^{k,t,x}_{k+1} + (D_{k,k} + \bar{D}_{k,k})^T\mathbb{E}_k(Z^{k,t,x}_{k+1}w_k) \tag{7}$$

and the convexity condition

$$\begin{aligned} 0 \le\ & \mathbb{E}_k\big[u^T_k(R_{k,k} + \bar{R}_{k,k})u_k\big] \\ & + \sum_{\ell=k}^{N-1}\mathbb{E}_k\Big[(Y^{k,\bar{u}_k}_\ell)^T Q_{k,\ell}Y^{k,\bar{u}_k}_\ell \\ & + (\mathbb{E}_k Y^{k,\bar{u}_k}_\ell)^T\bar{Q}_{k,\ell}\mathbb{E}_k Y^{k,\bar{u}_k}_\ell\Big] \\ & + \mathbb{E}_k\big[(Y^{k,\bar{u}_k}_N)^T G_k Y^{k,\bar{u}_k}_N\big] + (\mathbb{E}_k Y^{k,\bar{u}_k}_N)^T\bar{G}_k\mathbb{E}_k Y^{k,\bar{u}_k}_N \end{aligned} \tag{8}$$

In the above, $Y^{k,\bar{u}_k}$ *and* $X^{t,x,*}$ *are given, respectively, by*

$$\left\{\begin{array}{l} Y^{k,\bar{u}_k}_{\ell+1} = A_{k,\ell}Y^{k,\bar{u}_k}_\ell + \bar{A}_{k,\ell}\mathbb{E}_k Y^{k,\bar{u}_k}_\ell \\ \qquad + \big(C_{k,\ell}Y^{k,\bar{u}_k}_\ell + \bar{C}_{k,\ell}\mathbb{E}_k Y^{k,\bar{u}_k}_\ell\big)w_\ell, \quad \ell \in \mathbb{T}_{k+1}, \\ Y^{k,\bar{u}_k}_{\ell+1} = (B_{k,k} + \bar{B}_{k,k})\bar{u}_k + (D_{k,k} + \bar{D}_{k,k})\bar{u}_k w_k, \\ Y^{k,\bar{u}_k}_k = 0, \end{array}\right.$$

and

$$\left\{\begin{array}{l} X^{t,x,*}_{k+1} = (A_{k,k} + \bar{A}_{k,k})X^{t,x,*}_k + (B_{k,k} + \bar{B}_{k,k})u^{t,x,*}_k \\ \qquad + \big[(C_{k,k} + \bar{C}_{k,k})X^{t,x,*}_k + (D_{k,k} + \bar{D}_{k,k})u^{t,x,*}_k\big]w_k, \\ X^{t,x,*}_t = x, \quad k \in \mathbb{T}_t. \end{array}\right.$$

In this case, $(X^{t,x,*}, u^{t,x,*})$ *given in (ii) is an open-loop equilibrium pair.*

Though the means of the state and control appear in both the system dynamics and the cost functional, the stationary condition (7) does not contain the means. This is different from the case of static mean-field LQ optimal control [17]. Here, the static mean-field LQ optimal control is a variant of Problem (LQ) where the conditional expectation operator $\mathbb{E}_t$ is replaced by the expectation operator $\mathbb{E}$ and the matrices in the system and cost functional are independent of the initial time. Another interesting phenomenon is that the system (9) for the equilibrium pair doesn't contain the mean-field terms too.

To simplify the notations, let

$$\left\{\begin{array}{l} \mathcal{A}_{k,\ell} = A_{k,\ell} + \bar{A}_{k,\ell}, \quad \mathcal{B}_{k,\ell} = B_{k,\ell} + \bar{B}_{k,\ell}, \\ \mathcal{C}_{k,\ell} = C_{k,\ell} + \bar{C}_{k,\ell}, \quad \mathcal{D}_{k,\ell} = D_{k,\ell} + \bar{D}_{k,\ell}, \\ \mathcal{Q}_{k,\ell} = Q_{k,\ell} + \bar{Q}_{k,\ell}, \quad \mathcal{R}_{k,\ell} = R_{k,\ell} + \bar{R}_{k,\ell}, \\ \mathcal{G}_k = G_k + \bar{G}_k, \quad k \in \mathbb{T}_t, \ \ell \in \mathbb{T}_k. \end{array}\right. \tag{9}$$

