A Unified Framework for Fault Tolerant Quantum Filtering and Fault Detection

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Abstract: A quantum-classical probability space model is developed for a class of open quantum systems where the system dynamics involve both classical and quantum random variables. This model provides a unified framework for analyzing open quantum systems subject to stochastic faults. A fault tolerant quantum filter and a fault detection equation are simultaneously derived for this class of open quantum systems using a reference probability approach. A two-level open quantum system subject to Poisson-type faults is presented to illustrate the proposed method. These results have the potential for the systematic development of new fault tolerant control theory for quantum systems.

Key Words: Open quantum systems; quantum-classical probability model; fault tolerant quantum filtering; fault detection

1 Introduction

From the fundamental postulates of quantum mechanics, one is not allowed to make noncommutative observations for quantum systems in a single realization or experiment. Any quantum measurement yields in principle only partial information about the system. This fact makes the theory of quantum filtering extremely useful in measurement based feedback control of quantum systems, especially in the field of quantum optics [1]. A very early approach to quantum filtering was presented in a series of papers by Belavkin in the 1980s [2–4]. In physics community, the theory of quantum filtering was also independently developed in the early 1990s [5], named "quantum trajectory theory" in the context of quantum optics.

Particular emphasis is given to the work by Bouten *et al.* [6] where quantum probability theory was used in a rigorous way and the quantum filtering equations for a laser-atom interaction setup in quantum optics were derived using a quantum reference probability method. A basic idea in quantum probability theory is an isomorphic equivalence between a commutative subalgebra of quantum operators on a Hilbert space and a classical (Kolmogorov) probability space through the spectral theorem, from which any probabilistic quantum operation within the commutative subalgebra can be associated with a classical random variable. The complete quantum probability model is treated as the noncommutative counterpart of Kolmogorov's axiomatic characterisation of classical probability. Similar to the classical case [13], the optimal estimate of any observable is given by its quantum expectation conditioned on the history of continuous nondemolition quantum measurements of the electromagnetic field. The quantum filter was derived in terms of $It\hat{o}$ stochastic differential equations using a reference probability method [13].

In practice, classical randomness may be introduced directly into the system dynamics of open quantum systems (e.g., classical random variables in the Hamiltonians [9–11]). However, many probabilistic operations for random quantum observables and classical random variables are not well defined in the framework of quantum probability theory built on a *deterministic* type commutative-algebra-normal-state structure. It is desirable to develop a unified framework that can be used to analyze a quantum system where classical and quantum randomnesses coexist. In this paper, we establish a new quantum-classical probability model that is built on a random commutative algebra equipped with a new normal state. The quantum-classical probability space is described by a quadruple, and quantum probability space and classical probability space can be considered as its special cases. When we concentrate on a class of open quantum systems with stochastic faults, the quantum-classical probability model provides a unified framework to simultaneously derive a fault tolerant quantum filter and a fault detection equation for this class of systems.

This paper is organized as follows. Section 2 describes the class of open quantum systems under consideration in this paper. A quantum-classical probability space model is presented in Section 3. In Section 4, the fault tolerant quantum filter and fault detection equations are simultaneously derived for open quantum systems using the quantum-classical probability space model. An example of two-level quantum systems with Poisson-type faults is illustrated. Section 5 concludes this paper.

2 Heisenberg Dynamics of Open Quantum Systems

In this work, we concentrate on an open quantum system that has been widely investigated in quantum optics [1, 7, 8]. The quantum system under consideration is a cloud of atoms in weak interaction with an external laser probe field which is continuously monitored by a homodyne detector [6, 19]. Such quantum systems can be described by quantum stochastic differential equations driven by quantum noises $B(t)$ and $B^{\dagger}(t)$ [1]. The dynamics of the quantum system are described by the following quantum stochastic differential equation[1]:

This work was supported by the Australian Research Council (DP130101658, FL110100020).

[1]We have assumed $\hbar$=1 by using atomic units in this paper.

$$dU(t) = \left\{ \left(-iH(t) - \frac{1}{2}L^\dagger L \right) dt + LdB^\dagger(t) - L^\dagger dB(t) \right\} U(t), \quad (1)$$

with initial condition $U(0) = I$ and $i = \sqrt{-1}$. In terms of the system states, if π_0 is a given system state, we write $\rho_0 = \pi_0 \otimes |\upsilon\rangle \langle\upsilon|$, where $|\upsilon\rangle$ represents the vacuum state. Here $U(t)$ describes the Heisenberg-picture evolution of the system operators. The Hilbert space for the composite system is given by $\mathcal{H}_\mathcal{S} \otimes \mathcal{E} = \mathcal{H}_\mathcal{S} \otimes \mathcal{E}_{t]} \otimes \mathcal{E}_{(t}$, with $\dim(\mathcal{H}_\mathcal{S}) = n < \infty$. The atomic observables are described by self-adjoint operators on $\mathcal{H}_\mathcal{S}$. Any system observable X at time t is given by$X(t) = j_t(X) = U^\dagger(t)(X \otimes I)U(t)$. The system operator L, together with the field operator $b(t) = \dot{B}(t)$ model the interaction between the system and the field. When the input field is in the vacuum state $|\upsilon\rangle$, one has [14]

$$dB(t)dB^\dagger(t) = dt,$$
$$dB^\dagger(t)dB(t) = dB(t)dB(t) = dB^\dagger(t)dB^\dagger(t) = 0.$$

It is noted that (1) is written in $It\hat{o}$ form, as will all stochastic differential equations in this paper.

In practice, the system Hamiltonian may change randomly because of, e.g., faulty control Hamiltonians that appear in system dynamics at random times [9, 10] or random fluctuations of the external electromagnetic field [11, 12]. In this case, the system Hamiltonian can be described by a Hermitian operator $H(F(t))$ that depends on some classical stochastic process $F(t)$. Using the quantum $It\hat{o}$ rule [16], one has $d(U^\dagger(t)U(t)) = d(U(t)U^\dagger(t)) = 0$, which implies that $U(t)$ is a *random unitary operator* which depends on the stochastic process $F(t)$. In this paper, for simplicity we still write $U(t)$ instead of the functional form $U(F, t)$. One can conclude that the commutativity of observables are preserved, that is, $[j_t(A), j_t(B)] = 0$ if $[A, B] = 0$ where A, B are two system observables in $\mathcal{H}_\mathcal{S}$. In addition, from (1) one can see that $U(t)$ depends on $B(t')$ and $B^\dagger(t')$, $0 \le t' < t$, since the increments $dB(t)$ and $dB^\dagger(t)$ point to the future evolution. Consequently,

$$[U(t), dB(t)] = [U(t), dB^\dagger(t)] = 0. \quad (2)$$

Similarly, the time evolution operator $U(t, s) = U(t)U^\dagger(s)$ from time s to time t depends only on the field operators $dB(s')$ and $dB^\dagger(s')$ with $s \le s' \le t$. Thus,

$$[U(t, s), B(\tau)] = [U(t, s), B^\dagger(\tau)] = 0, \tau \le s. \quad (3)$$

In quantum experiments, generally measurement is performed on the field. Using homodyne detectors, the observation process is given by $Y(t) = j_t(Q(t)) = U^\dagger(t)(I \otimes Q(t))U(t)$ where $Q(t) = B(t) + B^\dagger(t)$ is the real quadrature of the input field. The operator $Q(t)$ is commutative at different times, i.e., $[Q(t), Q(s)] = 0$. When the field is initialized in the vacuum state, $Q(t)$ is isomorphically equivalent to a real Wiener process [14]. Combing (2) and (3) with the fact that $[I \otimes Q(t), X \otimes I] = 0$, it is easy to show that: (i) $[Y(t), Y(s)] = 0$ at all times s, t and (ii) $[Y(s), X(t)] = 0, \forall s \le t$. These two properties guarantee that (i) $Y(t)$ can be continuously monitored and (ii) it is possible to obtain the conditional statistics of an observable $X(t)$ based on the history of $Y(t)$. In addition, by using the quantum $It\hat{o}$ rule, one has

$$dY(t) = U^\dagger(t)(L + L^\dagger)U(t)dt + dQ(t), \quad (4)$$

from which $Y(t)$ looks like $j_t(L+L^\dagger) = U^\dagger(t)(L+L^\dagger)U(t)$ with a noise $Q(t)$.

