Generalized Formation Control for Unicycles

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Abstract: This paper studies formation control of a group of unicycles, and proposes a distributed control law, depending only on the relative position and bearing angle of one nearby vehicle, to realize the circular formation and render more achievable general formations by designing the parameters in the control law. For circular formation, not only we can strictly prove that only two sets of equilibria are asymptotically stable, but also by choosing appropriate control parameters, determine the distances of two vehicles and the radius of the circle that all the unicycles converge to.

Key Words: formation control, unicycle, multi-agent systems, circular formation.

1 Introduction

Formation control is to make a group of vehicles move in an ordered manner along a desired reference trajectory or along a geometric path [6] and it has wide and promising applications in terrestrial, space and ocean exploration, surveillance and rescue missions as well as intelligent transportation. Take large-area surveillance for example, to monitor and gather information, it is required that a group of unmanned aerial vehicles to circle around a target or an area. Interest for formation control can be found in both linear [7, 16] and nonlinear multi-agent systems, including unicycles [5, 9, 15], spacecraft [13] and so on.

Formation control, especially circular formation of unicycles, which are often used to model a large class of mobile robots, has been extensively studied under different settings due to its various applications. [14] investigated the collective circular motions with all-to-all communications. However, all-to-all communication means increased costs in sensor, communication and maintenance, thus results in limited applications. [6, 2] relaxed such all-to-all communication condition to the communication network containing a static or switching spanning tree, respectively, for which the stability analysis was more challenging since it not only needed to consider the nonlinear dynamics of unicycles, kinematic pattern constraints, but also had to take the nonlinear feedbacks based on local interactions into consideration. Based on the earlier works, another question was raised: What is the minimum communication links for achieving circular formation? The answer to this question has been explored under two scenarios, namely with or without leaders. A family of controllers have been proposed to drive a single follower to the maneuvering leader and ultimately make it circulate around the target at the desired distance and speed in [3, 12], while [18] enabled a team of unicycle-type agents to circle on different orbits centered at the required target, maintaining specific circular formation at the same time. However, The leader-follower approach may be undesirable when controller for each follower heavily depends on the information of the leader system.

Under the leaderless scenario, some of the early contributions were given by [10, 11, 17], which required a minimum number of communication links, that is, n identical unicycles were ordered such that vehicle i pursued vehicle $i+1$ modulo n, namely $n+1$ was identified with 1. By giving each vehicle the same constant linear speed, [10] enabled a group of unicycles to form any generalized regular polygons through appropriated controller gain assignment and revealed which formations were stable and which were not. From the main result of [10], we can easily find that multiple regular polygons can be asymptotically stable. In contrast, by redesigning the linear speed of each unicycle, [11] only allowed one equilibrium polygon to be asymptotically stable. In [17], the cyclic pursuit control law for vehicle i had linear speed and angular speed proportional to the projection of vehicle $i+1's$ position on its forward direction and lateral direction, respectively. The asymptotic stability of two equilibrium polygons was also established through root locus analysis of a complex characteristic polynomial.

In this paper, we begin with studying the circular formation of unicycles similar to that in [10, 11, 17], but are not limited to it. A distributed control law, depending only on its bearing angle and the relative position of its pursuing vehicle, is proposed containing two design parameters. This paper, as the foundation of the future work, first considers the special case that each control law has the same parameters. And this special case alone can render some interesting results. By choosing the appropriate parameters in the control law, a group of dynamic unicycles can converge to a common circle, whose center is stationary but dependent on the initial conditions, and with a desired direction, they can travel on the circle. Like [17], our control law only allows two sets of equilibrium polygons to be asymptotically stable while others are unstable, and rigorous proof has been established for such result. Different from [17], first, the desired ordering and spacing between vehicles can be uniquely decided by the control parameter. Second, we can achieve other more generalized formations by designing the control parameters. One special example is to change the control parameters and enable vehicles stop at a desired position in the circle while the orientation of each vehicle points directly to the pursuing one. Finally we want to point out that by designing different parameters for each vehicle, the more sophisticated control law can be used to investigate more general formations for a group of unicycles, and the possibility of such application has also been illustrated in Section 6.

The rest of this paper is organized as follows: in Section 2, we formulate our problem. In Section 3, we present the main result for our problem. Some examples are illustrated

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in Section 4. Finally, we close this paper with some concluding remarks in Section 5 and introduce the future work in Section 6.

The following notation will be used throughout this paper: given the column vectors a_i, $i = 1, \cdots, s$, we denote $\mathrm{col}(a_1, \cdots, a_n) = [a_1^T, \cdots, a_n^T]^T$. $\kappa = \upsilon(\theta_1, \theta_2, \cdots)$ denotes the number of variations in sign in $\theta_1, \theta_2, \cdots$.

2 Problem Formulation

Consider a collection of mobile agents modeled as unicycles:

$$\begin{aligned} \dot{x}_i &= v_i \cos(\theta_i) \\ \dot{y}_i &= v_i \sin(\theta_i) \\ \dot{\theta}_i &= w_i, \quad i = 1, \cdots, n \end{aligned} \tag{1}$$

where $\mathrm{col}(x_i, y_i) \in \mathbb{R}^2$ denotes the position of vehicle's center of mass in the plane, $\theta_i \in \mathbb{R}$ is the vehicle's orientation, and $u_i = \mathrm{col}(v_i, w_i) \in \mathbb{R}^2$ is the control input with v_i as the linear speed and w_i as the angular speed.

Associated with system (1) is a so-called digraph[1] $\mathcal{G}(t) = (\mathcal{V}, \mathcal{E})$ where $\mathcal{V} = \{1, ..., n\}$ with $i = 1, \cdots, n$ associated with the ith subsystem of (1), and $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$. The set $\mathcal{V}$ is called the node set of $\mathcal{G}$ and the set $\mathcal{E}$ is called the edge set of $\mathcal{G}$. Like in [10, 17], we first consider a special pursuit graph with edge set given by $\mathcal{E} = \{(1,2), \cdots, (i, i+1), \cdots, (n-1, n), (n, 1)\}$.

To facilitate analysis, we introduce the relative coordinates as shown in Fig. 1.

Let $\begin{bmatrix} \bar{x}_i \\ \bar{y}_i \\ \bar{\theta}_i \end{bmatrix} = R(\theta_{i+1}) \begin{bmatrix} x_i - x_{i+1} \\ y_i - y_{i+1} \\ \theta_i - \theta_{i+1} \end{bmatrix}$ where $R(\theta) = \begin{bmatrix} \cos(\theta) & \sin(\theta) & 0 \\ -\sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix}$. Let

$$r_i = \sqrt{\bar{x}_i^2 + \bar{y}_i^2}, \quad \alpha_i = \arctan(\frac{\bar{y}_i}{\bar{x}_i}) + \pi - \bar{\theta}_i, \quad \beta_i = \bar{\theta}_i - \pi$$

with r_i denoting the distance between vehicle i and vehicle $i+1$, α_i denoting the bearing angle, i.e., the angle difference between vehicle i's heading and the line of sight which would take directly towards vehicle $i+1$, and β_i being the heading difference minus π. Note that we identify index $n+1$ with 1 and set $\alpha_i \in [-\pi, \pi)$ and $\beta_i \in [-\pi, \pi)$.

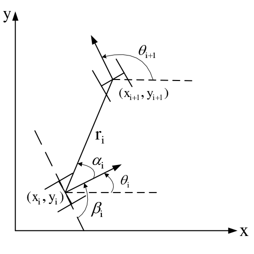

Fig. 1: Relative coordinates

Then based on (1) we can derive

$$\begin{aligned} \dot{r}_i &= -v_i \cos(\alpha_i) - v_{i+1} \cos(\alpha_i + \beta_i) \\ \dot{\alpha}_i &= \frac{1}{r_i}(v_i \sin(\alpha_i) + v_{i+1} \sin(\alpha_i + \beta_i)) - w_i \\ \dot{\beta}_i &= w_i - w_{i+1}, \quad i = 1, \cdots, n \end{aligned} \tag{2}$$

[1]See Appendix in [4] for a summary of digraph.

