Stabilization of Nonlinearly Parameterized Discrete-Time Systems by NLS Algorithm

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Abstract: This paper addresses the challenging problem of designing an adaptive feedback strategy for the stabilization of a class of nonlinearly parameterized uncertain systems in discrete time. The nonlinear least squares (NLS) algorithm is applied to estimate the unknown parameters, and a sufficient condition is proposed to ensure the strong consistency of the estimator. Based on the NLS algorithm, the global stability of the system is achieved under the output feedback design.

Key Words: Stabilization, NLS algorithm, Discrete-time systems, Nonlinearly parameterized systems

1 Introduction

An interesting phenomenon occurs when one attempts to control systems with output nonlinearity growing faster than linearity, where similarities of adaptive control between continuous- and discrete-time cases will no longer exist. It is generally known that a large class of continuous-time nonlinear parametric systems, regardless of how fast the growth rate is, can be globally stabilized by the nonlinear damping or back-stepping approach in adaptive control (e.g., [5, 6]). However, fundamental difficulties arise for the discrete-time case. These difficulties are caused by the inherent limitations of the feedback principle in dealing with uncertainties (see [3, 8–13, 15, 22]), which means that the discrete-time systems with the uncertainties beyond the feedback capability cannot be stabilized by any feedback control law, no matter how hard one may try.

Accordingly, a natural question is: within the capability of the feedback mechanism, how can one design a stabilizing feedback control in the presence of nonlinear parameterization? As traditional least squares and gradient-estimation-based controllers will encounter essential difficulties in implementation, nonlinear parametrization in discrete time arouses a formidable obstacle in studying adaptive stabilization of such uncertain systems. Indeed, there are only a few results dedicated to this topic [7, 10, 12, 17]. Specifically, an approach based on the Implicit Function Theorem is suggested in [12] to stabilize a class of nonlinearly parameterized uncertain systems, which are contaminated with bounded disturbances. Notwithstanding the parallel result developed in the deterministic case, the adaptive stabilization under the stochastic framework is rather an intractable issue. For instance, the problem of stabilizing a nonlinear discrete-time uncertain system, whose parameters are linearly parameterized, is relatively easy under the deterministic framework. However, the adaptive stabilization of its counterpart with random parameters and Gaussian noises had been an open problem for more than a decade until it was solved in [14]. When the linear parametrization extends to the nonlinear parametrization, as one would expect, the problem will become much more complicated, especially for the case where multiple parameters are involved.

Seeking an answer to the proposed question, we apply the nonlinear least squares (NLS) algorithm in this paper to estimate the unknown parameters of the nonlinearly parameterized stochastic systems. The estimator can thus be applied to a rich class of systems, where the nonlinear parametric system functions could be in various forms, including nonconvex functions and functions with growth rates faster than linearity. The parameters to be identified are either deterministic or stochastic, and their uncertain domains need not even be bounded. It is worth pointing out that the NLS algorithm degenerates to be the standard LS algorithm for the specific case where the model is linearly parameterized. With the help of this estimator, one can set about tackling the proposed problem.

As a starting point, we will investigate a class of nonlinearly parameterized uncertain systems with a scalar-valued parameter and Gaussian noise. For the multiple-parameter case, the analysis will in general be extremely difficult. Now, as indicated in [3] and [12], for any nonlinearly parameterized system with its sensitivity function having a polynomial growth rate, the feedback principle will be out of its capability whenever the exponent of the growth rate is equal to or faster than 4. Moreover, the global stabilization of the nonlinearly parameterized system with the sensitivity function ([12], [13]) growing slower than linearity has already been resolved in [10], where an adaptive switching controller is employed. All these works suggest that the rest of the work should be directed towards the case where the exponent, which characterizes the growth rate of the sensitivity function, is between 1 and 4. For such systems, we will prove that there indeed exists a feedback control law, based on the NLS estimator, such that the related closed-loop uncertain system is globally stable under an appropriate constraint on the sensitivity function.