3 Quantum-Classical Probability Space

Let $(\Omega, \mathcal{F}, \mathcal{P})$ be a complete classical probability space with a right continuous and complete filtration $\{\mathcal{F}_t\}_{t\ge0}$ of sub-σ fields of $\mathcal{F}$. In the sequel, $\mathbb{E}\{\cdot\}$ denotes the mathematical expectation operator with respect to the given probability measure $\mathcal{P}$.

When classical random variables are introduced into the Hamiltonian of an open quantum system, the standard quantum filter will fail to produce (least mean square) optimal estimates of the system states. The quantum probability theory, which is built on a deterministic type commutative-algebra-normal-state structure, cannot provide a simple formulation for applications with many probabilistic operations on random observables and classical random variables. This brings difficulties in applying the quantum probability theory to analyzing quantum systems that evolve randomly, like the case we considered in Section 2.

In this section, we introduce a new quantum-classical probability space which can deal with both classical and quantum randomnesses in a unified framework. To begin with, we briefly introduce the quantum probability theory. Let $\mathcal{H}$ be a complex Hilbert space and $\mathscr{B}(\mathcal{H})$ be the set of all bounded operators in $\mathcal{H}$. We first discuss the case that $\dim(\mathcal{H}) = n < \infty$. It is known that the foundations of quantum mechanics can be also formulated in a similar language to the classical Kolmogorov's probability theory [14]. The basic ideas are as follows. Based on the spectral theorem [15], any self-adjoint operator A admits a spectral decomposition $A = \sum_{j=1}^n a_j P_{A_j}$, where $\{a_j\} \subset \mathbb{R}$ are the eigenvalues of A and $\{P_{A_j}\}$ are the corresponding orthogonal projection operators which form a resolution of the identity, i.e., $P_{A_j}P_{A_k} = \delta_{jk}P_{A_k}$ and $\sum_{j=1}^n P_{A_j} = I$. For any continuous function $f : \mathbb{R} \to \mathbb{C}$, one has $f(A) = \sum_{j=1}^n f(a_j)P_{A_j}$. Thus the set $\mathscr{A} = \{X : X = f(A), \forall f : \mathbb{R} \to \mathbb{C}\}$ forms a commutative $*-$algebra generated by A. That is, arbitrary linear combinations, products and adjoints of operators in $\mathscr{A}$ are still in $\mathscr{A}$, $I \in \mathscr{A}$ and all elements of $\mathscr{A}$ commute. A mapping $\mathbb{P} : \mathscr{A} \to \mathbb{C}$ is called a normal state on $\mathscr{A}$ if it is positive and normalized, i.e., $\mathbb{P}(X) \ge 0$ if $X \ge 0$ and $\mathbb{P}(I) = 1$. From Theorem 7.1.12 in [22], there is always a density operator ρ such that $\mathbb{P}(X) = \mathrm{Tr}(\rho X)$. The following lemma can be obtained.

Lemma 2.1 [6] (Equivalence theorem, finite-dimensional case). Let $\mathscr{A}$ be a commutative $*-$algebra of operators on a finite-dimensional Hilbert space $\mathcal{H}$, and let $\mathbb{P}$ be a normal state on $\mathscr{A}$. There is a classical probability space $(\Omega', \mathcal{F}', \mathcal{P}')$ and a $*-$isomorphism[2] ι from $\mathscr{A}$ onto the

[2] A $*-$isomorphism ι is a linear bijection with $\iota(XY) = \iota(X)\iota(Y)$ and $\iota(X^\dagger) = \iota(X)^\dagger$. Here ι depends only on a unitary operator U by which all elements of the algebra $\mathscr{A}$ can be diagonalized. One can always find such an operator U since all elements of $\mathscr{A}$ commute.

set of measurable functions on Ω, and moreover $\mathbb{P}(X) = \mathbb{E}_{\mathcal{P}'}(\iota(X)), \forall X \in \mathscr{A}$.

Thus a commutative $*-$algebra structure is completely equivalent to a classical probability space. The pair $(\{P_{A_j}\}, \mathbb{P})$ acts the same as $(\mathcal{F}', \mathcal{P}')$. What makes quantum probability model different from classical probability model is the existence of non-commutative observables. In classical probability, in every realization any event is either true or false, regardless of how many events we choose to observe and the order of observations. However, in quantum probability, given a prior observation of an event P, any subsequent events that do not commute with P become physical meaningless within the same realization. Consequently, the joint statistics are only defined among commuting observables.

Quantum probability theory is built on the $*-$algebra structure of deterministic operators on a Hilbert space. In many physical situations, however, quantum systems may evolve randomly because of the interaction with classical random processes, as discussed in Section 2. In this case, the system evolution is described by a random unitary operator $U_R(t)$ that depends on some classical random variable (vector) R, and any system observable A at time t will be in the form of a *random observable* $A(R,t) = U_R^\dagger(t) A U_R(t)$. In this paper, we assume that R is defined on a probability space $(\Omega, \mathcal{F}, \mathcal{P})$ and takes values in a finite set $\{R_1, ..., R_{n_r}\}$. In each single quantum measurement of the system observable, we actually go through two realizations: (i) the choice of a sample point $\omega \in \Omega$ and (ii) the quantum measurement performed on a deterministic observable $A(R(\omega), t)$. As a result, given a system state ρ, the average observed value of $A(R,t)$ should be $\tilde{\mathbb{P}}(A(R,t))$, where $\tilde{\mathbb{P}}$ is a linear map $\tilde{\mathbb{P}}(\cdot) = \mathbb{E}\{\mathrm{Tr}\{\rho(\cdot)\}\} : \mathscr{B}(\mathcal{H}) \to \mathbb{R}$.

The measurement results of $A(R,t)$ contains information of the random variable R, which makes it natural to ask if the joint statistics of $A(R,t)$ and some classical random variables depending on R can be well defined. This is indeed possible because we can equivalently treat any classical random variable $\nu(R)$ as a random observable $\nu(R)I$ under $\tilde{\mathbb{P}}$ on the Hilbert space, since $\mathbb{E}_{\mathcal{P}}(e^{it\nu(R)}) = \tilde{\mathbb{P}}(e^{it\nu(R)I})$ for any density operator ρ. In other words, $\nu(R)$ and $\nu(R)I$ share the same character function. Without loss of generality, here we suppose ν is a scalar function. The results obtained can be extended directly to the multi-dimensional case. It is clear that $\nu(R)I$ commutes with all quantum operators in $\mathcal{H}$ (this is exactly a property of classical random variables). Then we have the following result.