The purpose of this paper is to propose a new control strategy for a group of unicycles that not only realizes circular formation as in [10, 11, 17], but also provides possibility for achieving more general and stable formations. We will consider the control law in the following form:

$$v_i = f_1(r_i, \alpha_i), \quad w_i = f_2(r_i, \alpha_i) \tag{3}$$

where f_1 and f_2 are sufficiently smooth functions to be specified later.

Circular formation problem: Given a group of unicycles (2), find a control law of the form (3), such that for the closed-loop system, there exist locally asymptotically stable equilibria $\{r_i = \bar{r}, \alpha_i = \bar{\alpha}, \beta_i = \bar{\beta}\}$ for $i = 1, \cdots, n$ with $\bar{r} > 0$, $\bar{\alpha} \in [-\pi, \pi)$ and $\bar{\beta} \in [-\pi, \pi)$.

3 Main Result

In this section, we will propose our distributed control law, which only depends on its bearing angle and the relative position of its pursuing vehicle, and meanwhile, analyze the stability of the closed-loop system in details.

It is noted that the coordinate transformation given by (2) is not invertible, and results in the following constraints:

$$\begin{aligned} g_1(\xi) &= r_1 \sin(\alpha_1) + r_2 \sin(\alpha_2 + \pi - \beta_1) + \cdots \\ &\quad + r_n \sin(\alpha_n + (n-1)\pi - \beta_1 - \beta_2 - \cdots - \beta_{n-1}) = 0 \\ g_2(\xi) &= r_1 \cos(\alpha_1) + r_2 \cos(\alpha_2 + \pi - \beta_1) + \cdots \\ &\quad + r_n \cos(\alpha_n + (n-1)\pi - \beta_1 - \beta_2 - \cdots - \beta_{n-1}) = 0 \\ g_3(\xi) &= \sum_{i=1}^{n} \beta_i - n\pi + 2k\pi = 0, \quad k = 1, \cdots, n \end{aligned} \tag{4}$$

with $\xi_i = \mathrm{col}(r_i, \alpha_i, \beta_i)$, $i = 1, \cdots, n$ and $\xi = \mathrm{col}(\xi_1, \cdots, \xi_n)$.

We consider the following control law for the formation control of unicycles:

$$\begin{aligned} v_i &= \frac{1}{\cos(\theta_0)}(r_i \cos(\alpha_i - \theta_0) - d) \\ w_i &= \frac{1}{d\cos(\theta_0)}(r_i \sin(\alpha_i) - d\sin(\theta_0)) \end{aligned} \tag{5}$$

with $d > 0$ and $\theta_0 \in [-\pi, \pi)$, $\cos(\theta_0) \neq 0$.

Remark 3.1 *This control law for vehicle i only depends on the relative position and bearing angle of its pursuing vehicle $i+1$, which is inspired by the cyclic pursuit control law in [17]. However, our control law contains two design parameters d and θ_0, thus is more sophisticated than that in [17]. Here we first consider designing the same d and θ_0 for each vehicle as a special case. If assigning each agents with different d and θ_0, these vehicles can achieve more general formations as shown in Section 6.*

Under the distributed control law (5), the closed-loop system is

$$
\begin{aligned}
\dot{r}_i &= -\frac{1}{\cos(\theta_0)}(r_i\cos(\alpha_i-\theta_0)-d)\cos(\alpha_i) \\
&\quad -\frac{1}{\cos(\theta_0)}(r_{i+1}\cos(\alpha_{i+1}-\theta_0)-d)\cos(\alpha_i+\beta_i) \\
\dot{\alpha}_i &= \frac{1}{r_i\cos(\theta_0)}((r_i\cos(\alpha_i-\theta_0)-d)\sin(\alpha_i) \\
&\quad +(r_{i+1}\cos(\alpha_{i+1}-\theta_0)-d)\sin(\alpha_i+\beta_i)) \\
&\quad -\frac{1}{d\cos(\theta_0)}(r_i\sin(\alpha_i)-d\sin(\theta_0)) \\
\dot{\beta}_i &= \frac{1}{d\cos(\theta_0)}(r_i\sin(\alpha_i)-r_{i+1}\sin(\alpha_{i+1}))
\end{aligned} \tag{6}
$$

Each subsystem in (6) can be written as $\dot{\xi}_i = f(\xi_i, \xi_{i+1})$ with $\xi_{n+1} = \xi_1$.

3.1 Equilibria

Theorem 3.1 *Assume $r_i \neq 0$, for $i = 1, \cdots, n$,*

$$
\begin{aligned}
&\{\xi | r_i = \frac{d}{2}(f(\theta_0) \pm \sqrt{f^2(\theta_0)-8}), \alpha_i = \frac{k\pi}{n}, \beta_i = \pi - \frac{2k\pi}{n}\} \\
&\qquad 1 \le k < n \\
&\{\xi | r_i = \frac{d}{2}(f(\theta_0) \pm \sqrt{f^2(\theta_0)-8}), \alpha_i = \frac{k\pi}{n} - \pi, \beta_i = \pi - \frac{2k\pi}{n}\} \\
&\qquad 1 \le k < n \\
&\{\xi | r_i = d, \alpha_i = \theta_0, \sum_{i=1}^{n}\beta_i - n\pi + 2k\pi = 0\}, \ \ 1 \le k \le n
\end{aligned} \tag{7}
$$

with $f(\theta_0) = \frac{\sin(\theta_0)}{\sin(\alpha_i)} + 2\cos(\alpha_i - \theta_0) > 2\sqrt{2}$, are equilibria of system (6).

Proof: In order to get the equilibria of the multi-agent system, let $\dot{r}_i = 0$, $\dot{\alpha}_i = 0$, and $\dot{\beta}_i = 0$, $i = 1, \cdots, n$, Then, from (6),

$$
\begin{aligned}
\dot{r}_i &= -\frac{1}{\cos(\theta_0)}(r_i\cos(\alpha_i-\theta_0)-d)\cos(\alpha_i) \\
&\quad -\frac{1}{\cos(\theta_0)}(r_{i+1}\cos(\alpha_{i+1}-\theta_0)-d)\cos(\alpha_i+\beta_i) = 0 \\
\dot{\alpha}_i &= \frac{1}{r_i\cos(\theta_0)}((r_i\cos(\alpha_i-\theta_0)-d)\sin(\alpha_i) \\
&\quad +(r_{i+1}\cos(\alpha_{i+1}-\theta_0)-d)\sin(\alpha_i+\beta_i)) \\
&\quad -\frac{1}{d\cos(\theta_0)}(r_i\sin(\alpha_i)-d\sin(\theta_0)) = 0 \\
\dot{\beta}_i &= \frac{1}{d\cos(\theta_0)}(r_i\sin(\alpha_i)-r_{i+1}\sin(\alpha_{i+1})) = 0
\end{aligned} \tag{8}
$$

However, it is extremely difficult to solve (8) directly.

Part I: As is in [17], we first consider the case that all the agents are uniformly distributed in a circle. Then due to the rotational symmetry property of cyclic pursuit, we can assume

$$
r_i = \bar{r}, \quad \alpha_i = \bar{\alpha}, \quad \beta_i = \bar{\beta} \tag{9}
$$

with $\bar{\alpha} \in [-\pi, \pi)$ and $\bar{\beta} \in [-\pi, \pi)$. When (9) holds, from the third constraint, we can obtain $\bar{\beta} = \pi - \frac{2k\pi}{n}$, $k = \{1, 2, \cdots, n\}$.

$$
\begin{aligned}
&-\frac{1}{\cos(\theta_0)}(\cos(\bar{\alpha})+\cos(\bar{\alpha}+\bar{\beta}))(\bar{r}\cos(\bar{\alpha}-\theta_0)-d) = 0 \\
&\frac{1}{\bar{r}\cos(\theta_0)}((\sin(\bar{\alpha})+\sin(\bar{\alpha}+\bar{\beta}))(\bar{r}\cos(\bar{\alpha}-\theta_0)-d)) \\
&-\frac{1}{d\cos(\theta_0)}(\bar{r}\sin(\bar{\alpha})-d\sin(\theta_0)) = 0
\end{aligned} \tag{10}
$$

case 1: $\cos(\bar{\alpha}) + \cos(\bar{\alpha} + \bar{\beta}) = 0$

- If $k = n$, then $\bar{\beta} = -\pi$. From the first constraint of (4), $\sum_{i=1}^{n} r_i \sin(\alpha_i) = 0$, thus $\alpha_i = 0$ or $-\pi$. From the second constraint of (4), we can obtain $\pm\sum_{i=1}^{n} r_i = 0$. Thus, $k \neq n$.
- If $k \neq n$, substitute $\bar{\beta} = \pi - \frac{2k\pi}{n}$ into $\cos(\bar{\alpha}) + \cos(\bar{\alpha} + \bar{\beta}) = 0$, we can get $\bar{\alpha} = \frac{k\pi}{n}$ or $\bar{\alpha} = \frac{k\pi}{n} - \pi$. From the second equation of (10), one can get $\bar{r} = \frac{d}{2}(f(\theta_0) \pm \sqrt{f^2(\theta_0) - 8})$ with $f(\theta_0) = \frac{\sin(\theta_0)}{\sin(\alpha_i)} + 2\cos(\alpha_i - \theta_0)$.