2 Stabilization of Nonlinear Feedback System

To tackle the problem proposed in Introduction, the nonlinearly parameterized uncertain system with a scalar-valued parameter is studied in this paper as a starting point. Now,

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consider

$$y_{t+1} = f(\theta, \varphi_t) + h(u_t, \varphi_t) + w_{t+1}, \tag{1}$$

where $\theta \in \mathbb{R}$ is an unknown parameter, $y_t, u_t, w_t \in \mathbb{R}$ are the system output, input and noise signals, respectively. $\varphi_t = (y_t, y_{t-1}, \ldots, y_{t-m+1}) \in \mathbb{R}^m$ is the output regressor with the initial vector $\varphi_0 = (y_0, y_{-1}, \ldots, y_{-m+1})$ being independent of noise $\{w_t\}$. Furthermore, $f : \mathbb{R} \times \mathbb{R}^m \to \mathbb{R}$ and $h : \mathbb{R} \times \mathbb{R}^m \to \mathbb{R}$ are two known smooth mappings with

$$\frac{\partial h(u, x)}{\partial u} \neq 0, \quad u \in \mathbb{R}, x \in \mathbb{R}^m.$$

The global stabilization is guaranteed under the following assumptions:

A1 The noise $\{w_t\}$ is an i.i.d sequence with a standard normal distribution $N(0, 1)$.

A2 Parameter $\theta \in \mathbb{R}^n$ is independent of noise $\{w_t\}$, either deterministic or stochastic.

A3 There are some constants $C_1, C_2, C_3, C_4 > 0$ such that for any $x \in \mathbb{R}^m$ and $\vartheta \in \mathbb{R}$,

$$C_1\|x\| \leq \Big|\frac{df(\vartheta, x)}{d\vartheta}\Big| \leq C_2\|x\|^b, \tag{2}$$

$$\sup_{\vartheta\in\mathbb{R}}\Big|\frac{df(\vartheta, x)}{d\vartheta}\Big| \leq C_3 \inf_{\vartheta\in\mathbb{R}}\Big|\frac{df(\vartheta, x)}{d\vartheta}\Big|, \tag{3}$$

$$\Big|\frac{d^2 f(\vartheta, x)}{d\vartheta^2}\Big| \leq C_4\Big|\frac{df(\vartheta, x)}{d\vartheta}\Big|, \tag{4}$$

where $b \in [1, 4)$ is a real number.

To achieve our goal, let

$$\bar{u}_t \triangleq h(u_t, \varphi_t). \tag{5}$$

By the Implicit Function Theorem, there is a differentiable function $d : \mathbb{R} \times \mathbb{R}^m \to \mathbb{R}$ such that (5) holds whenever

$$u_t = d(\bar{u}_t, \varphi_t).$$

Now, design

$$u_t = d(-f(\hat{\theta}_t, \varphi_t), \varphi_t), \tag{6}$$

where $\hat{\theta}_t$ is the NLS estimator to be studied in Section 3. By (5), the outputs of system (1) satisfy the following:

$$y_{t+1} = \tilde{f}_t(\theta) + w_{t+1}, \quad t \geq 0, \tag{7}$$

where $\tilde{f}_t(\theta) \triangleq f(\theta, \varphi_t) - f(\hat{\theta}_t, \varphi_t)$.

The main result is stated as below.

Theorem 2.1 *Under Assumptions A1–A3, the closed-loop system (1) and (6) is globally stable in the sense that*

$$\frac{1}{t}\left(\sum_{i=1}^{t} y_i^2\right) = O(1), \quad a.s.. \tag{8}$$

Remark 2.1 *In fact, Theorem 2.1 is also valid if the noise in Assumption A1 is replaced by a bounded i.i.d sequence.*

Remark 2.2 *If taking into account of the case where*

$$\Big|\frac{df(\vartheta, x)}{d\vartheta}\Big| = O(\|x\|),$$

we refer to [10], which demonstrates that the system is globally adaptively stabilizable, provided that the sensitivity function of the unknown parameter has a linear growth rate.