Lemma 2.2. For any given self-adjoint operator A, random unitary operator $U_R(t)$ and scalar function $\nu(R)$, the set of random self-adjoint operators $\tilde{\mathscr{A}} = \{X : X = f(\nu(R)A(R,t)), \forall f : \mathbb{R} \to \mathbb{C}\}$ forms a commutative $*-$algebra on $\mathcal{H}$ and has a normal state $\tilde{\mathbb{P}}$. In addition, any normal state on $\tilde{\mathscr{A}}$ can be written in the form of $\mathbb{E}\{\mathrm{Tr}\{\rho'(\cdot)\}\}$ for some density operator ρ'.

Proof. The normality of $\tilde{\mathbb{P}}$ can be directly verified. The first part of Lemma 2.2 can be obtained if $\tilde{\mathscr{A}}$ has a basis consisting of projection operators. From the spectral theorem, one has $A(R,t) = \sum_{j=1}^{n} a_j \tilde{P}_j(R,t)$ and $\nu(R) = \sum_{k=1}^{n_0} \nu_k \mathbf{1}_{\nu(R)=\nu_k}$, where $\tilde{P}_j(R,t) = U_R^\dagger(t) P_{A_j} U_R(t)$ and $\mathbf{1}_{\nu(R)=\nu_k}$ is the indicator function. Then any element in $\tilde{\mathscr{A}}$ can be written as a linear combination of the projection operators $\{\mathbf{1}_{\nu(R)=\nu_k} \tilde{P}_j(R,t)\}, k \in \{1,..,n_r\}, j \in \{1,...,n\}$. The conclusion of the second part follows directly from Theorem 7.1.12 in [8]. □

Remark 2.1. One can observe that both $\mathscr{A}$ and $U_R^\dagger(t)\mathscr{A}U_R(t)$ are commutative subalgebras of $\tilde{\mathscr{A}}$. In addition, $\tilde{\mathscr{A}}$ contains the $\sigma-$algebra generated by the classical random variable $\nu(R)$ as a special case.

From Lemmas 2.1 and 2.2, we have the following result.

Theorem 2.1. (General equivalence theorem, finite-dimensional case). Let $\tilde{\mathscr{A}}$ be a random commutative $*-$algebra of operators on the Hilbert space $\mathcal{H}$, equipped with a normal state $\tilde{\mathbb{P}}$. There exist a probability space $(\Omega', \mathcal{F}', \mathcal{P}')$ and a $*-$isomorphism ι from $\tilde{\mathscr{A}}$ onto the set of measurable functions on Ω', and moreover $\tilde{\mathbb{P}}(X) = \mathbb{E}_{\mathcal{P}'}(\iota(X)), \forall X \in \tilde{\mathscr{A}}$.

Thus a random commutative $*-$algebra $\tilde{\mathscr{A}}$ with a normal state $\tilde{\mathbb{P}}$ is completely equivalent to a classical probability space. In other words, when the discussion is restricted to a random commutative $*-$algebra, any probabilistic operation can be defined directly in terms of the associated classical probability space. In particular, we consider the conditional expectation which will be used in subsequent analysis.

Let $Y_s \in \tilde{\mathscr{A}}'$ be a self-adjoint operator. Then Y_s and $\tilde{\mathscr{A}}$ can generate a larger random commutative $*-$algebra, which is isomorphic to a classical probability space through a linear mapping ι, based on Theorem 2.1. By using classical probability theory, the conditional expectation can be defined as $\tilde{\mathbb{P}}(Y_s|\tilde{\mathscr{A}}) = \iota^{-1}(\mathbb{E}_{\mathcal{P}'}(\iota(Y_s)|\sigma\{\iota(\tilde{\mathscr{A}})\}))$. This discussion can be extended to any operator $Y \in \tilde{\mathscr{A}}'$. To be specific, Y can be written as $Y = Y_1 + iY_2$, where $Y_1 = \frac{Y+Y^\dagger}{2}$ and $Y_2 = \frac{i(Y^\dagger - Y)}{2}$. Thus, $\tilde{\mathbb{P}}(Y|\tilde{\mathscr{A}}) = \tilde{\mathbb{P}}(Y_1|\tilde{\mathscr{A}}) + i\tilde{\mathbb{P}}(Y_2|\tilde{\mathscr{A}})$, with $\tilde{\mathbb{P}}(Y_1|\tilde{\mathscr{A}})$ and $\tilde{\mathbb{P}}(Y_2|\tilde{\mathscr{A}})$ well defined.

Following the same idea in classical probability theory, the following definition of conditional expectation is given.

Definition 2.1. (Conditional expectation, finite-dimensional case) Consider a random commutative $*-$algebra $\tilde{\mathscr{A}}$ equipped with a normal state $\tilde{\mathbb{P}}$. The map $\tilde{\mathbb{P}}(\cdot|\tilde{\mathscr{A}}) : \tilde{\mathscr{A}}' \to \tilde{\mathscr{A}}$ is called (a version of) the conditional expectation from $\tilde{\mathscr{A}}'$ onto $\tilde{\mathscr{A}}$ if $\tilde{\mathbb{P}}(\tilde{\mathbb{P}}(X|\tilde{\mathscr{A}})Y) = \tilde{\mathbb{P}}(XY)$ for all $X \in \tilde{\mathscr{A}}', Y \in \tilde{\mathscr{A}}$.

When the Hilbert space is of finite dimension, an explicit expression for conditional expectation can be obtained. Let $\{\tilde{P}_j\}$ be the set of basis projection operators of $\tilde{\mathscr{A}}$ and $X \in \tilde{\mathscr{A}}'$. Then a version of the conditional expectation is given by

$$\tilde{\mathbb{P}}(X|\tilde{\mathscr{A}}) = \sum_{\tilde{\mathbb{P}}(\tilde{P}_j) \neq 0} \frac{\tilde{\mathbb{P}}(\tilde{P}_j X)}{\tilde{\mathbb{P}}(\tilde{P}_j)} \tilde{P}_j. \tag{5}$$

We here discuss two special cases of the expression (5).

Case 1. Suppose $\tilde{\mathscr{A}}$ is a deterministic commutative $*-$algebra, e.g., $\tilde{\mathscr{A}} = \mathscr{A}$, and $X \in \tilde{\mathscr{A}}'$ is a deterministic operator. Then $\{\tilde{P}_j\}$ is also deterministic and one has

$$\tilde{\mathbb{P}}(X|\tilde{\mathscr{A}}) = \sum_{\mathbb{P}(\tilde{P}_j) \neq 0} \frac{\mathbb{P}(\tilde{P}_j X)}{\mathbb{P}(\tilde{P}_j)} \tilde{P}_j, \tag{6}$$

where $\mathbb{P}(\cdot) = \mathrm{Tr}\{\rho(\cdot)\}$. In this case, (5) reduces to the quantum conditional expectation in Equation (2.10) of [6].