Thus, for case 1,

$$
\begin{aligned}
&\{\xi | r_i = \frac{d}{2}(f(\theta_0) \pm \sqrt{f^2(\theta_0)-8}), \alpha_i = \frac{k\pi}{n}, \beta_i = \pi - \frac{2k\pi}{n}\} \\
&\qquad 1 \le k < n \\
&\{\xi | r_i = \frac{d}{2}(f(\theta_0) \pm \sqrt{f^2(\theta_0)-8}), \alpha_i = \frac{k\pi}{n} - \pi, \beta_i = \pi - \frac{2k\pi}{n}\} \\
&\qquad 1 \le k < n
\end{aligned} \tag{11}
$$

case 2: $r\cos(\bar{\alpha} - \theta_0) - d = 0$

From (10),

$$
\begin{aligned}
\bar{r}\cos(\bar{\alpha}-\theta_0) &= d \\
\bar{r}\sin(\bar{\alpha}) &= d\sin(\theta_0)
\end{aligned} \tag{12}
$$

Thus,

$$
\begin{aligned}
&\bar{r}\sin(\bar{\alpha}) = \bar{r}\cos(\bar{\alpha}-\theta_0)\sin(\theta_0) \\
&\Rightarrow \cos(\theta_0)\sin(\bar{\alpha}-\theta_0) = 0
\end{aligned} \tag{13}
$$

Since $\cos(\theta_0) \neq 0$ and $\bar{\alpha}, \theta_0 \in [-\pi, \pi)$, then $\bar{\alpha} = \theta_0$ or $\theta_0 \pm \pi$.

- $\bar{\alpha} = \theta_0$, then $\bar{r} = d$;
- $\bar{\alpha} = \theta_0 \pm \pi$, then $\bar{r}\cos(\pm\pi) = d$, i.e., $\bar{r} = -d < 0$.

Thus, for case 2,

$$
\{\xi | r_i = d, \alpha_i = \theta_0, \beta_i = \pi - \frac{2k\pi}{n}\}, \ \ 1 \le k \le n \tag{14}
$$

Part II: When the rotational symmetry property of cyclic pursuit does not hold, that is, without the assumption of (9), we can still find other solutions satisfying (8). For example, if $r_i\cos(\alpha_i - \theta_0) - d = 0$, $i = 1, \cdots, n$, we can get $r_i\sin(\alpha_i) = d\sin(\theta_0)$ from $\dot{\alpha}_i = 0$, then

$$
\{\xi | r_i = d, \alpha_i = \theta_0, \sum_{i=1}^{n}\beta_i - n\pi + 2k\pi = 0\}, \ \ 1 \le k \le n \tag{15}
$$

which covers (14).

Remark 3.2 *Among all the equilibria given by (7), the first two sets can achieve circular formation. For the equilibria $\{\xi | r_i = \frac{d}{2}(f(\theta_0) \pm \sqrt{f^2(\theta_0) - 8}), \alpha_i = \frac{k\pi}{n}, \beta_i = \pi - \frac{2k\pi}{n}\}$, $1 \le k < n$, in order to satisfy $r_i > 0$, $i = 1, \cdots, n$, $f(\theta_0) > 2\sqrt{2}$. From Fig. 2, we can see that we can always find a proper θ_0 such that $f(\theta_0) > 2\sqrt{2}$. A special case is $\theta_0 = \frac{k\pi}{n}$, $\theta_0 \neq \frac{\pi}{2}$. In that case, $f(\theta_0) = 3 > 2\sqrt{2}$, then $r_i = d$ or $2d$. We have the similar argument for $\{\xi | r_i = \frac{d}{2}(f(\theta_0) \pm \sqrt{f^2(\theta_0) - 8}), \alpha_i = \frac{k\pi}{n} - \pi, \beta_i = \pi - \frac{2k\pi}{n}\}$, $1 \le k < n$.*

If $\theta_0 = \frac{k\pi}{n}$ or $\frac{k\pi}{n} - \pi$, the equilibria given by (11) become

$$
\begin{aligned}
&\{\xi | r_i = d \text{ or } 2d, \alpha_i = \frac{k\pi}{n}, \beta_i = \pi - \frac{2k\pi}{n}\}, \ \ 1 \le k < n \\
&\{\xi | r_i = d \text{ or } 2d, \alpha_i = \frac{k\pi}{n} - \pi, \beta_i = \pi - \frac{2k\pi}{n}\}, \ \ 1 \le k < n
\end{aligned} \tag{16}
$$

The physical meaning of the equilibria given by the first equation of (16) is shown in Fig. 3 with $n = 8$.

It is noted that from geometry, multiple vehicles can not occupy the same point in the coordinate frame, thus it is required $r_i > 0$, $i = 1, \cdots, n$, that is, it is physically infeasible if n and k in (7)

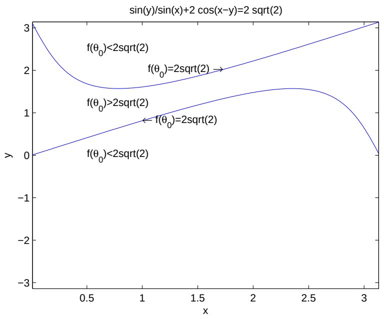

Fig. 2: $x := \frac{k\pi}{n}, k = 1, \cdots, n-1, y := \theta_0, f(\theta_0) = 2\sqrt{2}$

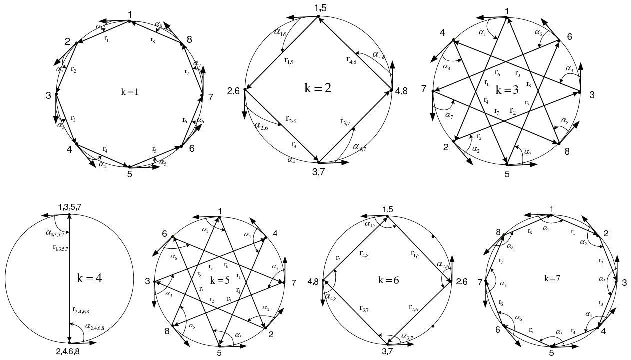

Fig. 3: $\{\xi | r_i = d$ or $2d, \alpha_i = \frac{k\pi}{n}, \beta_i = \pi - \frac{2k\pi}{n}\}$, $k = 1, \cdots, 7$

are not coprime. In Fig. 3, $k = 2, 4, 6$ are not feasible. It is also noted that $\alpha_i > 0$ corresponds to counterclockwise rotation of the system's pursuit pattern at equilibria and $\alpha_i < 0$ corresponds to clockwise rotation.