Recall the pseudo-inverse of a matrix. By [20], for a given matrix $M \in \mathbb{R}^{n\times m}$, there exists a unique matrix in $\mathbb{R}^{m\times n}$ denoted by $M^\dagger$ such that

$$\left\{\begin{array}{l} MM^\dagger M = M, \ \ M^\dagger MM^\dagger = M^\dagger, \\ (MM^\dagger)^T = MM^\dagger, \ \ (M^\dagger M)^T = M^\dagger M. \end{array}\right. \tag{10}$$

This $M^\dagger$ is called the Moore-Penrose inverse of M. The following lemma is from [1].

Lemma 2.1 *Let matrices L, M and N be given with appropriate size. Then,* $LXM = N$ *has a solution* X *if and only if* $LL^\dagger NMM^\dagger = N$*. Moreover, the solution of* $LXM = N$ *can be expressed as* $X = L^\dagger NM^\dagger + Y - L^\dagger LYMM^\dagger$*, where* Y *is a matrix with appropriate size.*

The following theorem is concerned with the necessary condition on the existence of the open-loop equilibrium pair for a fixed initial pair.

Theorem 2.2 *Let Problem (LQ) for the initial pair* (t, x) *admit an open-loop equilibrium pair. Then, the following set of LDEs*

$$\left\{\begin{array}{l} \left\{\begin{array}{l} P_{k,\ell} = Q_{k,\ell} + A^T_{k,\ell}P_{k,\ell+1}A_{k,\ell} \\ \qquad + C^T_{k,\ell}P_{k,\ell+1}C_{k,\ell}, \\ \mathcal{P}_{k,\ell} = \mathcal{Q}_{k,\ell} + \mathcal{A}^T_{k,\ell}\mathcal{P}_{k,\ell+1}\mathcal{A}_{k,\ell} \\ \qquad + \mathcal{C}^T_{k,\ell}P_{k,\ell+1}\mathcal{C}_{k,\ell}, \\ P_{k,N} = G_k, \ \ \mathcal{P}_{k,N} = \mathcal{G}_k, \\ \ell \in \mathbb{T}_k, \end{array}\right. \\ \mathcal{R}_{k,k} + \mathcal{B}^T_{k,k}\mathcal{P}_{k,k+1}\mathcal{B}_{k,k} + \mathcal{D}^T_{k,k}P_{k,k+1}\mathcal{D}_{k,k} \ge 0, \\ k \in \mathbb{T}_t \end{array}\right. \tag{11}$$

is solvable in the sense that the following inequalities hold

$$\left\{\begin{array}{l} \mathcal{R}_{k,k} + \mathcal{B}^T_{k,k}\mathcal{P}_{k,k+1}\mathcal{B}_{k,k} + \mathcal{D}^T_{k,k}P_{k,k+1}\mathcal{D}_{k,k} \ge 0, \\ k \in \mathbb{T}_t. \end{array}\right. \tag{12}$$

Furthermore, we have

$$\left\{\begin{array}{l} Z^{k,t,x}_\ell = P_{k,\ell}\big(X^{k,t,*}_\ell - \mathbb{E}_k X^{k,t,*}_\ell\big) + \mathcal{P}_{k,\ell}\mathbb{E}_k X^{k,t,x}_\ell \\ \qquad + T_{k,\ell}\big(X^{t,x,*}_\ell - \mathbb{E}_k X^{t,x,*}_\ell\big) + \mathcal{T}_{k,\ell}\mathbb{E}_k X^{t,x,*}_\ell, \\ \ell \in \mathbb{T}_k, \end{array}\right.$$

and an open-loop equilibrium control is given by

$$u^{t,x,*}_k = -\mathcal{W}^\dagger_k\mathcal{H}_k X^{t,x,*}_k, \ \ k \in \mathbb{T}_t. \tag{13}$$