Case 2. Set $A \equiv I$, then $\tilde{\mathscr{A}}$ in Lemma 2.2 reduces to a $\sigma-$field generated by a classical random variable. Let $X = xI$ with x being a random variable on $(\Omega, \mathcal{F}, \mathcal{P})$. Then one has

$$\begin{aligned}\tilde{\mathbb{P}}(X|\tilde{\mathscr{A}}) &= \sum_{\mathbb{E}(\mathbf{1}_{\nu(R)=\nu_j})\neq 0} \frac{\mathbb{E}(x\mathbf{1}_{\nu(R)=\nu_j})}{\mathbb{E}(\mathbf{1}_{\nu(R)=\nu_j})}\mathbf{1}_{\nu(R)=\nu_j} \\ &= \mathbb{E}(x|\nu(R)), \end{aligned} \tag{7}$$

which is exactly the classical conditional expectation [13].

Thus the defined conditional expectation contains both classical and quantum conditional expectations as special cases. The above analysis can be directly extended to infinite-dimensional Hilbert spaces and we go directly to the following fundamental theorems and definitions without mentioning the details.

Theorem 2.2. (Equivalence theorem). Let $\tilde{\mathscr{C}}$ be a random commutative von Neumann algebra equipped with a normal state $\tilde{\mathbb{P}}$. Then there exist a probability space $(\Omega', \mathcal{F}', \mathcal{P}')$ and a $*-$isomorphism ι from $\tilde{\mathscr{C}}$ onto the algebra of bounded measurable complex functions on Ω', such that $\tilde{\mathbb{P}}(X) = \mathbb{E}_{\mathcal{P}'}(\iota'(X)), X \in \tilde{\mathscr{C}}$.

Proof. Theorem 2.2 follows from Theorem 3.3 in [6]. □

Definition 2.2. (Quantum-classical probability space) A quantum-classical probability space is a quadruple $(\Omega, \mathcal{F}, \tilde{\mathscr{N}}, \tilde{\mathbb{P}})$, where $\tilde{\mathscr{N}}$ is a von Neumann algebra.

Definition 2.3. (Conditional expectation) Let $\tilde{\mathscr{C}}$ be a commutative von Neumann algebra equipped with a normal state $\tilde{\mathbb{P}}$. Then the map $\tilde{\mathbb{P}}(\cdot|\tilde{\mathscr{C}})$ is called (a version of) the conditional expectation from $\tilde{\mathscr{C}}'$ onto $\tilde{\mathscr{C}}$, if $\tilde{\mathbb{P}}(\tilde{\mathbb{P}}(X|\tilde{\mathscr{C}})Y) = \tilde{\mathbb{P}}(XY)$ for all $X \in \tilde{\mathscr{C}}'$ and $Y \in \tilde{\mathscr{C}}$.

Theorem 2.3. The conditional expectation of Definition 2.3 exists and is unique with probability one (any two versions P and Q of $\tilde{\mathbb{P}}(\cdot|\tilde{\mathscr{C}})$ satisfies $\|P - Q\|_{\tilde{\mathbb{P}}} = 0$, where $\|X\|_{\tilde{\mathbb{P}}} = \tilde{\mathbb{P}}(X^\dagger X)$). Moreover, $\tilde{\mathbb{P}}(X|\tilde{\mathscr{C}})$ is the least mean square estimate of X given $\tilde{\mathscr{C}}$ in the sense that $\|X - \tilde{\mathbb{P}}(X|\tilde{\mathscr{C}})\| \leq \|X - Y\|$ for all $Y \in \tilde{\mathscr{C}}$.

Proof. Theorem 2.3 follows from Theorem 3.16 in [6]. □

Remark 2.2. The elementary properties of classical conditional expectation, for example, linearity, positivity, the tower property and "taking out what is known", still hold for the above defined conditional expectation. Proofs follow directly from classical cases and are omitted here.

We end this section with a quantum-classical Bayes formula.

Theorem 2.4. (Quantum-classical Bayes formula) Consider a quantum-classical probability space $(\Omega, \mathcal{F}, \tilde{\mathscr{N}}, \tilde{\mathbb{P}})$ and let $\tilde{\mathscr{C}} \subset \tilde{\mathscr{N}}$ be a commutative von Neumann algebra. Suppose a new probability measure $\mathcal{Q}$ is defined by $d\mathcal{Q} = \Lambda d\mathcal{P}$, where the $\mathcal{F}-$measurable random variable Λ is the Radon-Nikon derivative. Choose $V \in \tilde{\mathscr{C}}'$ such that $V^\dagger V > 0$ and $\tilde{\mathbb{P}}(\Lambda V^\dagger V) = 1$. Then we can define on $\tilde{\mathscr{C}}'$ a new normal state $\tilde{\mathbb{Q}}$ by $\tilde{\mathbb{Q}}(X) = \tilde{\mathbb{P}}(\Lambda V^\dagger XV)$ and

$$\tilde{\mathbb{Q}}(X|\tilde{\mathscr{C}}) = \frac{\tilde{\mathbb{P}}(\Lambda V^\dagger XV/\tilde{\mathscr{C}})}{\tilde{\mathbb{P}}(\Lambda V^\dagger V/\tilde{\mathscr{C}})}, \qquad \forall X \in \tilde{\mathscr{C}}'. \tag{8}$$

Remark 2.3. Theorem 2.4 contains both quantum Bayes formula [6] and classical Bayes formula [13] as special cases.

4 Fault Tolerant Quantum Filtering and Fault Detection

4.1 Fault tolerant quantum filter and fault detection equation

Recall the quantum systems described in Section 2. In the laser-atom interaction realization, the laser field is often treated in a classical way and it generates an electromagnetic field at the position of the atom. Then the laser-atom interaction can be described by a dipole interaction Hamiltonian which depends on the intensity of the classical electromagnetic field [11]. Therefore, if the macroscopic laser device suffers from a fault, e.g., it produces a faulty electromagnetic field, then an unexpected additional Hamiltonian will be introduced into the quantum system. In this case, the system Hamiltonian will be given by $H(F(t))$ where $F(t)$ is the fault process. In practice, the system may transit from different faulty modes at random times. This makes it desirable to model the fault process on a probability space $(\Omega, \mathcal{F}, \mathcal{P})$ by a continuous-time Markov chain $\{F(t)\}_{t\geq 0}$ adapted to $\{\mathcal{F}_t\}_{t\geq 0}$ [21, 23, 24]. The state space of $F(t)$ is often chosen to be the finite set $\mathbb{S} = \{e_1, e_2, ..., e_N\}$ (for some positive integer N) of canonical unit vectors in $\mathbb{R}^N$. Let $p_t = (p_t^1, p_t^2, ..., p_t^N)^T$ be the probability distribution of $F(t)$, i.e., $p_t^k = \mathcal{P}(F(t) = e_k), k = 1, 2, ..., N$ and suppose the Markov process $F(t)$ has a so-called Q matrix or transition rate matrix $\Pi = (a_{jk}) \in \mathbb{R}^{N\times N}$. Then p_t satisfies the forward Kolmogorov equation $\frac{dp_t}{dt} = \Pi p_t$. Because Π is a Q matrix, we have $a_{jj} = -\sum_{j\neq k} a_{jk}$, and $a_{jk} \geq 0, j \neq k$. Then $F(t)$ is a corlol process that satisfies the following stochastic differential equation:

$$dF(t) = \Pi F(t)dt + dM(t), \tag{9}$$

where $M(t)$ is an $\{\mathcal{F}_t\}$ martingale such that $\sup_{0\leq t\leq T} \mathbb{E}(|M(t)|^2) < \infty$.

