Stability Conditions and Phase Transition for Kalman Filtering over Markovian Channels

Junfeng Wu[1], Guodong Shi[2], Brian D. O. Anderson[2], Karl Henrik Johansson[1]

1. ACCESS Linnaeus Center, School of Electrical Engineering, Royal Institute of Technology, Stockholm, Sweden
E-mail: {junfengw, kallej}@kth.se

2. Research School of Engineering, The Australian National University, Canberra, Australia
E-mail: {guodong.shi, brian.anderson}@anu.edu.au

Abstract: This paper investigates the stability of Kalman filtering over Gilbert-Elliott channels where the random packet drop follows a time-homogeneous two-state Markov chain whose state transition is determined by a pair of failure and recovery rates. First, we establish a relaxed condition guaranteeing peak-covariance stability described by an inequality in terms of the spectral radius of the system matrix and transition probabilities of the Markov chain. We show that this condition can be rewritten as a linear matrix inequality feasibility problem. Next, we prove that the peak-covariance stability implies mean-square stability, if the system matrix has no defective eigenvalues on the unit circle. This implication holds for any random packet drop process, and is thus not restricted to Gilbert-Elliott channels. We prove that there exists a critical curve in the failure-recovery rate plane, below which the Kalman filter is mean-square stable and above is unstable for some initial values. Finally, a lower bound for this critical failure rate is obtained making use of the relationship we establish between the two stability criteria, based on an approximate relaxation of the system matrix.

Key Words: Kalman filtering; estimation; stochastic system; Markov processes; stability

1 Introduction

1.1 Background and Related Works

Wireless communications are being widely used nowadays in sensor networks and networked control systems for a large spectrum of applications, such as environmental monitoring, health care, smart building operation, intelligent transportation and power grids. New challenges accompany the considerable advantages wireless communications offer in these applications, one of which is how channel fading and congestion, influence the performance of estimation and control. In the past decade, this fundamental question has inspired various significant results focusing on the interface of control and communication, and has become a central theme in the study of networked sensor and control systems [1–8].

State estimation, based on collecting measurements of the system output from sensors deployed in the field is embedded in many networked control applications and is often implemented recursively using the fundamental Kalman filter. The study of the interplay between a Kalman filtering and a lossy communication channel is pioneered in the seminal work [9], where Sinopoli et al. modeled the statistics of intermittent observations by an independent and identically distributed (i.i.d.) Bernoulli random process and studied how packet losses affect the state estimation. It was proved that there exists a critical arrival probability for packets, below which the expected prediction error covariance matrix is no longer uniformly bounded [9]. Tremendous research has since then been devoted to further stability analysis of Kalman filtering or the closed-loop control systems over i.i.d. packet lossy packet networks in [10–12].

To capture the temporal correlation of realistic communication channels, the Gilbert-Elliott model [13, 14] that describes time-homogeneous Markovian packet losses has been introduced to partially address this problem. The so-called peak-covariance stability was introduced with a focus on the stability of the error covariance matrix at certain stopping times of the Markovian packet process in [15, 16], which turned out to be rather useful also for the mean-square stability analysis [16]. Improvements to these results appeared in [17, 18]. Besides the two widely adopted stability notions, weak convergence of Kalman filtering with packet losses, i.e., that error covaraince matrix converges to a limit distribution, were investigated in [19–21] for i.i.d., semi-Markov, and Markov drop models, respectively.

1.2 Our Contributions

This paper aims to answer the following two fundamental questions that arise for Kalman filtering over lossy channels:

[Q1] What is the essential relation between peak-covariance and mean-square stabilities for general linear time-invariant (LTI) systems?

[Q2] Does the phase transition for mean-square stability with i.i.d. packet losses continue to exist with Markovian channels?

For **[Q1]**, we prove that peak-covariance stability implies mean-square stability if the system matrix has no defective eigenvalues on the unit circle. Remarkably enough this implication holds for *arbitrary* random packet drop process that allows peak-covariance stability to be defined. This answer bridges two stability criteria in the literature, and offers a tool for studying mean-square stability of the Kalman filter through its peak-covariance stability, where in fact we can easily bypass the no-defective-eigenvalue assumption for general LTI systems using an approximation method.

For **[Q2]**, we prove that there is a critical $p - q$ curve, with p being the failure rate and q being the recovery rate of the Gilbert-Elliott channel, below which the expected pre-

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diction error covariance matrices are uniformly bounded and unbounded above. This result is proved via a novel *coupling* argument, and to the best of our knowledge, this is the first time phase transition is established for Kalman filtering over Markovian channels.

We also derived a relaxed condition guaranteeing peak-covariance stability, described by an inequality in terms of the spectral radius of the system matrix and transition probabilities of the Markov chain, rather than an infinite sum of matrix norms as in [15–17]. We show that this condition can be recast as a linear matrix inequality (LMI) feasibility problem. These conditions are theoretically and numerically shown to be less conservative than those in the literature. Making use of the above results we derived for the relation between peak-covariance and mean-square stabilities and the existence of the critical curve, we manage to present a lower bound for the critical failure rate that holds for *general* LTI systems under Markovian packet drops. We believe these results add to the fundamental understanding of Kalman filtering under random packet drops.

1.3 Paper Organization and Notations

The remainder of the paper is organized as follows. Section 2 presents the problem setup. Section 3 focuses on the peak-covariance stability. Section 4 studies the relationship between the peak-covariance and mean-square stabilities, the critical $p-q$ curve, and presents a sufficient condition for mean-square stability of general LTI systems. Two numerical examples in Section 5 demonstrate the effectiveness of our approach compared with the literature. Finally we provide some concluding remarks in Section 6[1].