Here, $\{T_{k,\ell}, \ell \in \mathbb{T}_k, k \in \mathbb{T}_t\}$, $\{\mathcal{T}_{k,\ell}, \ell \in \mathbb{T}_k, k \in \mathbb{T}_t\}$, $\{\mathcal{W}_k, k \in \mathbb{T}_t\}$ *and* $\{\mathcal{H}_k, k \in \mathbb{T}_t\}$ *are given, respectively, by*

$$\left\{\begin{array}{l} \left\{\begin{array}{l} T_{k,\ell} = A^T_{k,\ell}T_{k,\ell+1}\mathcal{A}_{\ell,\ell} + C^T_{k,\ell}T_{k,\ell+1}\mathcal{C}_{\ell,\ell} \\ \quad - \big(A^T_{k,\ell}P_{k,\ell+1}B_{k,\ell} + A^T_{k,\ell}T_{k,\ell+1}\mathcal{B}_{\ell,\ell} \\ \quad + C^T_{k,\ell}P_{k,\ell+1}D_{k,\ell} + C^T_{k,\ell}T_{k,\ell+1}\mathcal{D}_{\ell,\ell}\big)\mathcal{W}^\dagger_\ell\mathcal{H}_\ell, \\ \mathcal{T}_{k,\ell} = \mathcal{A}^T_{k,\ell}\mathcal{T}_{k,\ell+1}\mathcal{A}_{\ell,\ell} + \mathcal{C}^T_{k,\ell}T_{k,\ell+1}\mathcal{C}_{\ell,\ell} \\ \quad - \big(\mathcal{A}^T_{k,\ell}\mathcal{P}_{k,\ell+1}\mathcal{B}_{k,\ell} + \mathcal{A}^T_{k,\ell}\mathcal{T}_{k,\ell+1}\mathcal{B}_{\ell,\ell} \\ \quad + \mathcal{C}^T_{k,\ell}P_{k,\ell+1}\mathcal{D}_{k,\ell} + \mathcal{C}^T_{k,\ell}T_{k,\ell+1}\mathcal{D}_{\ell,\ell}\big)\mathcal{W}^\dagger_\ell\mathcal{H}_\ell, \\ T_{k,N} = 0, \ \ \mathcal{T}_{k,N} = 0, \\ \ell \in \mathbb{T}_k, \end{array}\right. \\ k \in \mathbb{T}_t, \end{array}\right. \tag{14}$$

and

$$\begin{cases} \mathcal{W}_k = \mathcal{R}_{k,k} + \mathcal{B}_{k,k}^T\big(\mathcal{P}_{k,k+1} + \mathcal{T}_{k,k+1}\big)\mathcal{B}_{k,k} \\ \qquad\quad + \mathcal{D}_{k,k}^T\big(P_{k,k+1} + T_{k,k+1}\big)\mathcal{D}_{k,k}, \\ \mathcal{H}_k = \mathcal{B}_{k,k}^T\big(\mathcal{P}_{k,k+1} + \mathcal{T}_{k,k+1}\big)\mathcal{A}_{k,k} \\ \qquad\quad + \mathcal{D}_{k,k}^T\big(P_{k,k+1} + T_{k,k+1}\big)\mathcal{C}_{k,k}. \end{cases}$$

Proof. Denote the righthand side of (8) by $\widehat{J}(k,0;\bar{u}_k)$. We then have