One goal of this paper is to derive equations of the fault tolerant quantum filter and fault detection for this class of open quantum systems. To be specific, we use a reference probability approach to find the least-mean-square estimates of a system observable $X \in \mathscr{B}(\mathcal{H})$ at time t and the fault process $F(t)$ for the quantum system under consideration, given the observation process $Y(s), 0 \leq s \leq t$. This can be accomplished if we can obtain the following estimates:

$$\sigma_t^j(X) = \tilde{\mathbb{P}}(\langle F(t), e_j\rangle\, U^\dagger(t)XU(t)|\mathscr{Y}_t), \tag{10}$$

where $\mathscr{Y}_t$ is the commutative von Neumann algebra generated by $Y(s)$ up to time t, and $< \cdot, \cdot >$ is the inner product in $\mathbb{R}^N$. From the previous analysis, one has $< F(t), e_j > U^\dagger(t)XU(t) \in \mathscr{Y}_t'$, which guarantees that the conditional expectation (10) is well defined.

It follows from (3) that for $\forall s \leq t$,

$$\begin{aligned} U^\dagger(t)Q(s)U(t) &= U^\dagger(s)U^\dagger(t,s)Q(s)U(t,s)U(s) \\ &= U^\dagger(s)Q(s)U(s) = Y(s), \end{aligned} \tag{11}$$

which implies that $\mathscr{Y}_t$ can be rewritten as $\mathscr{Y}_t = U^\dagger(t)\mathscr{Q}_tU(t)$ where $\mathscr{Q}_t$ is the commutative von Neumann algebra generated by $Q(s)$ up to time t. From quantum probability theory, we know that $Q(t)$ under the vacuum state is equivalent to a classical Wiener process [14]. This fact makes it simpler to design a quantum filter in terms of $Q(t)$ because it is convenient to manipulate $Q(t)$ using the quantum $It\hat{o}$ formula [16]. Next, we will use a quantum analog of the classical change-of-measure technique to obtain an explicit expression for $\sigma_t^j(X)$.

Define an operator $V(t)$ that satisfies the quantum stochastic differential equation

$$dV(t) = \left\{ \left(-iH(F(t)) - \frac{1}{2}L^\dagger L \right) dt + LdQ(t) \right\} V(t), \quad (12)$$

with $V(0) = I$. Then $V(t) \in \mathscr{Q}_t'$ and we have the following lemma.

Lemma 3.1. For any system observable $X \in \mathscr{B}(\mathcal{H})$, the conditional expectation in (10) can be rewritten as

$$\sigma_t^j(X) = U^\dagger(t) \frac{\tilde{\mathbb{P}}(\langle F(t), e_j \rangle V^\dagger(t) X V(t) | \mathscr{Q}_t)}{\tilde{\mathbb{P}}(V^\dagger(t)V(t)) | \mathscr{Q}_t)} U(t). \quad (13)$$

Proof. See Appendix. □

Write

$$\pi_t^j(X) = U^\dagger(t) \tilde{\mathbb{P}}(\langle F(t), e_j \rangle V^\dagger(t) X V(t) | \mathscr{Q}_t) U(t), \quad (14)$$

which is the unnormalized conditional expectation. Since $\sum_{j=1}^N \langle F(t), e_j \rangle = 1$, we have

$$\sigma_t^j(X) = \frac{\pi_t^j(X)}{\sum_{k=1}^N \pi_t^k(I)}. \quad (15)$$

An explicit expression for $\pi_t^j(X)$ can now be obtained.

Theorem 3.1. (Unnormalized fault tolerant quantum filtering equation) The unnormalized conditional expectation $\pi_t^j(X)$ satisfies the following quantum stochastic differential equation:

$$\begin{aligned} d\pi_t^j(X) = & \left(\sum_{k=1}^N a_{jk} \pi_t^k(X) + \pi_t^j(\mathscr{L}_{L,H(e_j)}(X)) \right) dt \\ & + \pi_t^j(XL + L^\dagger X) dY(t), \end{aligned} \quad (16)$$

where the so-called Lindblad generator is given by

$$\mathscr{L}_{L,H}(X) = i[H, X] + L^\dagger X L - \frac{1}{2}(L^\dagger L X + X L^\dagger L).$$

Proof. By using the $It\hat{o}$ product rule and from (9) and (12), we obtain

$$\begin{aligned} & \langle F(t), e_j \rangle V^\dagger(t) X V(t) \\ = & \langle F(0), e_j \rangle X + \int_0^t \langle \Pi F(s), e_j \rangle V^\dagger(s) X V(s) ds \\ & + \left\langle \int_0^t V^\dagger(s) X V(s) dM(s), e_j \right\rangle \\ & + \int_0^t \langle F(s), e_j \rangle d(V^\dagger(s) X V(s)). \end{aligned} \quad (17)$$

Taking conditional expectation with respect to $\mathscr{Q}_t$ on both sides of (17) while using the mutual independence of $\{Q(t), M(t), F(0)\}$, we obtain

$$\begin{aligned} & \tilde{\mathbb{P}}(\langle F(t), e_j \rangle V^\dagger(t) X V(t) | \mathscr{Q}_t) \\ = & \tilde{\mathbb{P}}(\langle F(0), e_j \rangle X) \\ & + \tilde{\mathbb{P}} \left(\int_0^t \langle \Pi F(s), e_j \rangle V^\dagger(s) X V(s) ds | \mathscr{Q}_t \right) \\ & + \tilde{\mathbb{P}} \left(\int_0^t \langle F(s), e_j \rangle V^\dagger(s) (\mathscr{L}_{L,H(F(s))}(X) \right. \\ & \left. + XL + L^\dagger X) V(s) ds | \mathscr{Q}_t \right) \\ = & \tilde{\mathbb{P}}(\langle F(0), e_j \rangle X) \\ & + \int_0^t \tilde{\mathbb{P}}(\langle \Pi F(s), e_j \rangle V^\dagger(s) X V(s) | \mathscr{Q}_s) ds \\ & + \int_0^t \tilde{\mathbb{P}}(\langle F(s), e_j \rangle V^\dagger(s) (\mathscr{L}_{L,H(e_j)}(X) \\ & + XL + L^\dagger X) V(s) | \mathscr{Q}_s) ds. \end{aligned} \quad (18)$$

In addition,

$$\begin{aligned} & \langle \Pi F(s), e_j \rangle = \langle F(s), \Pi^T e_j \rangle \\ = & \left\langle F(s), \sum_{k=1}^N a_{jk} e_k \right\rangle = \sum_{k=1}^N a_{jk} \langle F(s), e_k \rangle. \end{aligned} \quad (19)$$

Let $h_t^j(X) = \tilde{\mathbb{P}}(\langle F(t), e_j \rangle V^\dagger(t) X V(t) | \mathscr{Q}_t)$, then $\pi_t^j(X) = U^\dagger(t) h_t^j(X) U(t)$. From (18) and (19), $h_t^j(X)$ satisfies the following stochastic differential equation:

$$\begin{aligned} dh_t^j(X) = & \left(\sum_{k=1}^N a_{jk} h_t^k(X) + h_t^j(\mathscr{L}_{L,H(e_j)}(X)) \right) dt \\ & + h_t^j(XL + L^\dagger X) dQ(t). \end{aligned} \quad (20)$$