Notations: $\mathbb{N}$ is the set of positive integers. $\mathbb{S}_+^n$ is the set of n by n positive semi-definite matrices over the complex field. For a matrix X, $\sigma(X)$ denotes the spectrum of X and λ_X denotes the eigenvalue of X that has the largest magnitude. X^*, X' and $\overline{X}$ are the Hermitian conjugate, transpose and complex conjugate of X. Moreover, $\|\cdot\|$ means the 2-norm of a vector or the induced 2-norm of a matrix. $\otimes$ is the Kronecker product of two matrices. The indicator function of a subset $\mathcal{A}\subset\Omega$ is a function $1_{\mathcal{A}}:\Omega\to\{0,1\}$, where $1_{\mathcal{A}}(\omega)=1$ if $\omega\in\mathcal{A}$, otherwise $1_{\mathcal{A}}(\omega)=0$. For random variables, $\sigma(\cdot)$ is the σ-algebra generated by the variables.

2 Kalman Filtering over Markovian Channel

Consider an LTI system:

$$x_{k+1} = Ax_k + w_k, \tag{1}$$

$$y_k = Cx_k + v_k, \tag{2}$$

where $A\in\mathbb{R}^{n\times n}$ is the system matrix and $C\in\mathbb{R}^{m\times n}$ is the observation matrix, $x_k\in\mathbb{R}^n$ is the process state vector and $y_k\in\mathbb{R}^m$ is the observation vector, $w_k\in\mathbb{R}^n$ and $v_k\in\mathbb{R}^m$ are zero-mean Gaussian random vectors with auto-covariance $\mathbb{E}[w_k w_j{}']=\delta_{kj}Q$ $(Q\geq 0)$, $\mathbb{E}[v_k v_j{}']=\delta_{kj}R$ $(R>0)$, $\mathbb{E}[w_k v_j{}']=0$ $\forall j,k$. Here δ_{kj} is the Kronecker delta function with $\delta_{kj}=1$ if $k=j$ and $\delta_{kj}=0$ otherwise. The initial state x_0 is a zero-mean Gaussian random vector that is uncorrelated with w_k and v_k and has covariance $\Sigma_0\geq 0$. We assume that (C,A) is detectable and $(A,Q^{1/2})$ is stabilizable. By applying a similarity transformation, the unstable and stable modes of the considered LTI system can be decoupled. An open-loop prediction of the stable mode always has a bounded estimation error covariance, therefore, this mode does not play any key role in the stability issues considered in this paper. Without loss of generality, we assume that

(A1) *All of the eigenvalues of A have magnitudes not less than one.*

Certainly A is nonsingular, (C,A) is observable and $(A,Q^{1/2})$ is controllable.

We consider an estimation scheme where the raw measurements of the sensor $\{y_k\}_{k\in\mathbb{N}}$ are transmitted to the estimator via an erasure communication channel over which packets may be dropped randomly. Denote by $\gamma_k\in\{0,1\}$ the arrival of y_k at time k: y_k arrives error-free at the estimator if $\gamma_k=1$; otherwise $\gamma_k=0$. Whether γ_k takes value 0 or 1 is assumed to be known by the receiver at time k. Define $\mathcal{F}_k$ as the filtration generated by all the measurements received by the estimator up to time k, i.e., $\mathcal{F}_k\triangleq\sigma(\gamma_t y_t,\gamma_t;1\leq t\leq k)$ and $\mathcal{F}=\sigma\left(\cup_{k=1}^{\infty}\mathcal{F}_k\right)$. We will use a triple $(\Omega,\mathcal{F},\mathbb{P})$ to denote the probability space capturing all the randomness in the model.

To describe the temporal correlation of realistic communication channels, we assume the Gilbert-Elliott channel [13, 14], where the packet loss process is a time-homogeneous two-state Markov chain. To be precise, $\{\gamma_k\}_{k\in\mathbb{N}}$ is the state of the Markov chain with initial condition, without loss of generality, $\gamma_1=1$. The transition probability matrix for the Gilbert-Elliott channel is given by

$$\mathbf{P}=\begin{bmatrix}1-q & q\\ p & 1-p\end{bmatrix}, \tag{3}$$

where $p\triangleq\mathbb{P}(\gamma_{k+1}=0|\gamma_k=1)$ is called the failure rate, and $q\triangleq\mathbb{P}(\gamma_{k+1}=1|\gamma_k=0)$ is called the recovery rate. Assume that

(A2) *The failure and recovery rates satisfy $p,q\in(0,1)$.*

The estimator computes $\hat{x}_{k|k}$, the minimum mean-squared error estimate, and $\hat{x}_{k+1|k}$, the one-step prediction, according to $\hat{x}_{k|k}=\mathbb{E}[x_k|\mathcal{F}_k]$ and $\hat{x}_{k+1|k}=\mathbb{E}[x_{k+1}|\mathcal{F}_k]$. Let $P_{k|k}$ and $P_{k+1|k}$ be the corresponding estimation and prediction error covariance matrices, i.e., $P_{k|k}=\mathbb{E}[(x_k-\hat{x}_{k|k})(\cdot)'|\mathcal{F}_k]$ and $P_{k+1|k}=\mathbb{E}[(x_{k+1}-\hat{x}_{k+1|k})(\cdot)'|\mathcal{F}_k]$. They can be computed recursively via a modified Kalman filter [9]. The recursions for $\hat{x}_{k|k}$ and $\hat{x}_{k+1|k}$ are omitted here. To study the Kalman filtering system's stability, we focus on the the prediction error covariance matrix $P_{k+1|k}$, which is recursively computed as

$$\begin{aligned}P_{k+1|k} =& AP_{k|k-1}A'+Q\\ &-\gamma_k AP_{k|k-1}C'(CP_{k|k-1}C'+R)^{-1}CP_{k|k-1}A'.\end{aligned}$$

It can be seen that $P_{k+1|k}$ inherits the randomness of $\{\gamma_t\}_{1\leq t\leq k}$. In what follows, we focus on characterizing the impact of $\{\gamma_k\}_{k\in\mathbb{N}}$ on $P_{k+1|k}$. To simplify notations in the

[1] Due to hard space limitation we refer readers to [22] for detailed proofs of most of the claimed results.

sequel, let $P_{k+1} \triangleq P_{k+1|k}$, and define the functions h, g, h^k and g^k: $\mathbb{S}_+^n \to \mathbb{S}_+^n$ as follows:

$$h(X) \triangleq AXA' + Q, \tag{4}$$

$$g(X) \triangleq AXA' + Q - AXC'(CXC' + R)^{-1}CXA', \tag{5}$$

$h^k(X) \triangleq \underbrace{h \circ h \circ \cdots \circ h}_{k \text{ times}}(X)$ and $g^k(X) \triangleq \underbrace{g \circ g \circ \cdots \circ g}_{k \text{ times}}(X)$, where $\circ$ denotes the function composition.