$$\begin{aligned} &\widehat{J}(k,0;\bar{u}_k) \\ &= \sum_{\ell=k}^{N-1} \mathbb{E}_k\big[(Y_\ell^{k,\bar{u}_k})^T Q_{k,\ell} Y_\ell^{k,\bar{u}_k} + (\mathbb{E}_k Y_\ell^{k,\bar{u}_k})^T \bar{Q}_{k,\ell}\mathbb{E}_k Y_\ell^{k,\bar{u}_k}\big] \\ &\quad + \bar{u}_k^T \mathcal{R}_{k,k}\bar{u}_k + \mathbb{E}_k\big[(Y_N^{k,\bar{u}_k})^T G_k Y_N^{k,\bar{u}_k}\big] \\ &\quad + (\mathbb{E}_k Y_N^{k,\bar{u}_k})^T \bar{G}_k \mathbb{E}_k Y_N^{k,\bar{u}_k} \\ &= \sum_{\ell=k}^{N-1} \mathbb{E}_k\big[(Y_\ell^{k,\bar{u}_k})^T Q_{k,\ell} Y_\ell^{k,\bar{u}_k} + (\mathbb{E}_k Y_\ell^{k,\bar{u}_k})^T \bar{Q}_{k,\ell}\mathbb{E}_k Y_\ell^{k,\bar{u}_k} \\ &\quad + (Y_{\ell+1}^{k,\bar{u}_k})^T P_{\ell+1} Y_{\ell+1}^{k,\bar{u}_k} - (Y_\ell^{k,\bar{u}_k})^T P_\ell Y_\ell^{k,\bar{u}_k} \\ &\quad + (\mathbb{E}_k Y_{\ell+1}^{k,\bar{u}_k})^T \bar{P}_{\ell+1}\mathbb{E}_k Y_{\ell+1}^{k,\bar{u}_k} - (\mathbb{E}_k Y_\ell^{k,\bar{u}_k})^T \bar{P}_\ell \mathbb{E}_k Y_\ell^{k,\bar{u}_k}\big] \\ &\quad + \bar{u}_k^T \mathcal{R}_{k,k}\bar{u}_k \\ &= \sum_{\ell=k+1}^{N-1} \mathbb{E}_k\big[(\mathbb{E}_k Y_\ell^{k,\bar{u}_k})^T\big(\mathcal{Q}_{k,\ell} + \mathcal{A}_{k,\ell}^T \mathcal{P}_{k,\ell+1}\mathcal{A}_{k,\ell} \\ &\quad + \mathcal{C}_{k,\ell}^T P_{k,\ell+1}\mathcal{C}_{k,\ell} - \mathcal{P}_{k,\ell}\big)\mathbb{E}_k Y_\ell^{k,\bar{u}_k} \\ &\quad + (Y_\ell^{k,\bar{u}_k} - \mathbb{E}_k Y_\ell^{k,\bar{u}_k})^T\big(Q_{k,\ell} + A_{k,\ell}^T P_{k,\ell+1} A_{k,\ell} \\ &\quad + C_{k,\ell}^T P_{k,\ell+1} C_{k,\ell} - P_{k,\ell}\big)(Y_\ell^{k,\bar{u}_k} - \mathbb{E}_k Y_\ell^{k,\bar{u}_k})\big] \\ &\quad + \bar{u}_k^T\big(\mathcal{R}_{k,k} + \mathcal{B}_{k,k}^T \mathcal{P}_{k,k+1}\mathcal{B}_{k,k} + \mathcal{D}_{k,k}^T P_{k,k+1}\mathcal{D}_{k,k}\big)\bar{u}_k \\ &= \bar{u}_k^T\big(\mathcal{R}_{k,k} + \mathcal{B}_{k,k}^T \mathcal{P}_{k,k+1}\mathcal{B}_{k,k} + \mathcal{D}_{k,k}^T P_{k,k+1}\mathcal{D}_{k,k}\big)\bar{u}_k. \end{aligned}$$

By this and (8), we have the solvability of (11).

Let $u^{t,x,*}$ be an open-loop equilibrium control. Then, from Theorem 2.1, for any $k \in \mathbb{T}_t$, the FBSΔE (6) admits a solution and (7) holds. As

$$Z_N^{N-1,t,x} = G_{N-1}X_N^{N-1,t,x} + \bar{G}_{N-1}\mathbb{E}_{N-1}X_N^{N-1,t,x},$$

we have by (7)

$$\begin{aligned} 0 &= \mathcal{R}_{N-1,N-1}u_{N-1}^{t,x,*} \\ &\quad + \mathcal{B}_{N-1,N-1}^T\big[\mathcal{G}_{N-1}\mathbb{E}_{N-1}X_N^{t,x,*} + g_{N-1}\big] \\ &\quad + \mathcal{D}_{N-1,N-1}^T G_{N-1}\mathbb{E}_{N-1}(X_N^{t,x,*}w_{N-1}). \end{aligned}$$