Remark 2.3 *When $b \geq 4$, the stability can no longer be generally guaranteed for system (1) by [19]. If the nonlinear system in [19] is considered under Assumption A1, then for any feedback control law, there always exists some set with a positive probability, on which the outputs diverge exponentially.*

3 Asymptotic Properties of NLS Algorithm

A first step towards exploring the stabilization of nonlinearly parameterized systems raised in previous section is to analyze the asymptotic properties of the NLS algorithm (see [16], [18]), which is applied to the parameter-estimation. For this purpose, consider the general stochastic regression discrete-time model

$$z_{t+1} = f(\theta, x_t) + w_{t+1}, \tag{9}$$

where $\theta \in \mathbb{R}^n$ is an unknown parameter vector and $w_t \in \mathbb{R}$ are unobserved disturbances. Outputs $z_t \in \mathbb{R}$ are the observed responses to the design levels $x_t \in \mathbb{R}^m$ and $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ is a known smooth mapping. The objective in this section is to establish the strong consistency of the NLS estimator for the unknown parameter θ of system (9).

3.1 Nonlinear Least Square Estimation

The NLS estimate $\hat{\theta}_t$ that minimizes

$$S_t(\theta) \triangleq \sum_{i=1}^{t}(z_i - f(\theta, x_{i-1}))^2$$

is typically computed by solving the following equation:

$$\nabla S_t(\theta) = -2\sum_{i=1}^{t}(z_i - f(\theta, x_{i-1}))\nabla f(\theta, x_{i-1}) = 0. \tag{10}$$

Such an estimate can be expressed by the Implicit Function Theorem, and it turns out to be the standard LS algorithm for the specific case where the model is linearly parameterized. The expression of the estimator is elaborated as follows.

First, for any $t \geq 0$, define $X_t = (x_0, \ldots, x_t)^T$, $Z_t = (z_1, \ldots, z_t)^T$, $W_t = (w_1, \ldots, w_t)^T$ and $F_t(\theta) = (f(\theta, x_0), f(\theta, x_1), \ldots, f(\theta, x_t))^T$. Up to time $t \geq 1$, one has the equation

$$F_{t-1}(\theta) + (W_t - Z_t) = 0. \tag{11}$$

For each $t \geq 1$, define function $G_t : \mathbb{R}^n \times \mathbb{R}^t \to \mathbb{R}^n$ as

$$\begin{aligned} G_t(\theta, W_t) \;\; &\triangleq \;\; \left(\frac{\partial F_{t-1}(\theta)}{\partial \theta}\right)^T F_{t-1}(\theta) \\ &\quad + \;\; \left(\frac{\partial F_{t-1}(\theta)}{\partial \theta}\right)^T (W_t - Z_t), \end{aligned} \tag{12}$$

which is a smooth function with respect to variables θ and W_t. Here, X_{t-1} and Z_t are viewed as constants. By multiplying $\left(\frac{\partial F_{t-1}(\theta)}{\partial \theta}\right)^T$ on both sides of (11), it yields

$$G_t(\theta, W_t) = 0. \tag{13}$$

Without loss of generality, assume

$$\left(\frac{\partial F_{n-1}(\theta)}{\partial\theta}\right)^T \frac{\partial F_{n-1}(\theta)}{\partial\theta} > 0, \tag{14}$$

which, together with (11), yields that for any $t \geq n$,

$$\begin{aligned}\frac{\partial G_t(\theta, W_t)}{\partial\theta} &= \left(\frac{\partial F_{t-1}(\theta)}{\partial\theta}\right)^T \frac{\partial F_{t-1}(\theta)}{\partial\theta} \\ &\geq \left(\frac{\partial F_{n-1}(\theta)}{\partial\theta}\right)^T \frac{\partial F_{n-1}(\theta)}{\partial\theta} > 0.\end{aligned} \tag{15}$$

Thus, by the Implicit Function Theorem, there is a smooth function $g_t : \mathbb{R}^t \to \mathbb{R}^n$ such that

$$G_t(g_t(W_t), W_t) = 0 \tag{16}$$

and

$$\frac{\partial g_t(W_t)}{\partial W_t} = -\left(\frac{\partial G_t(\theta, W_t)}{\partial\theta}\right)^{-1}\left(\frac{\partial F_{t-1}(\theta)}{\partial\theta}\right)^T. \tag{17}$$