3 Peak-covariance Stability

In this section, we study the peak-covariance stability [16] of the Kalman filter. To this end, define

$$\begin{aligned}
\tau_1 &\triangleq \min\{k : k \in \mathbb{N}, \gamma_k = 0\},\\
\beta_1 &\triangleq \min\{k : k > \tau_1, \gamma_k = 1\},\\
&\vdots\\
\tau_j &\triangleq \min\{k : k > \beta_{j-1}, \gamma_k = 0\},\\
\beta_j &\triangleq \min\{k : k > \tau_j, \gamma_k = 1\}.
\end{aligned} \tag{6}$$

It is straightforward to verify that $\{\tau_j\}_{j\in\mathbb{N}}$ and $\{\beta_j\}_{j\in\mathbb{N}}$ are two sequences of stopping times because both $\{\tau_j \le k\}$ and $\{\beta_j \le k\}$ are $\mathcal{F}_k$−measurable; see [23] for details. Due to the strong Markov property and the ergodic property of the Markov chain defined by (3) (see [16]), the sequences $\{\tau_j\}_{j\in\mathbb{N}}$ and $\{\beta_j\}_{j\in\mathbb{N}}$ have finite values $\mathbb{P}$-almost surely. Then we can define the sojourn times at the state 1 and state 0 respectively by τ_j^* and β_j^* $\forall j \in \mathbb{N}$ as

$$\begin{aligned}
\tau_j^* &\triangleq \tau_j - \beta_{j-1},\\
\beta_j^* &\triangleq \beta_j - \tau_j,
\end{aligned}$$

where we define $\beta_0 = 1$. Let us denote the prediction error covariance matrix at the stopping time β_j by P_{β_j} and call it the peak covariance[2] at β_j. To study the stability of Kalman filtering with Markovian packet losses, we introduce the concept of peak-covariance stability [16] as follows:

Definition 1. *The Kalman filtering system with packet losses is said to be peak-covariance stable if* $\sup_{j\in\mathbb{N}} \mathbb{E}\|P_{\beta_j}\| < \infty$.

3.1 Stability Conditions

To analyze the peak-covariance stability, we introduce the observability index of the pair (C, A).

Definition 2. *The observability index I_o is defined as the smallest integer such that $[C', A'C', \ldots, (A^{I_o-1})'C']'$ has rank n. If $\mathsf{I}_o = 1$, the system (C, A) is called one-step observable.*

We have the following result.

[2] The definition of peak covariance was first introduced in [16], where the term "peak" was attributed to the fact that for an unstable scalar system P_k monotonically increases to reach a local maximum at time β_j. This maximum property does not necessarily hold for the multi-dimensional case.

Theorem 1. *Suppose the following two conditions hold:*

(i). $|\lambda_A|^2(1-q) < 1$;

(ii). $\exists\, K \triangleq [K^{(1)}, \ldots, K^{(\mathsf{I}_o-1)}]$*, where $K^{(i)}$'s are matrices with compatible dimensions, such that* $|\lambda_{H(K)}| < 1$*, where*

$$H(K) = qp\Big[(A \otimes A)^{-1} - (1-q)I\Big]^{-1} \cdot \sum_{i=1}^{\mathsf{I}_o-1} \overline{(A^i + K^{(i)}C^{(i)})} \otimes (A^i + K^{(i)}C^{(i)})(1-p)^{i-1}. \tag{7}$$

Then $\sup_{j\ge1} \mathbb{E}\|P_{\beta_j}\| < \infty$*, i.e., the Kalman filtering system is peak-covariance stable.*

Remark 1. In [18], the authors defined stability in stopping times as the stability of P_k at packet reception times. Note that $\{\beta_j\}_{j\in\mathbb{N}}$, at which the peak covariance is defined, can also be treated as the stopping times defined on packet reception times. Clearly, in scalar systems, the covariance is at maximum when the channel just recovers from failed transmissions; therefore peak covariances give an upper envelop of covariance matrices at packet reception times. For higher-order systems, the relation between them is still unclear.

Since the second condition in Theorem 1 is difficult to verify, in the following proposition we present another condition for peak-covariance stability, which is, despite being conservative, easy to check. The new condition is obtained by making all $K^{(i)}$'s in Theorem 1 take the value zero.

Proposition 1. *If the following condition is satisfied:*

$$pq|\lambda_A|^2 \sum_{i=1}^{\mathsf{I}_o-1} |\lambda_A|^{2i}(1-p)^{i-1} < 1 - |\lambda_A|^2(1-q), \tag{8}$$

then the Kalman filtering system is peak-covariance stable.

Theorem 1 and Proposition 1 establish a direct connection between λ_A (or $\lambda_{H(K)}$), p, q, the most essential aspects of the system dynamic and channel characteristics on the one hand, and peak-covariance stability on the other hand. These results cover the ones in [15–17], as is evident using the subadditivity property of matrix norm, and the fact that the spectral radius is the infimum of all possible matrix norms. To see this, one should notice that

$$\begin{aligned}
&|\lambda_{H(k)}|\\
\le& qp\Big\|\Big[(A\otimes A)^{-1} - (1-q)I\Big]^{-1}\Big\|\\
&\sum_{i=1}^{\mathsf{I}_o-1}(1-p)^{i-1}\Big\|\overline{(A^i + K^{(i)}C^{(i)})} \otimes (A^i + K^{(i)}C^{(i)})\Big\|\\
\le& qp\sum_{i=1}^{\infty}(1-q)^{i-1}\|A^i \otimes A^i\|\\
&\sum_{i=1}^{\mathsf{I}_o-1}(1-p)^{i-1}\Big\|\overline{(A^i + K^{(i)}C^{(i)})} \otimes (A^i + K^{(i)}C^{(i)})\Big\|\\
=& q\sum_{i=1}^{\infty}(1-q)^{i-1}\|A^i\|^2 p\sum_{i=1}^{\mathsf{I}_o-1}(1-p)^{i-1}\|A^i + K^{(i)}C^{(i)}\|^2,
\end{aligned}$$