Substituting $X_{N-1}^{t,x,*}$ and by Lemma 2.1, we have

$$u_{N-1}^{t,x,*} = -\mathcal{W}_{N-1}^\dagger \mathcal{H}_{N-1}X_{N-1}^{t,x,*} \triangleq \Psi_{N-1}X_{N-1}^{t,x,*},$$

where

$$\begin{cases} \mathcal{W}_{N-1} = \mathcal{R}_{N-1,N-1} + \mathcal{B}_{N-1,N-1}^T\mathcal{G}_{N-1}\mathcal{B}_{N-1,N-1} \\ \qquad\qquad + \mathcal{D}_{N-1,N-1}^T G_{N-1}\mathcal{D}_{N-1,N-1}, \\ \mathcal{H}_{N-1} = \mathcal{B}_{N-1,N-1}^T\mathcal{G}_{N-1}\mathcal{A}_{N-1,N-1} \\ \qquad\qquad + \mathcal{D}_{N-1,N-1}^T G_{N-1}\mathcal{C}_{N-1,N-1}. \end{cases}$$

Similarly, we have

$$\begin{aligned} Z_{N-1}^{N-2,t,x} &= \big(P_{N-2,N-1} + T_{N-2,N-1}\big)X_{N-1}^{t,x,*} \\ &\quad + \big(\bar{P}_{N-2,N-1} + \bar{T}_{N-2,N-1}\big)\mathbb{E}_{N-2}X_{N-1}^{t,x,*}. \end{aligned}$$

From (7), we have for $k = N-2$

$$\begin{aligned} 0 &= \mathcal{R}_{N-2,N-2}u_{N-2}^{t,x,*} \\ &\quad + \mathcal{B}_{N-2,N-2}^T\big[\big(\mathcal{P}_{N-2,N-1} + \mathcal{T}_{N-2,N-1}\big)\mathbb{E}_{N-2}X_{N-1}^{t,x,*}\big] \\ &\quad + \mathcal{D}_{N-2,N-2}^T\big(P_{N-2,N-1} + T_{N-2,N-1}\big) \\ &\quad \times \mathbb{E}_{N-2}\big(X_{N-1}^{t,x,*}w_{N-2}\big). \end{aligned}$$

Substituting $X_{N-2}^{t,x,*}$ and by Lemma 2.1, we have

$$u_{N-2}^{t,x,*} = -\mathcal{W}_{N-2}^\dagger \mathcal{H}_{N-2}X_{N-2}^{t,x,*} \triangleq \Psi_{N-2}X_{N-2}^{t,x,*},$$

where

$$\begin{cases} \mathcal{W}_{N-2} = \mathcal{R}_{N-2,N-2} \\ \quad + \mathcal{B}_{N-2,N-2}^T\big(\mathcal{P}_{N-2,N-1} + \mathcal{T}_{N-2,N-1}\big)\mathcal{B}_{N-2,N-2} \\ \quad + \mathcal{D}_{N-2,N-2}^T\big(P_{N-2,N-1} + T_{N-2,N-1}\big)\mathcal{D}_{N-2,N-2}, \\ \mathcal{H}_{N-2} = \mathcal{B}_{N-2,N-2}^T\big(\mathcal{P}_{N-2,N-1} + \mathcal{T}_{N-2,N-1}\big) \\ \quad \times \mathcal{A}_{N-2,N-2} + \mathcal{D}_{N-2,N-2}^T \\ \quad \times \big(P_{N-2,N-1} + T_{N-2,N-1}\big)\mathcal{C}_{N-2,N-2}. \end{cases}$$

Backwardly repeating above procedure, we can get (14) and (15). This completes the proof. □