Remark 3.3 *The equilibria given by (15) are decided by the parameters θ_0 and d, and it is noted that β_i can be different from β_j, $i \neq j$, which renders more generalized patterns formed by unicycles.*

3.2 Circular Formation

In order to realize circular formation, we consider stability of the equilibria given by (11). For this purpose, we first design the parameters θ_0 and d. As indicated by Remark 3.2, let $\theta_0 = \frac{k\pi}{n}$ for the first set of equilibria given by (11). Then linearize the model at the equilibria $\{\xi | r_i = 2d, \alpha_i = \frac{k\pi}{n}, \beta_i = \pi - \frac{2k\pi}{n}\}, 1 \le k < n$,

$$\dot{\xi}_i = A\xi_i + B\xi_{i+1} \tag{17}$$

with $q = \frac{k}{n}$, and

$$A = \begin{bmatrix} -1 & 2d\tan(q\pi) & d\tan(q\pi) \\ -\frac{\tan(q\pi)}{d} & -2 & -\frac{1}{2} \\ \frac{\tan(q\pi)}{d} & 2 & 0 \end{bmatrix}$$
$$B = \begin{bmatrix} 1 & 0 & 0 \\ \frac{\tan(q\pi)}{2d} & 0 & 0 \\ -\frac{\tan(q\pi)}{d} & -2 & 0 \end{bmatrix} \tag{18}$$

It can be written in the compact form as follows:

$$\dot{\xi} = \hat{A}\xi \tag{19}$$

with $\hat{A} = \mathrm{circ}(A, B, 0, \cdots, 0) = \begin{bmatrix} A & B & 0 & 0 & \cdots & 0 \\ 0 & A & B & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ B & 0 & \cdots & 0 & 0 & A \end{bmatrix}$.

By Lemma 4 in [10], the matrix $\hat{A}$ can be block diagonalized into $\mathrm{diag}(D_1, \cdots, D_n)$, where $D_i = A + \varphi^{i-1}B, i = 1, \cdots, n$, with $\varphi^{i-1} = e^{\frac{2\pi(i-1)}{n}}$ and the eigenvalues of $\hat{A}$ are the collection of all eigenvalues of

$$A + B,\ A + \varphi B,\ A + \varphi^2 B,\ \cdots,\ A + \varphi^{n-1}B$$

We first introduce the following lemmas.

Lemma 3.1 *The eigenvalues of $A + B$ are*

$$\lambda_1 = 0,\ \ \lambda_{2,3} = -1 \pm \sqrt{1 - \tan^2(q\pi)}$$

Proof: $A+B = \begin{bmatrix} 0 & 2d\tan(q\pi) & d\tan(q\pi) \\ -\frac{\tan(q\pi)}{2d} & -2 & -\frac{1}{2} \\ 0 & 0 & 0 \end{bmatrix}$. Then it is easy to verify that the eigenvalues of $A + B$ are

$$\lambda_1 = 0,\ \ \lambda_{2,3} = -1 \pm \sqrt{1 - \tan^2(q\pi)}\ \ \sharp$$

Lemma 3.2 *(Adopted from Theorem 3.16 (Complex Hermite) in [1])*
Consider a complex polynomial

$$P(\lambda) = \lambda^n + c_1\lambda^{n-1} + \cdots + c_{n-1}\lambda + c_n \tag{20}$$

where $c_i \in \mathcal{C}$, $i = 1, \cdots, n$. Define a hermitian matrix $H^e = [h_{ij}]$ as in Theorem 3.16 in [1].

1) *The complex polynomial $P(\lambda)$ is asymptotically stable if and only if H^e is positive definite.*
2) *Provided that the leading principal minors satisfy $H_i^e \neq 0$, for all i, then $P(\lambda)$ has κ and $n - \kappa$ roots with positive and negative real parts, where*

$$\kappa = \upsilon(1, H_1^e, H_2^e, \cdots, H_n^e) \tag{21}$$

Let

$$D_i = A + \varphi^{i-1}B = \begin{bmatrix} -1 + \varphi^{i-1} & 2d\tan(q\pi) & d\tan(q\pi) \\ \frac{\tan(q\pi)}{d}(-1 + \frac{\varphi^{i-1}}{2}) & -2 & -\frac{1}{2} \\ \frac{\tan(q\pi)}{d}(1 - \varphi^{i-1}) & 2(1 - \varphi^{i-1}) & 0 \end{bmatrix}$$

with $i = 1, \cdots, n$. Then the determinant of $sI_3 - D_i$ can be calculated as follows:

$$|sI_3 - D_i| = s^3 + (2 + w_i)s^2 + (3w_i + \tan^2(q\pi))s + w_i^2(1 + \tan^2(q\pi)) \tag{22}$$

with $w_i = 1 - \varphi^{i-1} = 1 - \cos(\frac{2\pi(i-1)}{n}) - j\sin(\frac{2\pi(i-1)}{n})$, $i = 1, \cdots, n$. Let $w_i = m_i + jn_i$ with $m_i = 1 - \cos(\frac{2\pi(i-1)}{n})$ and $n_i = -\sin(\frac{2\pi(i-1)}{n})$. Then,

$$|sI_3 - D_i| = s^3 + (2 + m_i + jn_i)s^2 + (3m_i - 1 + \frac{1}{\cos^2(q\pi)} + 3jn_i)s + (m_i^2 - n_i^2)\frac{1}{\cos^2(q\pi)} + j\frac{2m_in_i}{\cos^2(q\pi)} \tag{23}$$

Lemma 3.3 *For $k = 1$, $n \geq 3$,*

- *D_2 has one imaginary eigenvalue $j\tan(\frac{\pi}{n})$, and other two eigenvalues for D_2 have negative real parts;*
- *D_n has one imaginary eigenvalue $-j\tan(\frac{\pi}{n})$, and other two eigenvalues for D_n have negative real parts.*

Please refer to the appendix A for the proof.

Remark 3.4 *By Lemma 3.1 and 3.3, $\hat{A}$ has one eigenvalue at zero and a pair of pure imaginary eigenvalues. The zero eigenvalue is due to the linear algebraic constraint g_3 in (4), while the pair of imaginary eigenvalues are due to g_1 and g_2. This implies that any translation and rotation of the formation would still make g_1, g_2, g_3 hold. From this we can see that these three eigenvalues will not affect the stability of the formation. A more involved explanation can be found in Lemma 2 in [10].*

Lemma 3.4 *for $k = 1$, $|sI_3 - D_i|$ given by (23) is asymptotically stable for $i \in [3, n-1]$, $n \geq 4$.*

Please refer to the appendix B for the proof.

Theorem 3.2 *For $\theta_0 = \alpha_i$, $n \geq 3$, the equilibria $\{\xi | r_i = 2d, \alpha_i = \frac{\pi}{n}, \beta_i = \pi - \frac{2\pi}{n}\}$, and $\{\xi | r_i = 2d, \alpha_i = -\frac{\pi}{n}, \beta_i = \frac{2\pi}{n} - \pi\}$, consequently the corresponding circular formations for unicycles given by (1), are locally asymptotically stable.*

Proof: We first show that for $\theta_0 = \frac{\pi}{n}$, $n \geq 3$, the equilibria $\{\xi | r_i = 2d, \alpha_i = \frac{\pi}{n}, \beta_i = \pi - \frac{2\pi}{n}\}$ are asymptotically stable, and the similar argument can be applied to the equilibria$\{\xi | r_i = 2d, \alpha_i = -\frac{\pi}{n}, \beta_i = \frac{2\pi}{n} - \pi\}$. For this purpose, we have to study the eigenvalues of $\hat{A}$. By Lemma 4 in [10], the eigenvalues of $\hat{A}$ are the collection of all eigenvalues of $A + B$, $A + \varphi B$, $A + \varphi^2 B$, $\cdots$, $A + \varphi^{n-1} B$.

Next step, we will show that except the three eigenvalues on the imaginary axis, the real parts of the rest of the eigenvalues are negative.

- By Lemma 3.1, The eigenvalues of D_1 are

$$\lambda_1 = 0, \ \ \lambda_{2,3} = -1 \pm \sqrt{1 - \tan^2(q\pi)}$$

  Thus $Re(\lambda_{2,3}) < 0$;
- By Lemma 3.3, D_2 has one imaginary eigenvalue $j\tan(\frac{\pi}{n})$, and other two eigenvalues for D_2 have negative real parts for $n \geq 3$, so does D_n;
- By Lemma 3.4, $|sI_3 - D_i|$, $i = 3, \cdots, n-1$, is asymptotically stable for $n \geq 4$. Thus, the eigenvalues of D_i, $i = 3, \cdots, n-1$, have negative real parts for $n \geq 4$.