$$\Psi \triangleq \begin{bmatrix} X & \sqrt{p}(A'Y + C'F_1) & \cdots & \sqrt{p(1-p)^{I_o-2}}\left((A^{I_o-1})'Y + (C^{(i)})'F_{I_o-1}\right) \\ \sqrt{p}(YA + F_1^*C) & Y & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \sqrt{p(1-p)^{I_o-2}}\left(YA^{I_o-1} + F_{I_o-1}^*C^{(i)}\right) & 0 & \cdots & Y \end{bmatrix} \tag{11}$$

in which the first inequality follows from $|\lambda_{H(K)}| \leq \|H(K)\|$ and the submultiplicative property of matrix norms, and the last equality holds because, for a matrix X, $\|\overline{X}^i \otimes X^i\| = \sqrt{\lambda_{(\overline{X}^i(X')^i)\otimes(X^i(X^*)^i)}} = \lambda_{X^i(X^*)^i} = \|X^i\|^2$. Comparison with the related results in the literature is also demonstrated by Example I in Section 5.

3.2 LMI Interpretation

In Theorem 1, a quite heavy computational overhead may be incurred in searching for a satisfactory K. Although computationally-friendly, Proposition 1 only provides a comparably rough criterion. In this part, we continue to polish the result of Theorem 1 with a way to retaining its power but with less computational burden. Based on what we have established in Theorem 1, we present a criterion which reduces to solving an LMI feasibility problem. To do so, we first introduce a linear operator and then present the equivalence between statements related to the linear operator (including an LMI feasibility statement) and $|\lambda_{H(K)}| < 1$, any of which results in peak-covariance stability.

Consider the operator $\mathcal{L}_K : \mathbb{S}_+^n \to \mathbb{S}_+^n$ defined as

$$\mathcal{L}_K(X) = p\sum_{i=1}^{\mathsf{I}_o-1}(1-p)^{i-1}(A^i+K^{(i)}C^{(i)})^*\Phi_X(A^i+K^{(i)}C^{(i)}), \tag{9}$$

where Φ_X is the positive definite solution of the Lyapunov equation $(1-q)A'\Phi_X A + qA'XA = \Phi_X$ with $|\lambda_A|^2(1-q) < 1$, and $K = [K^{(1)}, \ldots, K^{(\mathsf{I}_o-1)}]$ with each matrix $K^{(i)}$ having compatible dimensions. It can be easily shown that $\mathcal{L}_K(X)$ is linear and non-decreasing on the positive semi-definite cone.

The following result holds.

Theorem 2. *Suppose $|\lambda_A|^2(1-q) < 1$. The following statements are equivalent:*

(i). There exists $K \triangleq [K^{(1)}, \ldots, K^{(\mathsf{I}_o-1)}]$ with each matrix $K^{(i)}$ having compatible dimensions such that $\lim_{k\to\infty}\mathcal{L}_K^k(X) = 0$ for any $X \in \mathbb{S}_+^n$;

(ii). There exists $K \triangleq [K^{(1)}, \ldots, K^{(\mathsf{I}_o-1)}]$ with each matrix $K^{(i)}$ having compatible dimensions such that $|\lambda_{H(K)}| < 1$;

(iii). There exist $K \triangleq [K^{(1)}, \ldots, K^{(\mathsf{I}_o-1)}]$ with each matrix $K^{(i)}$ having compatible dimensions and $P > 0$ such that $\mathcal{L}_K(P) < P$;

(iv). There exist $F_1, \ldots, F_{\mathsf{I}_o-1}$, $X > 0, Y > 0$ such that

$$\begin{bmatrix} Y & \sqrt{1-q}A'Y & \sqrt{q}A'X \\ \sqrt{1-q}YA & Y & 0 \\ \sqrt{q}XA & 0 & X \end{bmatrix} \geq 0 \tag{10}$$

and $\Psi > 0$ where Ψ is given in (11).

If any of the above statements holds, then $\sup_{j\geq 1}\mathbb{E}\|P_{\beta_j}\| < \infty$, i.e., the Kalman filtering system is peak-covariance stable.

To summarize, Theorem 2 makes it possible to check the sufficient condition of Theorem 1 through an LMI feasibility criterion. It can be expected that, given the ability to search for $K^{(i)}$'s on a positive semi-definite cone, Theorem 2 gives a less conservative condition than Proposition 1 does; this is demonstrated by Example I in Section 5.

Remark 2. In [15–17], the criteria for peak-covariance stability are difficult to check since some constants related to the operator g are hard to explicitly compute. A thorough numerical search may be computationally demanding. In contrast, the stability check of Theorem 2 uses an LMI feasibility problem, which can often be efficiently solved.

4 Mean-square Stability

In this section, we will discuss mean-square stability of Kalman filtering with Markovian packet losses.

Definition 3. *The Kalman filtering system with packet losses is mean-square stable if $\sup_{k\in\mathbb{N}}\mathbb{E}\|P_k\| < \infty$.*

4.1 From Peak-covariance Stability to Mean-square Stability

Note that the peak-covariance stability characterizes the filtering system at stopping times defined by (6), while mean-square stability characterizes the property of stability at all sampling times. In the literature, the relationship between the two stability notations is still an open problem. In this section, we aim to establish a connection between peak-covariance stability and mean-square stability. Firstly, we need the following definition for the defective eigenvalues of a matrix.