Theorem 2.3 *For any $t \in \mathbb{T}$ and any $x \in L^2_{\mathcal{F}}(t;\mathbb{R}^n)$, Problem (LQ) for the initial pair (t,x) admits an open-loop equilibrium pair if and only if*

$$\begin{cases} \begin{cases} P_{k,\ell} = Q_{k,\ell} + A_{k,\ell}^T P_{k,\ell+1}A_{k,\ell} \\ \qquad\quad + C_{k,\ell}^T P_{k,\ell+1}C_{k,\ell}, \\ \mathcal{P}_{k,\ell} = \mathcal{Q}_{k,\ell} + \mathcal{A}_{k,\ell}^T \mathcal{P}_{k,\ell+1}\mathcal{A}_{k,\ell} \\ \qquad\quad + \mathcal{C}_{k,\ell}^T P_{k,\ell+1}\mathcal{C}_{k,\ell}, \\ P_{k,N} = G_k, \ \ \mathcal{P}_{k,N} = \mathcal{G}_k, \\ \ell \in \mathbb{T}_k, \end{cases} \\ \mathcal{R}_{k,k} + \mathcal{B}_{k,k}^T\mathcal{P}_{k,k+1}\mathcal{B}_{k,k} + \mathcal{D}_{k,k}^T P_{k,k+1}\mathcal{D}_{k,k} \geq 0, \\ k \in \mathbb{T} \end{cases} \tag{15}$$

and the set of GDREs

$$\begin{cases} \begin{cases} T_{k,\ell} = A_{k,\ell}^T T_{k,\ell+1}\mathcal{A}_{\ell,\ell} + C_{k,\ell}^T T_{k,\ell+1}\mathcal{C}_{\ell,\ell} \\ \quad - \big(A_{k,\ell}^T P_{k,\ell+1}B_{k,\ell} + A_{k,\ell}^T T_{k,\ell+1}\mathcal{B}_{\ell,\ell} \\ \quad + C_{k,\ell}^T P_{k,\ell+1}D_{k,\ell} + C_{k,\ell}^T T_{k,\ell+1}\mathcal{D}_{\ell,\ell}\big)\mathcal{W}_\ell^\dagger\mathcal{H}_\ell, \\ \mathcal{T}_{k,\ell} = \mathcal{A}_{k,\ell}^T \mathcal{T}_{k,\ell+1}\mathcal{A}_{\ell,\ell} + \mathcal{C}_{k,\ell}^T T_{k,\ell+1}\mathcal{C}_{\ell,\ell} \\ \quad - \big(\mathcal{A}_{k,\ell}^T \mathcal{P}_{k,\ell+1}\mathcal{B}_{k,\ell} + \mathcal{A}_{k,\ell}^T \mathcal{T}_{k,\ell+1}\mathcal{B}_{\ell,\ell} \\ \quad + \mathcal{C}_{k,\ell}^T P_{k,\ell+1}\mathcal{D}_{k,\ell} + \mathcal{C}_{k,\ell}^T T_{k,\ell+1}\mathcal{D}_{\ell,\ell}\big)\mathcal{W}_\ell^\dagger\mathcal{H}_\ell, \\ T_{k,N} = 0, \ \ \mathcal{T}_{k,N} = 0, \\ \ell \in \mathbb{T}_k, \end{cases} \\ \mathcal{W}_k\mathcal{W}_k^\dagger\mathcal{H}_k - \mathcal{H}_k = 0, \\ k \in \mathbb{T} \end{cases} \tag{16}$$

are solvable, where

$$\begin{cases} \mathcal{W}_k = \mathcal{R}_{k,k} + \mathcal{B}_{k,k}^T\big(\mathcal{P}_{k,k+1} + \mathcal{T}_{k,k+1}\big)\mathcal{B}_{k,k} \\ \qquad\quad + \mathcal{D}_{k,k}^T\big(P_{k,k+1} + T_{k,k+1}\big)\mathcal{D}_{k,k}, \\ \mathcal{H}_k = \mathcal{B}_{k,k}^T\big(\mathcal{P}_{k,k+1} + \mathcal{T}_{k,k+1}\big)\mathcal{A}_{k,k} \\ \qquad\quad + \mathcal{D}_{k,k}^T\big(P_{k,k+1} + T_{k,k+1}\big)\mathcal{C}_{k,k}. \end{cases} \tag{17}$$