Thus, for $n \geq 3$, $\hat{A}$ has $3(n-1)$ eigenvalues with negative real parts. By Remark 3.4, it also has one eigenvalue at zero and a pair of pure imaginary eigenvalues, and these three eigenvalues will not affect the stability of the formation. Therefore, for $n \geq 3$, the equilibria $\{\xi | r_i = 2d, \alpha_i = \frac{\pi}{n}, \beta_i = \pi - \frac{2\pi}{n}\}$, are locally asymptotically stable. Similar argument can be achieved for equilibria $\{\xi | r_i = 2d, \alpha_i = -\frac{\pi}{n}, \beta_i = \frac{2\pi}{n} - \pi\}$. ♯

Lemma 3.5 *For $2 \leq k < n$, $\frac{k}{n} \neq \frac{1}{2}$, $n \geq 5$, $|sI_3 - D_2|$ has one root with positive real part.*

Please refer to the appendix C for the proof.

Theorem 3.3 *For $\theta_0 = \frac{k\pi}{n} \neq \frac{\pi}{2}$, $2 \leq k < n$, $n \geq 5$, the equilibria $\{\xi | r_i = 2d, \alpha_i = \frac{k\pi}{n}, \beta_i = \pi - \frac{2k\pi}{n}\}$ and $\{\xi | r_i = 2d, \alpha_i = \frac{k\pi}{n} - \pi, \beta_i = \pi - \frac{2k\pi}{n}\}$ are unstable.*

Proof: when $n \geq 5$, by Lemma 3.5, $|sI_3 - D_2|$ has one root with positive real part for $2 \leq k < n$, $\frac{k}{n} \neq \frac{1}{2}$, thus $\{\xi | r_i = 2d, \alpha_i = \frac{k\pi}{n}, \beta_i = \pi - \frac{2k\pi}{n}\}$ is unstable.
By the similar argument, we can also show that equilibria $\{\xi | r_i = 2d, \alpha_i = \frac{k\pi}{n} - \pi, \beta_i = \pi - \frac{2k\pi}{n}\}$ is unstable for $\theta_0 = \frac{k\pi}{n} \neq \frac{\pi}{2}$, $2 \leq k < n, n \geq 5$. ♯

Remark 3.5 *It is noted that combine Theorem 3.2 and 3.3, we can find only two sets of equilibria given by $\{\xi | r_i = 2d, \alpha_i = \frac{k\pi}{n}, \beta_i = \pi - \frac{2k\pi}{n}\}$ and $\{\xi | r_i = 2d, \alpha_i = \frac{k\pi}{n} - \pi, \beta_i = \pi - \frac{2k\pi}{n}\}$ are asymptotically stable for $n \geq 5$.*
Here we also consider special cases $n = 3$ and $n = 4$, $2 \leq k \leq n-1$ and we can find that the eigenvalues of D_i, $i = 1, \cdots, n$ have negative real parts except three on the imaginary axis. Thus, the equilibrium points $\{\xi | r_i = 2d, \alpha_i = \frac{2\pi}{3}, \beta_i = -\frac{\pi}{3}\}$ and $\{\xi | r_i = 2d, \alpha_i = \frac{3\pi}{4}, \beta_i = -\frac{\pi}{2}\}$ are also asymptotically stale.

3.3 Other Stable Formations

In this section, we will discuss the stability of the equilibria given by (15) by letting $\theta_0 = 0$. In that case, vehicle $i+1$ stays at $(x_i + d\cos(\theta_i), y_i + d\sin(\theta_i))$ while vehicle $i's$ orientation θ_i, $i = 1, \cdots, n$, points directly at vehicle $i+1$, which also can be represented by $\{(x, y, \theta) | x_{i+1} = x_i + d\cos(\theta_i), y_{i+1} = y_i + d\sin(\theta_i), i = 1, \cdots, n, x_{n+1} = x_1, y_{n+1} = y_1\}$.

Let $\tilde{x}_i = x_{i+1} - (x_i + d\cos(\theta_i))$ and $\tilde{y}_i = y_{i+1} - (y_i + d\sin(\theta_i))$, $i = 1, \cdots, n$. For a sufficiently small $\epsilon > 0$, define a ball $B_R = \{(x, y, \theta) || \tilde{x}_i| < \epsilon, \ |\tilde{y}_i| < \epsilon\}$. Then the expression for $\dot{\tilde{x}}_i$ and $\dot{\tilde{y}}_i$ is

$$\begin{aligned} \dot{\tilde{x}}_i &= -\tilde{x}_i + \cos(\theta_{i+1})(\tilde{x}_{i+1}\cos(\theta_{i+1}) + \tilde{y}_{i+1}\sin(\theta_{i+1})) \\ \dot{\tilde{y}}_i &= -\tilde{y}_i + \sin(\theta_{i+1})(\tilde{x}_{i+1}\cos(\theta_{i+1}) + \tilde{y}_{i+1}\sin(\theta_{i+1})) \end{aligned} \tag{24}$$

Theorem 3.4 *The equilibria given by $\{(x, y, \theta) | x_{i+1} = x_i + d\cos(\theta_i), y_{i+1} = y_i + d\sin(\theta_i), i = 1, \cdots, n, x_{n+1} = x_1, y_{n+1} = y_1\}$ are stable.*

Proof: In the ball B_R, define a continuous function

$$V(\tilde{x}, \tilde{y}) = \sum_{i=1}^{n} (\tilde{x}_i^2 + \tilde{y}_i^2) \tag{25}$$

with $\tilde{x} = \mathrm{col}(\tilde{x}_1, \cdots, \tilde{x}_n)$ and $\tilde{y} = \mathrm{col}(\tilde{y}_1, \cdots, \tilde{y}_n)$.

Next we will show that $\dot{V}(\tilde{x}, \tilde{y})$ is negative semi-definite in the ball B_R.

$$\begin{aligned}
&\dot{V}(\tilde{x}, \tilde{y}) = \sum_{i=1}^{n} (2\tilde{x}_i\dot{\tilde{x}}_i + 2\tilde{y}_i\dot{\tilde{y}}_i) \\
&= \sum_{i=1}^{n} (-2\tilde{x}_i^2 + 2\tilde{x}_i\tilde{x}_{i+1}\cos^2(\theta_{i+1}) + 2\tilde{x}_i\tilde{y}_{i+1}\cos(\theta_{i+1})\sin(\theta_{i+1}) \\
&- 2\tilde{y}_i^2 + 2\tilde{y}_i\tilde{x}_{i+1}\cos(\theta_{i+1})\sin(\theta_{i+1}) + 2\tilde{y}_i\tilde{y}_{i+1}\sin^2(\theta_{i+1})) \\
&= \sum_{i=1}^{n} (-\tilde{x}_i^2 - \tilde{x}_i^2\cos^2(\theta_{i+1}) - \tilde{x}_i^2\sin^2(\theta_{i+1}) \\
&+ 2\tilde{x}_i\tilde{x}_{i+1}\cos^2(\theta_{i+1}) + 2\tilde{x}_i\tilde{y}_{i+1}\cos(\theta_{i+1})\sin(\theta_{i+1}) \\
&- \tilde{y}_i^2 - \tilde{y}_i^2\cos^2(\theta_i) - \tilde{y}_i^2\sin^2(\theta_{i+1}) \\
&+ 2\tilde{y}_i\tilde{x}_{i+1}\cos(\theta_{i+1})\sin(\theta_{i+1}) + 2\tilde{y}_i\tilde{y}_{i+1}\sin^2(\theta_{i+1})) \\
&= \sum_{i=1}^{n} (-(\tilde{x}_i\cos(\theta_{i+1}) - \tilde{x}_{i+1}\cos(\theta_{i+1}))^2 \\
&- (\tilde{x}_i\sin(\theta_{i+1}) - \tilde{y}_{i+1}\cos(\theta_{i+1}))^2 \\
&- (\tilde{x}_{i+1}\sin(\theta_{i+1}) - \tilde{y}_i\cos(\theta_{i+1}))^2 \\
&- (\tilde{y}_i\sin(\theta_{i+1}) - \tilde{y}_{i+1}\sin(\theta_{i+1}))^2) \leq 0
\end{aligned} \tag{26}$$

By Theorem 4.1 in [8], $\tilde{x}_i = 0$, $\tilde{y}_i = 0$, $i = 1, \cdots, n$, is stable. Thus, the equilibria given by $\{(x, y, \theta) | x_{i+1} = x_i + d\cos(\theta_i), y_{i+1} = y_i + d\sin(\theta_i), i = 1, \cdots, n, x_{n+1} = x_1, y_{n+1} = y_1\}$, $i = 1, \cdots, n$, are stable.