Definition 4. *For $\lambda \in \sigma(A)$ where A is a matrix, if the algebraic multiplicity and the geometric multiplicity of λ are equal, then λ is called a semi-simple eigenvalue of A. If λ is not semi-simple, λ is called a defective eigenvalue of A.*

We are now able to present the following theorem indicating that as long as A has no defective eigenvalues on the unit circle, i.e., the corresponding Jordan block is 1×1, peak-covariance stability always implies mean-square stability. In fact, we are going to prove this connection for *general* random packet drop processes $\{\gamma_k\}_{k\in\mathbb{N}}$, and is thus not restricted to Gilbert-Elliott channels.

Theorem 3. *Let $\{\gamma_k\}_{k\in\mathbb{N}}$ be a random process over an* *underlying probability space $(\mathscr{S}, \mathcal{S}, \mu)$ with each γ_k taking its value in $\{0,1\}$. Suppose $\{\beta_j\}_{j\in\mathbb{N}}$ take finite values $\mu-$almost surely, and that A has no defective eigenvalues on the unit circle. Then the peak-covariance stability of the Kalman filter always implies mean-square stability, i.e., $\sup_{k\in\mathbb{N}} \mathbb{E}\|P_k\| < \infty$ whenever $\sup_{j\in\mathbb{N}} \mathbb{E}\|P_{\beta_j}\| < \infty$.*

Note that $\{\beta_j\}_{j\in\mathbb{N}}$ can be defined over any random packet loss processes, therefore the peak-covariance stability with packet losses that the filtering system is undergoing remains in accord with Definition 1.

Theorem 3 bridges the two stability notions of Kalman filtering with random packet losses in the literature. Particularly this connection covers most of the existing models for packet losses, e.g., i.i.d. model [9], bounded Markovian [24], Gilbert-Elliott [15], and finite-state channel [25]. Although $\sup_{k\in\mathbb{N}} \mathbb{E}\|P_k\|$ and $\sup_{j\in\mathbb{N}} \mathbb{E}\|P_{\beta_j}\|$ are not equal in general, this connection is built upon a critical understanding that, no matter to which inter-arrival interval between two successive β_j's the time k belongs, $\|P_k\|$ *is uniformly bounded from above by an affine function of the norm of the peak covariances at the starting and ending points thereof.* This point holds regardless of the model of packet loss process.

We also remark that there is some difficulty in relaxing the assumption that A has no defective eigenvalues on the unit circle in Theorem 3. This is due to the fact that A's defective eigenvalues on the unit circle will influence both the peak-covariance stability and mean-square stability in a nontrivial manner.

Remark 3. In [16], for a scalar model with i.i.d. packet losses, it has been shown that the peak-covariance stability is equivalent to mean-square stability, while for a vector system even with i.i.d. packet losses, the relationship between the two is unclear. In [17], the equivalence between the two stability notions was established for systems that are one-step observable, again for the i.i.d. case. Theorem 3 now fills the gap for a large class of vector systems under general random packet drops.

4.2 The Critical $p-q$ Curve

In this subsection, we first show that for a fixed q in the Gilbert-Elliott channel, there exists a critical failure rate p_c, such that if and only if the failure rate is below p_c, the Kalman filtering is mean-square stable. This conclusion is relatively independent of previous results, and the proof relies on a coupling argument.

Proposition 2. *Let the recovery rate q satisfy $|\lambda_A|^2(1-q) < 1$. Then there exists a critical value $p_c \in (0,1]$ for the failure rate in the sense that*

(i) $\sup_{k\in\mathbb{N}} \mathbb{E}\|P_k\| < \infty$ *for all* $\Sigma_0 \geq 0$ *and* $0 < p < p_c$ *;*

(ii) there exists $\Sigma_0 \geq 0$ *such that* $\sup_{k\in\mathbb{N}} \mathbb{E}\|P_k\| = \infty$ *for all* $p_c < p < 1$.

Proof. If $p = 0$, we have the standard Kalman filter, which evidently converges to a bounded estimation error covariance, which suggests that there exists a transition point for p beyond which the expected prediction error covariance matrices are not uniformly bounded[3]. It remains to show that with a given q this transition point is unique. Fix a $0 \leq p_1 < 1$ such that $\sup_{k\in\mathbb{N}} \mathbb{E}_{p_1}\|P_k\| < \infty\ \forall \Sigma_0 \geq 0$. It suffices to show that, for any $p_2 < p_1$, $\sup_{k\in\mathbb{N}} \mathbb{E}_{p_2}\|P_k\| < \infty$ for all $\Sigma_0 \geq 0$. To differentiate two Markov chains with different failure rate in (3), we use the notation $\{\gamma_k(p_i)\}_{k\in\mathbb{N}}$ instead to represent the packet loss process with $p = p_i$ in (3). We define a sequence of random vectors $\{(z_k, \tilde{z}_k)\}_{k\in\mathbb{N}}$ over a probability space $(\mathscr{G}, \mathcal{G}, \pi)$ with $\mathscr{G} = \{(0,0),(0,1),(1,1)\}^{\mathbb{N}}$. Put $\varphi_1(\{z_k, \tilde{z}_k\}_{k=1:t}) = \psi_{z_t} \circ \cdots \circ \psi_{z_1}(\Sigma_0)$ and $\varphi_2(\{z_k, \tilde{z}_k\}_{k=1:t}) = \psi_{\tilde{z}_t} \circ \cdots \circ \psi_{\tilde{z}_1}(\Sigma_0)$, where $\psi_z = zg + (1-z)h$ with h, g defined in (4), (5) and $z = \{0,1\}$. Since $z_k \leq \tilde{z}_k$ in $\mathscr{G}$, we have $\varphi_1(\{z_k, \tilde{z}_k\}_{k=1:t}) \geq \varphi_2(\{z_k, \tilde{z}_k\}_{k=1:t})$.