Now, define the estimate $\hat{\theta}_t$ at time $t \geq n$ for parameter θ as

$$\hat{\theta}_t = g_t(0). \tag{18}$$

Clearly, Equation (13) equals to (10) whenever $W_t = 0$, and hence $\hat{\theta}_t$ defined by (18) is exactly the NLS estimate at time t. Let $\tilde{\theta}_t$ denote the error of estimate $\hat{\theta}_t$, that is, $\tilde{\theta}_t \triangleq \theta - \hat{\theta}_t$. From (15) and (17), for any $t \geq n$, one has

$$\begin{aligned}\tilde{\theta}_t &= g_t(W_t) - g_t(0) = \frac{\partial g_t(W_t)}{\partial W_t}\Big|_{W_t = W_t^*} W_t \\ &= -\left(\left(\frac{\partial F_{t-1}(\theta)}{\partial\theta}\right)^T \frac{\partial F_{t-1}(\theta)}{\partial\theta}\right)^{-1} \\ &\quad \times \left(\frac{\partial F_{t-1}(\theta)}{\partial\theta}\right)^T\Big|_{\theta=\theta_t^*} W_t,\end{aligned} \tag{19}$$

where $\theta_t^* \triangleq g_t(W_t^*)$ and

$$W_t^* \in \prod_{i=1}^{t}[-|w_i|, |w_i|]. \tag{20}$$

3.2 Analysis of the NLS Algorithm

This subsection deals with the consistency analysis of the NLS estimator. The estimation error of θ is going to be discussed when the model is corrupted with Gaussian noise under Assumption A1.

To facilitate the analysis, we introduce some notations. For any $\vartheta \in \mathbb{R}^n$, let $P_t^{-1}(\vartheta) \triangleq \sum_{i=0}^{t-1}\phi_i(\vartheta)\phi_i^T(\vartheta)$, where $\phi_i(\vartheta) \triangleq \left(\frac{\partial f(\vartheta, x_i)}{\partial\vartheta}\right)^T$. Therefore, the trace of $P_t^{-1}(\vartheta)$, which is denoted as $r_{t-1}(\vartheta)$, satisfies $r_{t-1}(\vartheta) = \sum_{i=0}^{t-1}\|\phi_i(\vartheta)\|^2$, where $\|\cdot\|$ denotes the Euclidean norm. Moreover, let $(\Omega, \mathcal{F}, P)$ denote the underlying probability space. Define

$$\mathcal{F}_t \triangleq \sigma\{\theta, x_0, w_i, 1 \leq i \leq t\}, \tag{21}$$

which is a series of non-decreasing σ-fields. Suppose

$$x_t \in \mathcal{F}_t, \tag{22}$$

where the design vectors $x_t, t \geq 0$ in model (9) assume values either deterministic or stochastic. It is easy to verify that $\{w_t, \mathcal{F}_t\}$ is a martingale difference sequence, and

$$\sup_{t\geq 1} E(w_t^2|\mathcal{F}_{t-1}) < \infty, \quad \text{a.s.}. \tag{23}$$

Due to the limited space, the following proposition is provided without proof.

Proposition 3.1 *Under Assumptions A1 and A2, let*

$$\inf_{\vartheta\in\mathbb{R}^n}\lambda_{\min}(P_n^{-1}(\vartheta)) \geq 1$$

and (22) hold with x_0 being independent of $\{w_t\}$. If

$$\liminf_{t\to\infty}\frac{\inf_{\vartheta\in\mathbb{R}^n}\lambda_{\min}(P_t^{-1}(\vartheta))}{\max\{r_{t-1}, t\}} > 0, \quad \textit{a.s.}, \tag{24}$$

$$\left\|\frac{\phi_t(\vartheta)}{\partial\vartheta}\right\| \leq M\|\phi_t(\vartheta)\|, \quad \vartheta \in \mathbb{R}^n, t \geq 0, \tag{25}$$

where $r_t \triangleq \sup_{\vartheta\in\mathbb{R}^n} r_t(\vartheta)$ and $M > 0$ is some constant, then

$$\|\tilde{\theta}_t\| = O\left(\frac{r_{t-1}^{\frac{1}{2}}\log r_{t-1}\log^3 t}{\inf_{\vartheta\in\mathbb{R}^n}\lambda_{\min}(P_t^{-1}(\vartheta))}\right) \to 0, \quad \textit{a.s.}.$$