4 Examples

Consider a group unicycles given by (1) with $n = 7$. Let $d = 1$ and $\theta_0 = \frac{\pi}{7}$.

Fig. 4 shows the trajectories for all agents, which forms a regular heptagon and keeps this pattern while moving in a circle. Fig. 5 shows the distances between agent i and $i+1$, which satisfy $r_i = 2d$, $i = 1, \cdots, 7$, and displays $\beta_i = \frac{5\pi}{7}$.

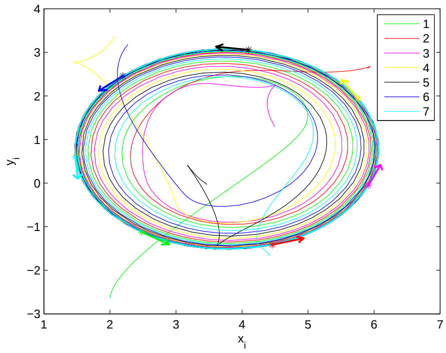

Fig. 4: trajectories for vehicle i

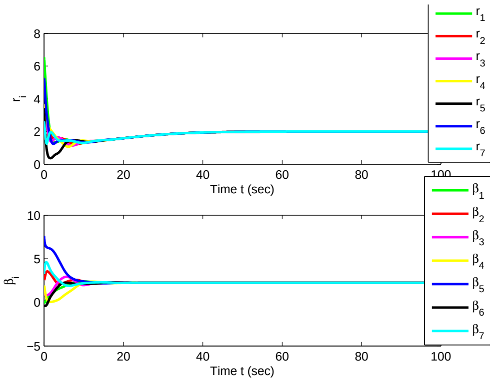

Fig. 5: r_i and β_i

An interesting phenomenon can be observed from the simulations is that after vehicles asymptotically approach $\{\xi | r_i = 2d = 2, \alpha_i = \frac{\pi}{7}, \beta_i = \frac{5\pi}{7}\}$, change parameters $d = 1$ and $\theta_0 = \frac{\pi}{7}$ to $d = 2$ and $\theta_0 = 0$, vehicles stop moving in the circle and the orientations for each vehicle θ_i directly point to vehicle $i+1$. That is because at that moment, these vehicles are near the stable equilibrium point given by $\{\xi | r_i = d = 2, \alpha_i = \theta_0 = 0, \sum_{i=1}^{7} \beta_i - 7\pi + 2k\pi = 0, 1 \le k \le 7\}$. Inspired by this property, we can enable vehicle i to stop at desired position in the circle. For example, let vehicle 1 in Fig. 4 stop at $(x_1 = 4, y_1 = 3)$ and θ_1 points to the location of vehicle 2. Simulation result can be found in Fig. 6.

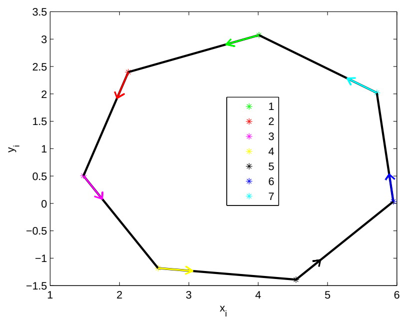

Fig. 6: $\{\xi | r_i = d = 2, \alpha_i = \theta_0 = 0, \beta_i = \frac{5\pi}{7}\}$

5 Conclusion

In this paper, we have studied the formation control of a group of unicycles. A distributed control law for vehicle i, depending only on its bearing angle and the relative position of vehicle $i+1$, has been proposed. By designing appropriate parameters for a group of vehicles, these vehicles can be uniformly spaced in a circle in order and rotate counterclockwise or clockwise, respectively. The distances between two vehicles can be uniquely decided by the control parameter. By changing design parameters, more generalized formations can be achieved. A very spacial case is to make these vehicles stop at a desired position and form a regular polygon.

6 Future work

By choosing different parameters for each vehicle in the control law (5), it is possible to form much more general formations by a group of unicycles. Our next step is to enable these vehicles to form convex $n-$gon by designing parameters θ_{0i} and d_i in the control law. For example, let $\theta_{0i} = \frac{2\pi}{7}$, $i = 1, \cdots, 6$, $\theta_{07} = \frac{3\pi}{7}$ and $d_i = 1$, $i = 1, 3, \cdots, 7$, $d_2 = 3$. From Fig. 7, we can find that

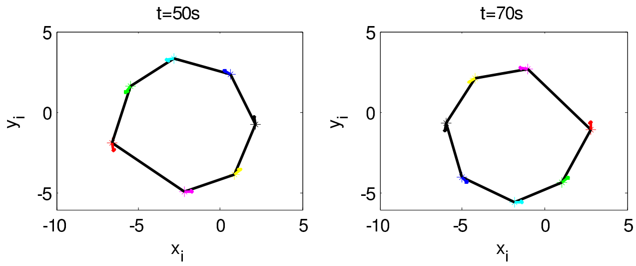

Fig. 7: Design different parameters

these vehicles form a heptagon with different sides and rotate in circles while keeping this pattern.

A Proof of Lemma 3.3

We first show for $k = 1$, D_2 has one imaginary eigenvalue $j\tan(\frac{\pi}{n})$ by substituting $s = j\tan(\frac{\pi}{n})$ into (22). Then one can achieve

$$
\begin{aligned}
&|j\tan(\frac{\pi}{n})I_3 - D_2| = -3\tan^2(\frac{\pi}{n}) + \tan^2(\frac{\pi}{n})\cos(\frac{2\pi}{n}) \\
&+ 3\tan(\frac{\pi}{n})\sin(\frac{2\pi}{n}) - 4\cos(\frac{2\pi}{n})\sin^2(\frac{\pi}{n}) \\
&+ 2\tan^2(\frac{\pi}{n})\cos^2(\frac{2\pi}{n}) - 2\tan^2(\frac{\pi}{n})\cos(\frac{2\pi}{n}) \\
&+ j(\tan^2(\frac{\pi}{n})\sin(\frac{2\pi}{n}) + 3\tan(\frac{\pi}{n}) - 3\tan(\frac{\pi}{n})\cos(\frac{2\pi}{n}) \\
&- 2\sin(\frac{2\pi}{n}) + 2\sin(\frac{2\pi}{n})\cos(\frac{2\pi}{n}) - 2\tan^2(\frac{\pi}{n})\sin(\frac{2\pi}{n}) \\
&+ 2\tan^2(\frac{\pi}{n})\sin(\frac{2\pi}{n})\cos(\frac{2\pi}{n}))
\end{aligned}
\tag{27}
$$

It can be verified both the real part and imaginary part are zero. Then $|j\tan(\frac{\pi}{n})I_3 - D_2| = 0$. Thus, for $k = 1$, $j\tan(\frac{\pi}{n})$is the imaginary eigenvalue of D_2.