In this case, an open-loop equilibrium control for the initial pair (t,x) is given in

$$u_k^{t,x,*} = -\mathcal{W}_k^{\dagger}\mathcal{H}_k X_k^{t,x,*},\ \ k\in\mathbb{T}_t. \tag{18}$$

Proof. Sufficiency. Introduce a dynamics

$$\begin{cases} \widetilde{X}_{k+1}^{t,x,*} = \big(\mathcal{A}_{k,k} - \mathcal{B}_{k,k}\mathcal{W}_k^{\dagger}\mathcal{H}_k\big)\widetilde{X}_k^{t,x,*} \\ \qquad\qquad + \big(\mathcal{A}_{k,k} - \mathcal{B}_{k,k}\mathcal{W}_k^{\dagger}\mathcal{H}_k\big)\widetilde{X}_k^{t,x,*} w_k, \\ \widetilde{X}_t^{t,x,*} = x,\ \ k\in\mathbb{T}_t, \end{cases} \tag{19}$$

and a control

$$\widetilde{u}_k^{t,x,*} = -\mathcal{W}_k^{\dagger}\mathcal{H}_k \widetilde{X}_k^{t,x,*},\ \ k\in\mathbb{T}_t. \tag{20}$$

Then, by reversing the first part of the proof of Theorem 2.2, we can show that for any $k\in\mathbb{T}_t$, the following FBSΔE admits an adapted solution

$$\begin{cases} \widetilde{X}_{\ell+1}^{k,t,x} = \big(A_{k,\ell}\widetilde{X}_\ell^{k,t,x} + \bar{A}_{k,\ell}\mathbb{E}_k\widetilde{X}_\ell^{k,t,x} \\ \qquad\qquad + B_{k,\ell}\widetilde{u}_\ell^{t,x,*} + \bar{B}_{k,\ell}\mathbb{E}_k\widetilde{u}_\ell^{t,x,*}\big) \\ \qquad\qquad + \big(C_{k,\ell}\widetilde{X}_\ell^{k,t,x} + \bar{C}_{k,\ell}\mathbb{E}_k\widetilde{X}_\ell^{k,t,x} \\ \qquad\qquad + D_{k,\ell}\widetilde{u}_\ell^{t,x,*} + \bar{D}_{k,\ell}\mathbb{E}_k\widetilde{u}_\ell^{t,x,*}\big)w_\ell, \\ \widetilde{Z}_\ell^{k,t,x} = A_{k,\ell}^T\mathbb{E}_\ell\widetilde{Z}_{\ell+1}^{k,t,x} + \bar{A}_{k,\ell}^T\mathbb{E}_k\widetilde{Z}_{\ell+1}^{k,t,x} \\ \qquad\qquad + C_{k,\ell}^T\mathbb{E}_\ell(\widetilde{Z}_{\ell+1}^{k,t,x}w_\ell) + \bar{C}_{k,\ell}^T\mathbb{E}_k(\widetilde{Z}_{\ell+1}^{k,t,x}w_\ell) \\ \qquad\qquad + Q_{k,\ell}\widetilde{X}_\ell^{k,t,x} + \bar{Q}_{k,\ell}\mathbb{E}_k\widetilde{X}_\ell^{k,t,x}, \\ \widetilde{X}_k^{k,t,x} = \widetilde{X}_k^{t,x,*},\ \ \widetilde{Z}_N^{k,t,x} = G_k\widetilde{X}_N^{k,t,x} + \bar{G}_k\mathbb{E}_k\widetilde{X}_N^{k,t,x}, \\ \ell\in\mathbb{T}_k \end{cases}$$

with property

$$\begin{aligned} \widetilde{Z}_\ell^{k,t,x} &= P_{k,\ell}(\widetilde{X}_\ell^{k,t,x} - \mathbb{E}_k\widetilde{X}_\ell^{k,t,x}) + \mathcal{P}_{k,\ell}\mathbb{E}_k\widetilde{X}_\ell^{k,t,x} \\ &\quad + T_{k,\ell}(\widetilde{X}_\ell^{t,x,*} - \mathbb{E}_k\widetilde{X}_\ell^{t,x,*}) + \mathcal{T}_{k,\ell}\mathbb{E}_k\widetilde{X}_\ell^{t,x,*}, \end{aligned}$$

and

$$0 = \mathcal{R}_{k,k}\widetilde{u}_k^{t,x,*} + \mathcal{B}_{k,k}^T\mathbb{E}_k\widetilde{Z}_{k+1}^{k,t,x} + \mathcal{D}_{k,k}^T\mathbb{E}_k(\widetilde{Z}_{k+1}^{k,t,x}w_k).$$