4 Proof of Theorem 2.1

The proof of this theorem is divided into a series of lemmas. Without loss of generality, assume $\|\varphi_0\| \geq \frac{1}{C_1}$. Now, for any $\vartheta \in \mathbb{R}$, define a random sequence $\bar{r}_k(\vartheta)$ as

$$\begin{cases}\bar{r}_0(\vartheta) = \phi_0^2(\vartheta) \\ \bar{r}_k(\vartheta) = \bar{r}_{k-1}(\vartheta) + \phi_{t_k}^2(\vartheta), \quad k = 1, 2, \ldots,\end{cases}$$

where the monotone random subscript t_k with $t_1 = 1$ satisfies for $k \geq 1$

$$\begin{cases}\dfrac{\phi_{t_{k+1}}^2(\vartheta)}{\bar{r}_k(\vartheta)} > \dfrac{\phi_{t_k}^2(\vartheta)}{\bar{r}_{k-1}(\vartheta)} \\ \dfrac{\phi_t^2(\vartheta)}{\bar{r}_k(\vartheta)} \leq \dfrac{\phi_{t_k}^2(\vartheta)}{\bar{r}_{k-1}(\vartheta)} \qquad \text{for any } t_k < t < t_{k+1}.\end{cases} \tag{26}$$

Since

$$\frac{\bar{r}_{t_{k+1}}(\vartheta)}{\bar{r}_k(\vartheta)} = 1 + \frac{\phi_{t_{k+1}}^2(\vartheta)}{\bar{r}_k(\vartheta)},$$

it is easy to see from (26) that

$$\frac{\bar{r}_{t_{k+1}}(\vartheta)}{\bar{r}_k(\vartheta)} \geq \frac{\bar{r}_k(\vartheta)}{\bar{r}_{k-1}(\vartheta)} \geq 1, \quad k = 1, 2, \ldots \tag{27}$$

and for any $k \geq 1$,

$$\inf_{\vartheta\in\mathbb{R}^n}\bar{r}_k(\vartheta) \geq \inf_{\vartheta\in\mathbb{R}^n}\bar{r}_0(\vartheta) \geq C_1^2\|\varphi_0\|^2 \geq 1. \tag{28}$$

Lemma 4.1 *Given $k \geq 1$, then for any $t \in [1, t_{k+1})$,*

$$\frac{r_t(\vartheta)}{r_{t-1}(\vartheta)} \leq \frac{\bar{r}_k(\vartheta)}{\bar{r}_{k-1}(\vartheta)}. \tag{29}$$

Proof. If $t \in (t_i, t_{i+1})$, where $1 \le i \le k$, then $t-1 \ge t_i$. According to (26), one has

$$\frac{r_t(\vartheta)}{r_{t-1}(\vartheta)} \le 1 + \frac{\phi_t^2(\vartheta)}{\bar{r}_i(\vartheta)} \le 1 + \frac{\phi_{t_i}^2(\vartheta)}{\bar{r}_{i-1}(\vartheta)}.$$

Furthermore, (27) leads to

$$\frac{\bar{r}_i(\vartheta)}{\bar{r}_{i-1}(\vartheta)} \le \frac{\bar{r}_k(\vartheta)}{\bar{r}_{k-1}(\vartheta)}, \tag{30}$$

which immediately implies (29).

As for $t = t_i, 1 \le i \le k$, since $t-1 = t_i - 1 \ge t_{i-1}$,

$$\frac{r_t(\vartheta)}{r_{t-1}(\vartheta)} \le 1 + \frac{\phi_t^2(\vartheta)}{\bar{r}_{i-1}(\vartheta)} = \frac{\bar{r}_{t_i}(\vartheta)}{\bar{r}_{i-1}(\vartheta)}.$$

Hence, (29) holds again by (30). □

Lemma 4.2 *Let $r_t(\vartheta) \triangleq \sum_{i=0}^t \phi_i^2(\vartheta)$, where $\phi_i(\vartheta) = \dfrac{\partial f(\vartheta, \varphi_i)}{\partial \vartheta}$ as defined before. Under Assumptions A1–A3, the growth rate of $r_t(\vartheta)$ satisfies*

$$\liminf_{t\to\infty} \frac{\inf_{\vartheta\in\mathbb{R}} r_t(\vartheta)}{t} > 0, \quad a.s. \tag{31}$$