When $p_2 + q \leq 1$, we let the evolution of $\{(z_k, \tilde{z}_k)\}_{k\in\mathbb{N}}$ follow the Markov chain in Fig. 1, whereby it can seen that $\pi(z_{k+1} = j | z_k = i)$'s for $i, j = \{0,1\}$ are constants independent of $\tilde{z}_k$'s, and conversely that $\pi(\tilde{z}_k = j | \tilde{z}_k = i)$'s for $i, j = \{0,1\}$ are constants independent of z_k's. Moreover,

$$\pi(z_{k+1} = j | z_k = i) = \mathbb{P}_{p_1}(\gamma_{k+1}(p_1) = j | \gamma_k(p_1) = i),$$

$$\pi(\tilde{z}_{k+1} = j | \tilde{z}_k = i) = \mathbb{P}_{p_2}(\gamma_{k+1}(p_2) = j | \gamma_k(p_2) = i)$$

for all $i, j = \{0,1\}$ and $k \in \mathbb{N}$. We assume the Markov chain starts at the stationary distribution. Then,

$$\begin{aligned}
\mathbb{E}^{\infty}_{p_1}\|P_k\| &= \int_{\Omega} \left\|\psi_{\gamma_k(p_1)} \circ \cdots \circ \psi_{\gamma_1(p_1)}(\Sigma_0)\right\| \mathrm{d}\mathbb{P}_{\mathrm{p}_1} \\
&= \int_{\mathscr{G}} \left\|\varphi_1(\{z_j, \tilde{z}_j\}_{j=1:k})\right\| \mathrm{d}\pi \\
&\geq \int_{\mathscr{G}} \left\|\varphi_2(\{z_j, \tilde{z}_j\}_{j=1:k})\right\| \mathrm{d}\pi \\
&= \int_{\Omega} \left\|\psi_{\gamma_k(p_2)} \circ \cdots \circ \psi_{\gamma_1(p_2)}(\Sigma_0)\right\| \mathrm{d}\mathbb{P}_{\mathrm{p}_2} \\
&= \mathbb{E}^{\infty}_{p_2}\|P_k\|,
\end{aligned}$$

where $\mathbb{E}^{\infty}$ means that the expectations is taken conditioned on the stationarily distributed γ_1.

When $p_2 + q > 1$, we allow the existence of negative measures in the Markov chain described in Fig. 1. By direct computation, the eigenvalues of transition probability matrix, denoted by $\mathbf{M} \in \mathbb{R}^{3\times 3}$, are $1-q-p_1$, $1-q-p_2$ and 1, respectively. As a result, $\mathbf{M}^k$ converges to a limit as k tends to infinity, indicating that the generalized Markov chain has a unique stationary distribution. Note that $\pi(z_1 = i_1, \ldots, z_t = i_t) = \mathbb{P}_{p_1}(\gamma_1 = i_1, \ldots, \gamma_t = i_t)$ and $\pi(\tilde{z}_1 = i_1, \ldots, \tilde{z}_t = i_t) = \mathbb{P}_{p_2}(\gamma_1 = i_1, \ldots, \gamma_t = i_t)$ for all $t \in \mathbb{N}$ and $i_1, \ldots, i_t \in \{0,1\}$. Thus, the inequality $\mathbb{E}^{\infty}_{p_1}\|P_k\| \geq \mathbb{E}^{\infty}_{p_2}\|P_k\|$ still proves true in this case.

Finally, by Lemma 2 in [18], we have $\sup_{k\in\mathbb{N}} \mathbb{E}_{p_2}\|P_k\| < \infty$, which completes the proof. □

It has been shown in [18] that a necessary condition for mean-square stability of the filtering system is $|\lambda_A|^2(1-q) < 1$, which is only related to the recovery rate q. For Gilbert-Elliot channels, a critical value phenomenon with respect to q is also expectable. Theorem 4 proves the existence

[3] Without loss of generality, if the Kalman filter is mean-square stable for any $p \in (0,1)$, the transition point for p is 1.

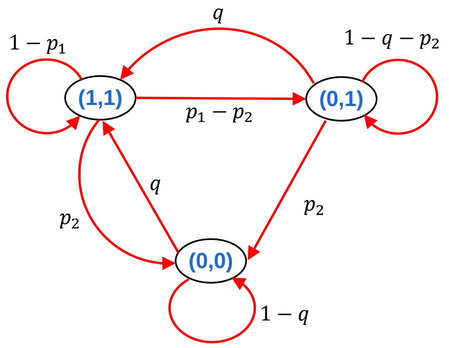

Fig. 1: The transitions of the Markov chain $\{(z_k, \tilde{z}_k)\}_{k\in\mathbb{N}}$ when $p_2 + q \leq 1$.

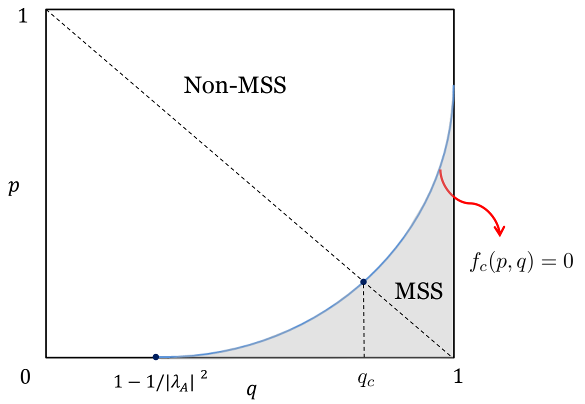

Fig. 2: The $p - q$ plane is divided into MSS (Mean-square Stability) and Non-MSS regions by the critical curve $f_c(p,q) = 0$. When $p + q = 1$, the Markovian packet loss process is reduced to an i.i.d. process. As a result, the intersection point of the curves $f_c(p,q) = 0$ and $p + q = 1$ gives the critical packet drop probability established in [9].

of the critical $p - q$ curve and Fig. 2 illustrates this critical curve in the $p - q$ plane.

Theorem 4. *There exists a critical curve defined by $f_c(p,q) = 0$, dividing $(0,1)^2$ into two disjoint regions such that:*

(i) If $(p,q) \in \{f_c(p,q) > 0\}$, then $\sup_{k\in\mathbb{N}} \mathbb{E}\|P_k\| < \infty$ for all $\Sigma_0 \geq 0$;

(ii) If $(p,q) \in \{f_c(p,q) < 0\}$, then there exists $\Sigma_0 \geq 0$ under which $\sup_{k\in\mathbb{N}} \mathbb{E}\|P_k\| = \infty$.