Furthermore, by (12) and (15) we have (8). From Theorem 2.1, Problem (LQ) for the initial pair (t,x) admits an open-loop equilibrium pair, and $(\widetilde{X}^{t,x,*},\widetilde{u}^{t,x,*})$ is an equilibrium pair.

Necessity. By Theorem 2.2 we need only to prove

$$\mathcal{W}_k\mathcal{W}_k^{\dagger}\mathcal{H}_k - \mathcal{H}_k = 0,\ \ k\in\mathbb{T}. \tag{21}$$

Consider Problem (LQ) for the initial pair $(N-1,x)$ with $x\in L^2_{\mathcal{F}}(N-1;\mathbb{R}^n)$. By the proof of Theorem 2.2, we have

$$0 = \mathcal{W}_{N-1}u_{N-1}^{N-1,x,*} + \mathcal{H}_{N-1}X_{N-1}^{N-1,x,*}. \tag{22}$$

Let e_i be a $\mathbb{R}^n$-valued vector with the i-th entry being 1 and other entries 0. Then, we have

$$0 = \mathcal{W}_{N-1}\big(u_{N-1}^{N-1,e_1,*},...,u_{N-1}^{N-1,e_n,*}\big) + \mathcal{H}_{N-1}\big(e_1,...,e_n\big).$$

Noting that $\big(e_1,...,e_n\big)$ is the identity matrix and by Lemma 2.1, we have $\mathcal{W}_{N-1}\mathcal{W}_{N-1}^{\dagger}\mathcal{H}_{N-1} - \mathcal{H}_{N-1} = 0$.

Considering Problem (LQ) for the initial pair $(N-2,x)$ with $x\in L^2_{\mathcal{F}}(N-2;\mathbb{R}^n)$, we can similarly prove

$$\mathcal{W}_{N-2}\mathcal{W}_{N-2}^{\dagger}\mathcal{H}_{N-2} - \mathcal{H}_{N-2} = 0.$$

Continuing above procedure, we then achieve the conclusion. □

Note that $P_{k,\ell},\mathcal{P}_{k,\ell},k\in\mathbb{T},\ell\in\mathbb{T}_{k+1}$, are symmetric. If $Q_{k,\ell},\bar{Q}_{k,\ell},R_{k,\ell},\bar{R}_{k,\ell}$ are selected such that

$$Q_{k,\ell},\ Q_{k,\ell}+\bar{Q}_{k,\ell},\ R_{k,\ell},\ R_{k,\ell}+\bar{R}_{k,\ell}\geq 0,\ k\in\mathbb{T},\ \ell\in\mathbb{T}_k,$$

then (15) is solvable. Furthermore, $\Theta_{k,\ell} = \{P_{k,\ell},\mathcal{P}_{k,\ell},T_{k,\ell},\mathcal{T}_{k,\ell},\pi_{k,\ell}\}$ are used to express $Z_\ell^{k,t,x}$. $\{P_{k,\ell},\mathcal{P}_{k,\ell}\}$ is then called the symmetric part of $\Theta_{k,\ell}$, and $\{T_{k,\ell},\mathcal{T}_{k,\ell}\}$ are viewed as the nonsymmetric part.

3 Conclusion

In this paper, the open-loop time-consistent equilibrium control is investigated for a kind of mean-field stochastic LQ problem. Necessary sufficient conditions are presented for the cases with a fixed initial pair and all the initial pairs. Furthermore, the GDREs and LDEs are introduced to characterize the open-loop equilibrium control. For future research, the closed-loop time-consistent solution should be studied.

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