Then we will show that for $k = 1$, D_2 has two eigenvalues with negative real parts, denoted by $s_2 = \alpha_2 + j\beta_2$ and $s_3 = \alpha_3 + j\beta_3$. Then,

$$
\begin{aligned}
|sI_3 - D_2| = s^3 &- (\alpha_2 + \alpha_3 + j(\beta_2 + \beta_3 + \tan(\frac{\pi}{n})))s^2 \\
&+ (\alpha_2\alpha_3 - \beta_2\beta_3 - \tan(\frac{\pi}{n})(\beta_2 + \beta_3) \\
&+ j(\alpha_2\beta_3 + \alpha_3\beta_2 + \tan(\frac{\pi}{n})(\alpha_2 + \alpha_3)))s \\
&- j\tan(\frac{\pi}{n})(\alpha_2\alpha_3 - \beta_2\beta_3 + j(\alpha_2\beta_3 + \alpha_3\beta_2))
\end{aligned}
\tag{28}
$$

Then, from (23) and (28), one can achieve

$$
\begin{aligned}
&\alpha_2 + \alpha_3 = \cos(\frac{2\pi}{n}) - 3, \quad \beta_2 + \beta_3 = \cos(\frac{2\pi}{n})\tan(\frac{\pi}{n}) \\
&\alpha_2\alpha_3 - \beta_2\beta_3 = 4(1 - \cos(\frac{2\pi}{n})) \\
&\alpha_2\beta_3 + \alpha_3\beta_2 = -4\cos(\frac{2\pi}{n})\tan(\frac{\pi}{n})
\end{aligned}
\tag{29}
$$

In order to show that $\alpha_2 < 0$ and $\alpha_3 < 0$, that is, to show

$$s^2 - (\alpha_2 + \alpha_3 + j(\beta_2 + \beta_3))s + \alpha_2\alpha_3 - \beta_2\beta_3 + j(\alpha_2\beta_3 + \alpha_3\beta_2) \tag{30}$$

is asymptotically stable. By Lemma 3.2, we have to show that

$$\begin{aligned} H^e &= \begin{bmatrix} c_1 + \bar{c}_1 & c_2 - \bar{c}_2 \\ -c_2 + \bar{c}_2 & c_1\bar{c}_2 + c_2\bar{c}_1 \end{bmatrix} \\ &= 2\begin{bmatrix} -(\alpha_2 + \alpha_3) & j(\alpha_2\beta_3 + \alpha_3\beta_2) \\ -j(\alpha_2\beta_3 + \alpha_3\beta_2) & h_e \end{bmatrix} \end{aligned} \tag{31}$$

with $h_e = -(\alpha_2+\alpha_3)(\alpha_2\alpha_3 - \beta_2\beta_3) - (\beta_2+\beta_3)(\alpha_2\beta_3+\alpha_3\beta_2)$, is positive definite. That is,

$$H^e = 2\begin{bmatrix} 3 - \cos(\frac{2\pi}{n}) & -4j\cos(\frac{2\pi}{n})\tan(\frac{\pi}{n}) \\ 4j\cos(\frac{2\pi}{n})\tan(\frac{\pi}{n}) & h_{e1} \end{bmatrix} \tag{32}$$

with $h_{e1} = 4(3 - \cos(\frac{2\pi}{n}))(1 - \cos(\frac{2\pi}{n})) + 4\cos^2(\frac{2\pi}{n})\tan^2(\frac{\pi}{n})$. The leading principle minors of $\frac{H^e}{2}$ are

$$\begin{aligned} &h_1 = 3 - \cos(\frac{2\pi}{n}) > 0 \\ &h_2 = 4(3 - \cos(\frac{2\pi}{n}))^2(1 - \cos(\frac{2\pi}{n})) + \\ &4(3 - \cos(\frac{2\pi}{n}))\cos^2(\frac{2\pi}{n})\tan^2(\frac{\pi}{n}) - 16\cos^2(\frac{2\pi}{n})\tan^2(\frac{\pi}{n}) \\ &= 8(9 - 6\cos(\frac{2\pi}{n}) + \cos^2(\frac{2\pi}{n}))\sin^2(\frac{\pi}{n}) \\ &- 8\cos^2(\frac{\pi}{n})\cos^2(\frac{2\pi}{n})\frac{\sin^2(\frac{\pi}{n})}{\cos^2(\frac{\pi}{n})} \\ &= 8(9 - 6\cos(\frac{2\pi}{n}))\sin^2(\frac{\pi}{n}) \\ &= 8(3 + 6(1 - \cos(\frac{2\pi}{n})))\sin^2(\frac{\pi}{n}) > 0, \quad \text{for } n \geq 3 \end{aligned} \tag{33}$$

Thus, by Lemma 3.2, (30) is asymptotically stable, that is, $\alpha_2, \alpha_3 < 0$.
By the similar procedure, we can also show that D_n has one imaginary eigenvalue $-j\tan(\frac{\pi}{n})$, and other two eigenvalues for D_n have negative real parts for $k = 1, n \geq 3$.

B Proof of Lemma 3.4

In order to show that for $i \in [3, n-1]$, $n \geq 4$, $|sI_3 - D_i|$ given by (23) is asymptotically stable, by Lemma 3.2, we need to show that the leading principal minors of (34) are all positive for $i \in [3, n-1], n \geq 4$.

$$H^e = \begin{bmatrix} c_1 + \bar{c}_1 & c_2 - \bar{c}_2 & c_3 + \bar{c}_3 \\ -c_2 + \bar{c}_2 & c_1\bar{c}_2 + c_2\bar{c}_1 - c_3 - \bar{c}_3 & c_3\bar{c}_1 - c_1\bar{c}_3 \\ c_3 + \bar{c}_3 & c_1\bar{c}_3 - c_3\bar{c}_1 & c_2\bar{c}_3 + c_3\bar{c}_2 \end{bmatrix} \tag{34}$$

with

$$\begin{aligned} c_1 &= 2 + m_i + jn_i \\ c_2 &= 3m_i - 1 + \frac{1}{\cos^2(\frac{k\pi}{n})} + 3jn_i \\ c_3 &= (m_i^2 - n_i^2)\frac{1}{\cos^2(\frac{k\pi}{n})} + j\frac{2m_in_i}{\cos^2(\frac{k\pi}{n})} \end{aligned}$$

The leading principal minors of $\frac{H^e}{2}$ are

$$\begin{aligned} &h_1 = 2 + m_i = 3 - cos(\frac{2\pi(i-1)}{n}) > 0 \\ &h_2 = (2 + m_i)((2 + m_i)(3m_i - 1 + \frac{1}{\cos^2(\frac{k\pi}{n})}) \\ &+ 3(2m_i - m_i^2) - (2m_i^2 - 2m_i)\frac{1}{\cos^2(\frac{k\pi}{n})}) - 9(2m_i - m_i^2) \\ &= (2 + m_i)(6m_i - 2 + 3m_i^2 - m_i + \frac{1}{\cos^2(\frac{k\pi}{n})}(2 + m_i) \\ &+ 6m_i - 3m_i^2 - \frac{1}{\cos^2(\frac{k\pi}{n})}(2m_i^2 - 2m_i)) - 18m_i + 9m_i^2 \\ &= \frac{1}{\cos^2(\frac{k\pi}{n})}(2 + m_i)(2 + 3m_i - 2m_i^2) + 20m_i^2 + 2m_i - 4 \end{aligned} \tag{35}$$

Since $\frac{1}{\cos^2(\frac{k\pi}{n})} > 1$ and $0 < m_i = 1 - \cos(\frac{2\pi(i-1)}{n}) \leq 2$ for $i \in [3, n-1]$, $n \geq 4$, and $2 + 3m_i - 2m_i^2 = -2(m_i - \frac{3}{4})^2 + 2 + \frac{9}{8} > 0$ for $m_i \in (0, 2]$, then,

$$\begin{aligned} h_2 &> 2(2 + 3m_i - 2m_i^2) + 20m_i^2 + 2m_i - 4 \\ &= 8m_i(2m_i + 1) > 0 \end{aligned} \tag{36}$$

For $k = 1$,

$$h_3 = \frac{1}{\cos^6(\frac{\pi}{n})}A_3(m_i) + \frac{1}{\cos^4(\frac{\pi}{n})}A_2(m_i) + \frac{1}{\cos^2(\frac{\pi}{n})}A_1(m_i) \tag{37}$$

where

$$\begin{aligned} A_3(m_i) &= 8m_i^6 - 32m_i^5 + 26m_i^4 + 22m_i^3 - 16m_i^2 - 8m_i \\ A_2(m_i) &= 56m_i^5 - 88m_i^4 - 66m_i^3 + 28m_i^2 + 16m_i \\ A_1(m_i) &= 80m_i^4 + 48m_i^3 - 12m_i^2 - 8m_i \end{aligned} \tag{38}$$

when $n = 4$, since $i \in [3, n-1]$, i only can be 3, and it can be calculated that $h_3(i = 3, n = 4) > 0$.
Then we consider the continuous function $h_3(\check{i}, \check{n})$ for $\check{i} \in [3, \check{n} - 1]$, $\check{n} \geq 5$.