Remark 4. If the packet loss process is an i.i.d. process, where $p + q = 1$ in the transition probability matrix defined in (3), Proposition 2 and Theorem 4 recover the result of Theorem 2 in [9]. It is worth pointing out that whether mean-square stability holds or not exactly on the curve $f_c(p,q) = 0$ is beyond the reach of the current analysis (even for the i.i.d. case with $p + q = 1$): such an understanding relies on the compactness of the stability or non-stability regions.

4.3 Mean-square Stability Conditions

We can now make use of the peak-covariance stability conditions we obtained in the last section, and the connection between peak-covariance stability and mean-square stability indicated in Theorem 3, to establish mean-square stability conditions for the considered Kalman filter. It turns out that the assumption requiring no defective eigenvalues on the unit circle, can be relaxed by an approximation method. We present the following result.

Theorem 5. *Let the recovery rate q satisfy $|\lambda_A|^2(1-q) < 1$. Then there holds $p_c \geq \underline{p}$, where*

$$\underline{p} \triangleq \sup\left\{p : \exists (K,P) \text{ s.t. } \mathcal{L}_K(P) < P, P > 0\right\}, \qquad (12)$$

i.e., for all $\Sigma_0 \geq 0$ and $0 < p < \underline{p}$, the Kalman filtering system is mean-square stable.

Remark 5. For second-order systems and certain classes of high-order systems, such as non-degenerate systems, necessary and sufficient conditions for mean-square stability have been derived in [18] and [26]. However, these results rely on a particular system structure and fail to apply to general LTI systems. It seems challenging to find an explicit description of necessary and sufficient conditions for mean-square stability of general LTI systems. Theorem 5 gives a stability criterion for general LTI systems.

5 Numerical Examples

In this section, we present two examples to demonstrate the theoretical results we established in Sections 3 and 4.

5.1 Example I: A Second-order System

To compare with the works in [15, 16], we will examine the same vector example considered therein. The parameters are specified as follows:

$$A = \begin{bmatrix} 1.3 & 0.3 \\ 0 & 1.2 \end{bmatrix}, \quad C = [1, 1],$$

$Q = I_{2\times 2}$ and $R = 1$.

First let us compare the sufficient condition we provide in Proposition 1 with the counterpart provided in [16]. Note that $|\lambda_A|^2(1-q) < 1$ is a necessary condition for mean-square stability. We take $q = 0.65$ as was done in [16]. As for the failure rate p, [16] concludes that $p < 0.04$ guarantees peak-covariance stability; while Proposition 1 requires $p < \frac{1-|\lambda_A|^2(1-q)}{|\lambda_A|^4 q}$, which generates the less conservative condition $p < 0.22$. Note that, for the channel with $(p,q) = (0.04, 0.65)$, $\mathbb{P}(\gamma_k = 0) = 0.0580$ when the packet loss process enters the stationary distribution, which means that the allowed long term packet loss rate is at most 5.80%. However, by choosing a larger p, Proposition 1 permits $\mathbb{P}(\gamma_k = 0) = 0.2529$ at the stationary distribution at most, i.e., the allowed long term packet dropout rate is 25.29%. Separately, we note that it is rather convenient to check the condition in Proposition 1 even with manual calculation; in contrast, it involves a considerable amount of numerical calculation to check the conditions in [16].

Then, we use the criterion established in Theorem 2 to check for the peak-covariance stability. We obtain that when $p = 1$ the LMI in 2) of Theorem 2 is still feasible.[4] It

[4] To satisfy the assumption (A2), we need to configure $p = 1 - \epsilon$ for an

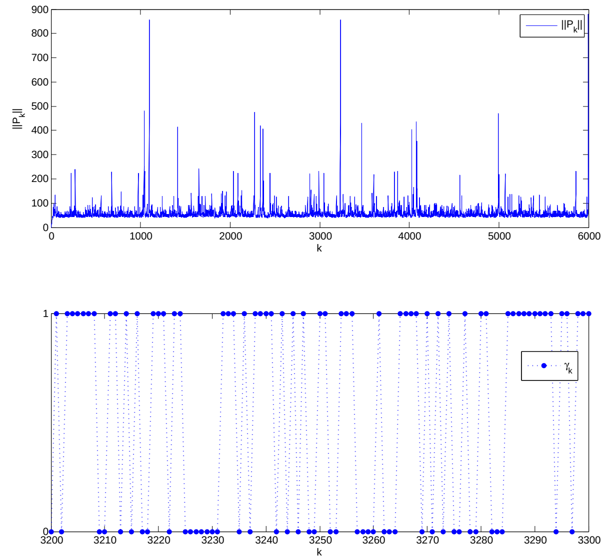

Fig. 3: A sample path of $\|P_k\|$ and γ_k with $p = 0.5$, $q = 0.65$ in Example I.

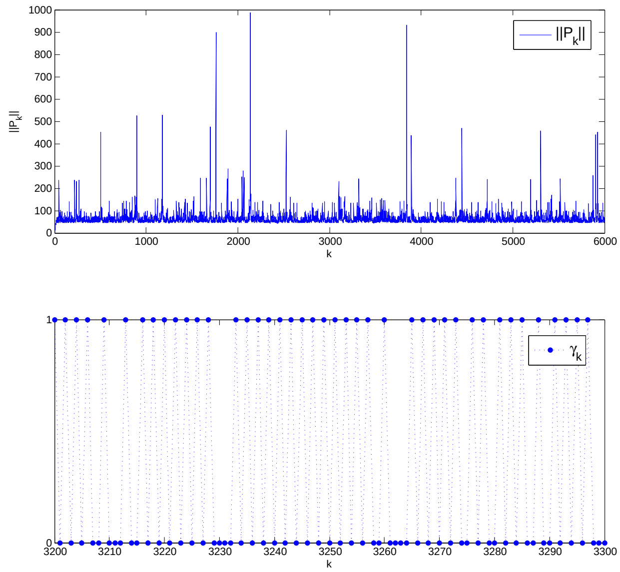

Fig. 4: A sample path of $\|P_k\|$ and γ_k with $p = 0.99$, $q = 0.65$ in Example I.

should be pointed that at least for the parameters specified in this example, the criterion of [16] only covers the Gilbert-Elliott models with failure rate lower than 4.5%. Fig. 3 and Fig. 4 illustrate sample paths of $\|P_k\|$ and γ_k with $(p, q) = (0.5, 0.65)$ and $(p, q) = (0.99, 0.65)$, respectively. The figures show that even a high value of p may not effect the peak-covariance stability in this example, showing that Theorem 2 provides a less conservative criterion than Proposition 1 or [16] does, a fact which is consistent with the theoretical analysis in Section 3.