From Definition 2.3, we know $h_t^j(X) \in \mathscr{Q}_t$. Using the $It\hat{o}$ formula, we have

$$\begin{aligned} d\pi_t^j(X) = & (U(t) + dU(t))^\dagger dh_t^j(X)(U(t) + dU(t)) \\ = & \left(\sum_{k=1}^N a_{jk} \pi_t^k(X) + \pi_t^j(\mathscr{L}_{L,H(e_j)}(X)) \right) dt \\ & + \pi_t^j(XL + L^\dagger X) dQ(t) \\ & + \pi_t^j(XL + L^\dagger X) U^\dagger(t)(L + L^\dagger) U(t) \\ = & \left(\sum_{k=1}^N a_{jk} \pi_t^k(X) + \pi_t^j(\mathscr{L}_{L,H(e_j)}(X)) \right) dt \\ & + \pi_t^j(XL + L^\dagger X) dY(t). \end{aligned} \quad (21)$$

□

Theorem 3.2. (Normalized fault tolerant quantum filtering equation) The normalized conditional expectation $\sigma_t^j(X)$ satisfies the following quantum stochastic differential equation:

$$\begin{aligned} d\sigma_t^j(X) = & (\sum_{k=1}^N a_{jk} \sigma_t^k(X) + \sigma_t^j(\mathscr{L}_{L,H(e_j)}(X))) dt + \\ & \left(\sigma_t^j(XL + L^\dagger X) - \sigma_t^j(X) \sum_{k=1}^N \sigma_t^k(L + L^\dagger) \right) dW(t), \end{aligned} \quad (22)$$

where $W(t) = Y(t) - \int_0^t \sum_{k=1}^N \sigma_s^k(L + L^\dagger) ds$ is called innovation process and is a Wiener process under $\tilde{\mathbb{P}}$.

Proof. From Theorem 3.1, we have

$$d\pi_t^j(I) = \sum_{k=1}^N a_{jk}\pi_t^k(I)dt + \pi_t^j(L+L^\dagger)dY(t), \quad (23)$$

since $\mathscr{L}_{L,H(e_j)}(I) = 0$.

In addition, it follows from the properties of the Q matrix

$$d\sum_{k=1}^N \pi_t^k(I) = \sum_{j=1}^N\sum_{k=1}^N a_{jk}\pi_t^k(I)dt + \sum_{k=1}^N \pi_t^k(L+L^\dagger)dY(t)$$
$$= \sum_{k=1}^N \pi_t^k(L+L^\dagger)dY(t). \quad (24)$$

Equation (15) can be rewritten as

$$\sum_{k=1}^N \pi_t^k(I)\sigma_t^j(X) = \pi_t^j(X). \quad (25)$$

Differentiating both sides of (25) based on the quantum $It\hat{o}$ rule yields

$$d\sum_{k=1}^N \pi_t^k(I)(\sigma_t^j(X) + d\sigma_t^j(X)) + \sum_{k=1}^N \pi_t^k(I)d\sigma_t^j(X)$$
$$= d\pi_t^j(X). (26)$$

It is noted that $[\sigma_t^j(X), dY(t)] = 0$ because $\sigma_t^j(X) \in \mathscr{Y}_t$. From (23)-(26), one has

$$\left(\sum_{k=1}^N \pi_t^k(I) + \sum_{k=1}^N \pi_t^k(L+L^\dagger)dY(t)\right) d\sigma_t^j(X)$$
$$= d\pi_t^j(X) - \sum_{k=1}^N \pi_t^k(L+L^\dagger)\sigma_t^j(X)dY(t). \quad (27)$$

From (16) and (25), one has

$$\left(\sum_{k=1}^N \pi_t^k(I)\right)^{-1} d\pi_t^j(X)$$
$$= \left(\sum_{k=1}^N a_{jk}\sigma_t^k(X) + \sigma_t^j(\mathscr{L}_{L,H(e_j)}(X))\right)dt$$
$$+\sigma_t^j(XL + L^\dagger X)dY(t). \quad (28)$$

Then dividing both sides of (27) by $\sum_{k=1}^N \pi_t^k(I)$ yields

$$\left(I + \sum_{k=1}^N \sigma_t^k(L+L^\dagger)dY(t)\right) d\sigma_t^j(X)$$
$$= \left(\sum_{k=1}^N a_{jk}\sigma_t^k(X) + \sigma_t^j(\mathscr{L}_{L,H(e_j)}(X))\right)dt$$
$$+ \left(\sigma_t^j(XL + L^\dagger X) - \sum_{k=1}^N \sigma_t^k(L+L^\dagger)\sigma_t^j(X)\right) dY(t). (29)$$

By multiplying both sides of (29) with $I - \sum_{k=1}^N \sigma_t^k(L + L^\dagger)dY(t)$, (22) can be obtained using the fact $dY(t)dY(t) = dt$.

Next, note $\sum_{k=1}^N \sigma_t^k(L + L^\dagger) = \tilde{\mathbb{P}}(U^\dagger(t)(L + L^\dagger)U(t)|\mathscr{Y}_t) \in \mathscr{Y}_t$. Thus one can prove that $W(t)$ is a commutative process which is equivalent to a classical stochastic process under $\tilde{\mathbb{P}}$ according to Theorem 2.2.

In addition, let $K \in \mathscr{Y}_s, s \leq t$, then

$$\tilde{\mathbb{P}}(\tilde{\mathbb{P}}(W(t)|\mathscr{Y}_s)K) = \tilde{\mathbb{P}}(W(t)K)$$
$$= \tilde{\mathbb{P}}\left(Y(t)K - \int_0^t \tilde{\mathbb{P}}(U^\dagger(\tau)(L+L^\dagger)U(\tau)|\mathscr{Y}_\tau))Kd\tau\right)$$
$$= \tilde{\mathbb{P}}\left(Y(t)K - \int_0^s \tilde{\mathbb{P}}(U^\dagger(\tau)(L+L^\dagger)U(\tau)|\mathscr{Y}_\tau))Kd\tau\right.$$
$$\left. - \int_s^t U^\dagger(\tau)(L+L^\dagger)U(\tau)d\tau K\right)$$
$$= \tilde{\mathbb{P}}(W(s)K) + \tilde{\mathbb{P}}((Q(t) - Q(s))K) = \tilde{\mathbb{P}}(W(s)K). (30)$$

Therefore, $\tilde{\mathbb{P}}(W(t)|\mathscr{Y}_s) = W(s), s \leq t$, which means $W(t)$ is a $\mathscr{Y}_t-$martingale. Finally, $dW(t)dW(t) = dY(t)dY(t) = dt$. Then $W(t)$ is a Wiener process using Levy's Theorem. □

Remark 3.1. Since our discussion is under the Heisenberg picture, $\tilde{\mathbb{P}}$ is fixed. Based on Theorem 2.2, (22) is a classical recursive stochastic differential equation driven by the classical Wiener process $W(t)$, and $Y(t)$ can be replaced by its classical observation process counterpart. As a result, (22) can be directly implemented on a classical signal processor.