Let $h_3(\check{i}, \check{n}) = 0$, $\check{i} \in [3, \check{n} - 1]$, $\check{n} \geq 5$. Then $\frac{1}{\cos^2(\frac{\pi}{\check{n}})}(\frac{1}{\cos^4(\frac{\pi}{\check{n}})}A_3(m_i) + \frac{1}{\cos^2(\frac{\pi}{\check{n}})}A_2(m_i) + A_1(m_i)) = 0$.

case 1: $A_3(m_i) \neq 0$, then $\frac{1}{\cos^2(\frac{\pi}{\check{n}})} = \frac{-A_2 \pm \sqrt{A_2^2 - 4A_1A_3}}{2A_3}$. It is noted that $\frac{1}{\cos^2(\frac{\pi}{\check{n}})} > 1$ and $\frac{1}{\cos^2(\frac{\pi}{\check{n}})}$ is real number for $\check{n} \geq 5$, thus

$$\begin{aligned} A_2^2 - 4A_1A_3 > 0 \\ A_1 + A_2 + A_3 < 0 \end{aligned} \tag{39}$$

However,

$$A_1 + A_2 + A_3 = 8m_i^6 + 24m_i^5 + 18m_i^4 + 4m_i^3 > 0$$

contradicting with $A_1 + A_2 + A_3 < 0$. Thus, $h_3(\check{i}, \check{n}) = 0$, $\check{i} \in [3, \check{n} - 1]$, $n \geq 5$, has no solution for $A_3(m_i) \neq 0$.

case 2: $A_3(m_i) = 0$, then $m_i = -\frac{1}{2}, -\frac{1}{2}, 0, 1, 2, 2$. Since $m_i \in (0, 2]$, $m_i = 1$ or $m_i = 2$.

- $m_i = 1$, then $A_2 = -54$, $A_3 = 108$, thus, $\frac{1}{\cos^2(\frac{\pi}{\check{n}})} = 2$, that is, $\frac{\pi}{\check{n}} = \frac{\pi}{4} + \rho\pi$ or $\frac{\pi}{\check{n}} = \frac{3\pi}{4} + \rho\pi$, $\rho = 0, 1, 2\cdots$. Then $\check{n} = \frac{4}{1+4\rho}$ or $\check{n} = \frac{4}{3+4\rho}$, contradicting with $\check{n} \geq 5$.
- $m_i = 2$, then $A_2 = 0$, $A_3 = 1600$, then $\frac{1}{\cos^2(\frac{\pi}{\check{n}})}A_2 + A_3 \neq 0$.

In conclusion, the continuous function $h_3(\check{i}, \check{n}) = 0$ has no solution for $\check{i} \in [3, \check{n} - 1]$, $\check{n} \geq 5$. Thus, $h_3(i, \check{n}) > 0$ for all $\check{i} \in [3, \check{n} - 1]$, $\check{n} \geq 5$ or $h_3(\check{i}, \check{n}) < 0$ for all $\check{i} \in [3, \check{n} - 1]$, $\check{n} \geq 5$. Substitute any $\check{i}, \check{n}$ satisfying $\check{i} \in [3, \check{n} - 1]$, $\check{n} \geq 5$, into $h_3(\check{i}, \check{n})$, one can obtain

$h_3(\check{i}, \check{n}) > 0$ for all $\check{i} \in [3, \check{n}-1]$, $\check{n} \geq 5$, thus, for the third leading principal of $\frac{H^e}{2}$, $h_3(i, n) > 0$ for all $i \in [3, n-1]$, $n \geq 5$, with i, n are integers.

Then the leading principal minors for H^e in (34) are all greater than 0 for $i \in [3, n-1]$, $n \geq 4$, thus by Lemma 3.2, $|sI_3 - D_i|$ is asymptotically stable.

C Proof of Lemma 3.5

By the similar proof of Lemma 3.4, one can easily obtain the leading principal minors $h_1 > 0$ and $h_2 > 0$ for $|sI_3 - D_2|$, $2 \leq k < n$, $i = 2$.

For $2 \leq \check{k} < \check{n}$, $\check{n} \geq 5$, we consider the continuous function

$$h_3(\check{k}, \check{n}) = \frac{1}{\cos^6(\frac{\check{k}\pi}{\check{n}})} A_3(m_2) + \frac{1}{\cos^4(\frac{\check{k}\pi}{\check{n}})} A_2(m_2) + \frac{1}{\cos^2(\frac{\check{k}\pi}{\check{n}})} A_1(m_2) \quad (40)$$

with $m_2 = 1 - \cos(\frac{2\pi}{\check{n}})$ and $A_3(m_2), A_2(m_2), A_1(m_2)$ given by (38).

Let $h_3(\check{k}, \check{n}) = 0$. Then $\frac{1}{\cos^2(\frac{\check{k}\pi}{\check{n}})}(\frac{1}{\cos^4(\frac{\check{k}\pi}{\check{n}})} A_3(m_2) + \frac{1}{\cos^2(\frac{\check{k}\pi}{\check{n}})} A_2(m_2) + A_1(m_2)) = 0$.

case 1: $A_3(m_2) \neq 0$,

$$\frac{1}{\cos^2(\frac{\check{k}\pi}{\check{n}})} = \frac{-A_2(m_2) \pm \sqrt{A_2^2(m_2) - 4A_1(m_2)A_3(m_2)}}{2A_3(m_2)} > 1$$

Thus $A_1(m_2)$, $A_2(m_2)$ and $A_3(m_2)$ also have to satisfy $A_1(m_2) + A_2(m_2) + A_3(m_2) < 0$, which contradicts with $A_1(m_2) + A_2(m_2) + A_3(m_2) = 8m_2^6 + 24m_2^5 + 18m_2^4 + 4m_2^3 > 0$.

case 2: $A_3(m_2) = 0$, $m_2 = 1, 2$ for $i = 2$, $2 \leq \check{k} < \check{n}$, $\check{n} \geq 5$.

- $m_2 = 1$,
$$1 - \cos(\frac{2\pi}{\check{n}}) = 1 \quad (41)$$
Thus, $\check{n} = \frac{4}{1+2t}$, $t = 0, 1, 2, \cdots$, contradicting with $\check{n} \geq 5$.
- $m_2 = 2$, $A_2(m_2) = 0$, $A_3(m_2) = 1600$, $h_3 \neq 0$.

In conclusion, the continuous function $h_3(\check{k}, \check{n}) = 0$ has no solution for $i = 2$, $2 \leq \check{k} < \check{n}$. Thus, $h_3(\check{k}, \check{n}) > 0$ for all $2 \leq \check{k} < \check{n}$ or $h_3(\check{k}, \check{n}) < 0$ for all $2 \leq \check{k} < \check{n}$. Substitute any $\check{k}, \check{n}$ satisfying $2 \leq \check{k} < \check{n}$, $\check{n} \geq 5$, into $h_3(\check{k}, \check{n})$, one can obtain $h_3(\check{k}, \check{n}) < 0$ for all $2 \leq \check{k} < \check{n}$, $\check{n} \geq 5$. Thus, $h_3(k, n) < 0$ for all $2 \leq k < n$, $n \geq 5$ with k, n are integers.

By Lemma 3.2, we can show

$$\kappa = \upsilon(1, h_1, h_2, h_3) = 1 \quad (42)$$

Thus for $2 \leq k < n$, $n \geq 5$, $|sI_3 - D_2|$ has one root with positive real part, and two roots with negative real parts.

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