5.2 Example II: A Third-order System

To compare the work in Section 4 with the result [18] and [26], we will use the following example, where the pa-

arbitrary small positive ϵ.

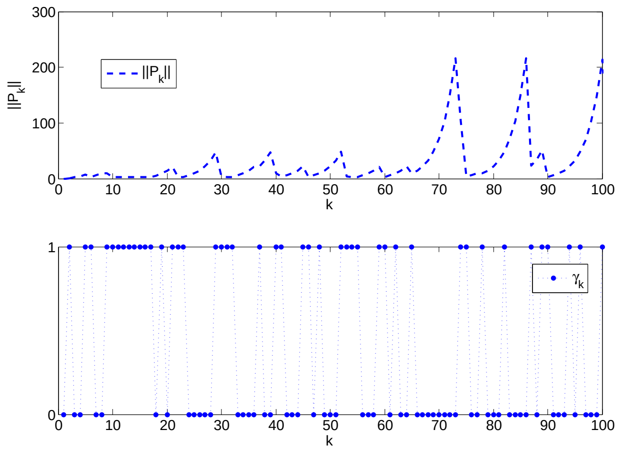

Fig. 5: A sample path of $\|P_k\|$ and γ_k with $p = 0.45$, $q = 0.5$ in Example II.

rameters are given by

$$A = \begin{bmatrix} 1.2 & 0 & 0 \\ 0 & 1.2 & 0 \\ 0 & 0 & -1.2 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}, \tag{13}$$

$Q = I_{3\times 3}$ and $R = I_{2\times 2}$. In [18,26,27], mean-square stability of Kalman filtering for so-called non-degenerate systems has been studied. Before proceeding, we introduce the definition.

Definition 5. *Consider a system (C, A) in diagonal standard form, i.e., $A = \operatorname{diag}(\lambda_1, \ldots, \lambda_n)$ and $C = [C_1, \ldots, C_n]$. A quasi-equiblock of the system is defined as a subsystem $(C_\mathcal{I}, A_\mathcal{I})$, where $\mathcal{I} \triangleq \{l_1, \ldots, l_i\} \subset \{1\ldots, n\}$, such that $A_\mathcal{I} = \operatorname{diag}(\lambda_{l_1}, \ldots, \lambda_{l_i})$ with $|\lambda_{l_1}| = \cdots = |\lambda_{l_i}|$ and $C_\mathcal{I} = [C_{l_1}, \ldots, C_{l_i}]$.*

Definition 6. *A diagonalizable system (C, A) is non-degenerate if every quasi-equiblock of the system is one-step observable. Conversely, it is degenerate if it has at least one quasi-equiblock that is not one-step observable.*

By definition, the system in (13) is observable but degenerate since $|\lambda_1| = |\lambda_2| = |\lambda_3|$ but (C, A) is not one-step observable. To the best of our knowledge, no tool has been established so far to study mean-square stability of such a system with Markovian packet losses. The results presented in Section 4 provide us a universal criterion for mean-square stability. Let us fix $q = 0.5$. We can conclude from Theorem 5 that if $p \le 0.465$ the Kalman filter is mean-square stable. Fig. 5 illustrates a sample path of $\|P_k\|$ and γ_k with $(p, q) = (0.45, 0.5)$. Fig. 6 illustrates that with $(p, q) = (0.99, 0.5)$ the expected prediction error covariance matrices diverge. One can verify that when $q = 0.5$ and $p = 1$ the criterion in Theorem 5 is violated as the LMI in Theorem 2 is infeasible.

6 Conclusions

We have investigated the stability of Kalman filtering over Gilbert-Elliott channels. Random packet drop follows a time-homogeneous two-state Markov chain where

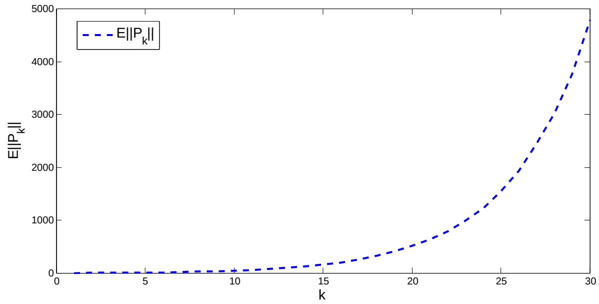

Fig. 6: Divergence of $\mathbb{E}\|P_k\|$ with $p = 0.99$, $q = 0.5$ in Example II.

the two states indicate successful or failed packet transmissions. We established a relaxed condition guaranteeing peak-covariance stability described by an inequality in terms of the spectral radius of the system matrix and transition probabilities of the Markov chain, and then showed that the condition can be reduced to an LMI feasibility problem. It was proved that peak-covariance stability implies mean-square stability if the system matrix has no defective eigenvalues on the unit circle. This connection holds for general random packet drop processes. We also proved that there exists a critical region in the $p - q$ plane such that if and only if the pair of recovery and failure rates falls into that region the expected prediction error covariance matrices are uniformly bounded. By fixing the recovery rate, a lower bound for the critical failure rate was obtained making use of the relationship between two stability criteria for general LTI systems. Numerical examples demonstrated significant improvement on the effectiveness of our appraoch compared with the existing literature.

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