Remark 3.2. The coupled system of stochastic differential equations (22) is the normalized conditional expectation of $\langle F(t), e_j\rangle U^\dagger(t)XU(t)$, given $\mathscr{Y}_t$. When $\pi_{jk} = 0, \forall j \neq k$, this system is decoupled and reduces to the well known quantum filtering equation of $U^\dagger(t)XU(t)$ given $\mathscr{Y}_t$ [4].

Normally, the open quantum system is defined on a finite dimensional Hilbert space $\mathcal{H}_s$. Noting that σ_t^j is linear, identity preserving and positive, it is a normal state on $\mathscr{Y}_t'$. From another point of view, it works as the expectation of $\langle F(t), e_j\rangle X$ with respect to some finite dimensional state on $\mathcal{H}_s$. Based on Lemma 2.2, there exists a density operator ρ' such that $\sigma_t^j(X) = \mathbb{E}\{\mathrm{Tr}\{\rho'(\langle F(t), e_j\rangle X)\}\} = \mathrm{Tr}\{\rho_t^j X\}$ with $\rho_t^j = \mathbb{E}(\langle F(t), e_j\rangle \rho')$. The following is a corollary of Theorem 3.2.

Corollary 3.1. Let ρ_t^j be the random operator that satisfies $\sigma_t^j(X) = \mathrm{Tr}(\rho_t^j X)$ for all system observable $X \in \mathscr{B}(\mathcal{H})$. Then ρ_t^j satisfies the following stochastic differential equation

$$d\rho_t^j = \left(\sum_{k=1}^N a_{jk}\rho_t^k + \mathscr{L}_{L,H(e_j)}^\dagger(\rho_t^j)\right)dt$$
$$+ \left(L\rho_t^j + \rho_t^j L^\dagger - \rho_t^j \sum_{k=1}^N \mathrm{Tr}(\rho_t^k(L+L^\dagger))\right)dW(t), \quad (31)$$

with $\rho_0^j = \mathbb{E}(\langle F(0), e_j\rangle)\pi_0$. Here $\mathscr{L}_{L,H(e_j)}^\dagger$ is the adjoint Lindblad generator:

$$\mathscr{L}_{L,H}^\dagger(X) = -i[H,X] + LXL^\dagger - \frac{1}{2}(L^\dagger LX + XL^\dagger L).$$

Note ρ_t^j is not a density matrix because it is not defined in terms of the conditional expectation of real system observables. In fact, we have

$$\tilde{\mathbb{P}}(U^\dagger(t)XU(t)|\mathscr{Y}_t) = \sum_{k=1}^{N} \sigma_t^k(X). \tag{32}$$

Let ρ_t be the random density matrix that satisfies $\tilde{\mathbb{P}}(U^\dagger(t)XU(t)|\mathscr{Y}_t) = \mathrm{Tr}(\rho_t X)$. We have

$$\rho_t = \sum_{k=1}^{N} \rho_t^k, \text{ with } \mathrm{Tr}(\rho_t) = 1 \text{ and } \rho_0 = \pi_0. \tag{33}$$

From Corollary 3.1, ρ_t satisfies

$$\begin{aligned} &d\rho_t \\ &= \left(\sum_{k=1}^{N} i[H(e_k), \rho_t^k] + L\rho_t L^\dagger - \frac{1}{2}L^\dagger L\rho_t - \frac{1}{2}\rho_t L^\dagger L \right) dt \\ &\quad + (L\rho_t + \rho_t L^\dagger - \rho_t \mathrm{Tr}((L+L^\dagger)\rho_t) dW(t). \end{aligned} \tag{34}$$

Equation (34) is the *fault tolerant quantum stochastic master equation*.

In addition, the estimate of the fault process is given by

$$\begin{aligned} \hat{F}(t) &= \sum_{k=1}^{N} e_k \tilde{\mathbb{P}}(\langle F(t), e_k\rangle |\mathscr{Y}_t) \\ &= \sum_{k=1}^{N} e_k \sigma_t^k(I) = \sum_{k=1}^{N} e_k \mathrm{Tr}(\rho_t^k). \end{aligned} \tag{35}$$

From Corollary 3.1, $\hat{F}(t)$ satisfies

$$\hat{F}(t) = \Pi \hat{F}(t)dt + G(t)dW(t), \tag{36}$$

where $G(t) = \sum_{k=1}^{N} e_k \mathrm{Tr}((L+L^\dagger)\rho_t^k) - \hat{F}(t)\mathrm{Tr}((L+L^\dagger)\rho_t)$. Equation (36) is the corresponding *fault detection equation*.

4.2 Application to Two-level Quantum Systems

Two-level quantum systems (qubits) play a significant role in quantum information processing. For a two-level system, the filter equations reduce to a finite set of stochastic differential equations. In this case, $\mathcal{H}_s = \mathbb{C}^2$. We select the coupling strength operator $L = \sigma_-$ and the free Hamiltonian $H_0 = \sigma_z$.

Assume that a fault occurs at time T, at which time a new Hamiltonian $H_f = \sigma_y$ is introduced into the system. Following [23], we assume that $f(t)$ is a Poisson process with rate λ, stopped at its first jump time T. That is,

$$f(t) = \begin{cases} 0, \text{ if } t < T \\ 1, \text{ if } t \geq T \end{cases} \tag{37}$$

and T is an exponential random variable with probability distribution

$$P(T \leq t) = 1 - e^{-\lambda t}. \tag{38}$$

From [21], the process $M(t) = f(t) - \lambda \min(t, T)$ is a martingale and the process $f(t)$ satisfies

$$df(t) = \lambda(1 - f(t))dt + dM(t). \tag{39}$$

Also, we consider $f(0) = 0$ only (because $f(t)$ stops at its first jump). Let $F(t) = [1 - f(t), f(t)]'$. Then $F(t)$ takes values in $\{e_1, e_2\}$ and satisfies

$$dF(t) = \begin{bmatrix} -\lambda & 0 \\ \lambda & 0 \end{bmatrix} F(t) + \begin{bmatrix} -1 \\ 1 \end{bmatrix} dM(t). \tag{40}$$

Using this, the coupled quantum filtering equations are given by

$$\begin{cases} d\rho_t^1 = \left(-\rho_t^1 + \mathscr{L}^\dagger_{L,H_0}(\rho_t^1)\right) dt \\ + \left(L\rho_t^1 + \rho_t^1 L^\dagger - \rho_t^1 \sum_{k=1}^2 \mathrm{Tr}(\rho_t^k(L+L^\dagger))\right) dW(t), \\ d\rho_t^2 = \left(\rho_t^1 + \mathscr{L}^\dagger_{L,H_0+H_f}(\rho_t^2)\right) dt \\ + \left(L\rho_t^2 + \rho_t^2 L^\dagger - \rho_t^2 \sum_{k=1}^2 \mathrm{Tr}(\rho_t^k(L+L^\dagger))\right) dW(t). \end{cases}$$

Write

$$\begin{cases} \rho_t^1 = \frac{1}{2}(\alpha(t)I + x_1(t)\sigma_x + y_1(t)\sigma_y + z_1(t)\sigma_z), \\ \rho_t^2 = \frac{1}{2}((1-\alpha(t))I + x_2(t)\sigma_x + y_2(t)\sigma_y + z_2(t)\sigma_z), \end{cases}$$

where $\sigma_j, j \in \{x, y, z\}$ are the Pauli matrices as follows:

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \text{ and } \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Then we obtain seven coupled equations for the seven coefficients related to the fault tolerant quantum stochastic master equation:

$$\begin{cases} d\alpha(t) = -\alpha(t)dt + (x_1(t) - \alpha(t)(x_1(t) + x_2(t)))dW(t) \\ dx_1(t) = -(\frac{3}{2}x_1(t) + 2y_1(t))dt \\ \qquad +(\alpha(t) + z_1(t) - x_1(t)(x_1(t) + x_2(t)))dW(t) \\ dy_1(t) = (2x_1(t) - \frac{3}{2}y_1(t))dt - (x_1(t) + x_2(t))y_1(t)dW(t) \\ dz_1(t) = -(\alpha(t) + 2z_1(t))dt \\ \qquad -(x_1(t) + (x_1(t) + x_2(t))z_1(t))dW(t) \\ dx_2(t) = (x_1(t) - \frac{1}{2}x_2(t) - 2y_2(t) + 2z_2(t))dt \\ \qquad +(1 - \alpha(t) - x_2(t)(x_1(t) + x_2(t)) + z_2(t))dW(t) \\ dy_2(t) = (y_1(t) + 2x_2(t) - \frac{1}{2}y_2(t))dt \\ \qquad -(x_1(t) + x_2(t))y_2(t)dW(t) \\ dz_2(t) = (-1 + \alpha(t) + z_1(t) - 2x_2(t) - z_2(t))dt \\ \qquad -(x_2(t) + (x_1(t) + x_2(t))z_2(t))dW(t) \end{cases}$$

The fault detection equation is given by

$$d\hat{F}(t) = \Pi \hat{F}(t)dt + G(t)dW(t). \tag{41}$$

where $G(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} - \hat{F}(t)(x_1(t) + x_2(t))$. The innovation process $W(t)$ is given by $W(t) = y(t) - \int_0^t x_1(s) + x_2(s)ds$.

5 Conclusions

In this paper, a unified framework is established to analyze fault tolerant filtering and fault detection for a class of laser-atom open quantum systems. In this framework, a quantum-classical probability space model is developed to enable us to deal with both classical and quantum randomnesses. By describing the stochastic fault process as a finite-state jump Markov chain and using a reference probability approach, a set of coupled stochastic differential equations satisfied by the conditional system state and fault process estimates are given. The obtained results have been applied to two-level quantum systems under Poisson type faults. The proposed approaches provide a new avenue to deal with various cases where classical randomness appear in quantum systems.

Appendix

Proof of Lemma 3.1. Let $\tilde{\mathbb{Q}}^t$ be a normal state as $\tilde{\mathbb{Q}}^t(X) = \tilde{\mathbb{P}}(U^\dagger(t)XU(t))$.

Let $K(t)$ be any element of $\mathscr{Y}_t$, then $K(t) = U^\dagger(t)K_o(t)U(t)$ for some $K_o(t) \in \mathscr{Q}_t$. Note the scalar valued function $\langle F(t), e_j\rangle \in \mathscr{Q}_t'$ and $X \in \mathscr{Q}_t'$. We have

$$\begin{aligned}
&\tilde{\mathbb{P}}(\tilde{\mathbb{P}}(\langle F(t), e_j\rangle U^\dagger(t)XU(t)|\mathscr{Y}_t)K)\\
&= \tilde{\mathbb{P}}(\langle F(t), e_j\rangle U^\dagger(t)XU(t)K(t))\\
&= \tilde{\mathbb{P}}(\langle F(t), e_j\rangle U^\dagger(t)XK_o(t)U(t))\\
&= \tilde{\mathbb{Q}}^t(\langle F(t), e_j\rangle XK_o(t))\\
&= \tilde{\mathbb{Q}}^t(\tilde{\mathbb{Q}}^t(\langle F(t), e_j\rangle XK_o(t)|\mathscr{Q}_t))\\
&= \tilde{\mathbb{Q}}^t(\tilde{\mathbb{Q}}^t(\langle F(t), e_j\rangle X|\mathscr{Q}_t)K_o(t))\\
&= \tilde{\mathbb{P}}(U^\dagger(t)\tilde{\mathbb{Q}}^t(\langle F(t), e_j\rangle X|\mathscr{Q}_t)K_o(t)U(t))\\
&= \tilde{\mathbb{P}}(U^\dagger(t)\tilde{\mathbb{Q}}^t(\langle F(t), e_j\rangle X|\mathscr{Q}_t)U(t)K(t)). \qquad (42)
\end{aligned}$$

Letting $K(t) = (\tilde{\mathbb{P}}(\langle F(t), e_j\rangle U^\dagger(t)XU(t)|\mathscr{Y}_t) - U^\dagger(t)\tilde{\mathbb{Q}}^t(\langle F(t), e_j\rangle X|\mathscr{Q}_t)U(t))^\dagger$ yields

$$\begin{aligned}
&\tilde{\mathbb{P}}(\langle F(t), e_j\rangle U^\dagger(t)XU(t)|\mathscr{Y}_t)\\
&= U^\dagger(t)\tilde{\mathbb{Q}}^t(\langle F(t), e_j\rangle X|\mathscr{Q}_t)U(t) \qquad (43)
\end{aligned}$$

almost surely under $\tilde{\mathbb{P}}$

In addition, suppose the system is initialized at $\pi_0 = \sum\limits_k p_k |\alpha_k\rangle\langle\alpha_k|$ and we define a curve $|\psi_k(t)\rangle = U(t)(|\alpha_k\rangle \otimes |\upsilon\rangle)$. Using the fact that $dB(t)|\upsilon\rangle = 0$, one obtains (see Equation (6.13) in [17])

$$d|\psi_k(t)\rangle = \{(-iH(F(t)) - \frac{1}{2}L^\dagger L)dt + LdQ(t)\}|\psi_k(t)\rangle. \quad (44)$$

In other words, $U(t)(|\alpha_k\rangle\otimes|\upsilon\rangle) = V(t)(|\alpha_k\rangle\otimes|\upsilon\rangle)$ since $U(0) = V(0) = I$. After some mathematical manipulation, one obtains $\mathrm{Tr}(\rho_0 U^\dagger(t)XU(t)) = \mathrm{Tr}(\rho_0 V^\dagger(t)XV(t))$ which leads to

$$\tilde{\mathbb{P}}(\langle F(t), e_j\rangle U^\dagger(t)XU(t)) = \tilde{\mathbb{P}}(\langle F(t), e_j\rangle V^\dagger(t)XV(t)). (45)$$

Then we can apply Theorem 2.4 with $\Lambda = 1$, $\tilde{A} = \langle F(t), e_j\rangle X \in \mathscr{Q}_t'$, $V = V(t)$ and $\tilde{\mathscr{C}} = \mathscr{Q}_t$ and obtain

$$\tilde{\mathbb{Q}}^t(\langle F(t), e_j\rangle X|\mathscr{Q}_t) = \frac{\tilde{\mathbb{P}}(\langle F(t), e_j\rangle V^\dagger(t)XV(t)|\mathscr{Q}_t)}{\tilde{\mathbb{P}}(V^\dagger(t)V(t)|\mathscr{Q}_t)}. \quad (46)$$

Lemma 3.1 can be concluded combing (43) and (46). □

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