Proof: Observe that system (1) has the form of (9) if we let $z_t = y_t - h(u_t, \varphi_t)$ and $x_t = \varphi_t$, therefore estimator (18) can be applied accordingly. Now, by (7), one has that for any $i \ge 1$,

$$y_i^2 = \tilde{f}_{i-1}^2(\theta) + w_i^2 + 2\tilde{f}_{i-1}(\theta) w_i.$$

As a consequence,

$$\sum_{i=1}^t y_i^2 = \sum_{i=1}^t \left(\tilde{f}_{i-1}^2(\theta) + w_i^2\right) + 2\sum_{i=1}^t \tilde{f}_{i-1}(\theta) w_i. \tag{32}$$

Since θ and φ_0 are independent of $\{w_t\}$, it is easy to verify that $\tilde{f}_{i-1}(\theta) \in \mathcal{F}_{i-1}$, where $\mathcal{F}_{i-1}$ is defined by (21) with $x_0 = \varphi_0$, is independent of w_i for all $i \ge 1$. Thus, by [1, Theorem 2.8], one has

$$\sum_{i=1}^t \tilde{f}_{i-1}(\theta) w_i = o\left(\sum_{i=1}^t \tilde{f}_{i-1}^2(\theta)\right) + O(1), \quad \text{a.s..} \tag{33}$$

Therefore, by (32) and (33), for any sufficiently large t,

$$\begin{aligned} \frac{1}{t}\sum_{i=1}^t y_i^2 &= \frac{1}{t}\left((1+o(1))\sum_{i=1}^t \tilde{f}_{i-1}^2(\theta) \right. \\ &\quad \left. + \sum_{i=1}^t w_i^2 + O(1)\right) \\ &\ge \frac{1}{t}\sum_{i=1}^t w_i^2 \to 1, \quad \text{a.s.,} \end{aligned}$$

which yields

$$\liminf_{t\to\infty} \frac{1}{t}\sum_{i=1}^t y_i^2 \ge 1, \quad \text{a.s..} \tag{34}$$

Finally, by Assumption A3, one has

$$\inf_{\vartheta\in\mathbb{R}} r_t(\vartheta) \ge C_1^2 \sum_{i=0}^t y_i^2,$$

and hence the desired result follows immediately by (34). □

Lemma 4.3 *Let $\vartheta \in \mathbb{R}$ be a constant such that $t_k \to \infty$ almost surely in (26). Then, under Assumptions A1–A3,*

$$P\left\{\bar{r}_k(\vartheta) < \left(2^{\log^2 t_k} t_k^{-4}\right)^{\frac{1}{2b}}, \ i.o.\right\} = 1.$$

Proof. Fix $\vartheta \in \mathbb{R}$. First, given $k \ge 1$, for any $i \in [1, t_{k+1})$, Lemma 4.1 yields

$$\frac{r_i(\vartheta)}{r_{i-1}(\vartheta)} \le \frac{\bar{r}_k(\vartheta)}{\bar{r}_{k-1}(\vartheta)}. \tag{35}$$

Consequently, by (27) and (28), one has

$$\begin{aligned} r_{i-2}(\vartheta) &= r_0(\vartheta)\prod_{j=1}^{i-2} \frac{r_j(\vartheta)}{r_{j-1}(\vartheta)} \\ &\le \bar{r}_0(\vartheta)\left(\frac{\bar{r}_k(\vartheta)}{\bar{r}_{k-1}(\vartheta)}\right)^{i-2} \\ &\le \bar{r}_k^{i-1}(\vartheta). \end{aligned} \tag{36}$$

Now, by Assumption A1, for any i that is sufficiently large,

$$w_i^2 \le \log^2 i, \quad \text{a.s..} \tag{37}$$

If furthermore $i \in [1, t_{k+1})$, it can be derived from (37), (7), (3) and Proposition 3.1 that for some θ'_{i-1},

$$\begin{aligned} y_i^2 &\le 2\tilde{f}_{i-1}^2(\theta) + 2w_i^2 \\ &= 2\phi_{i-1}^2(\theta'_{i-1})\tilde{\theta}_{i-1}^2 + 2w_i^2 \\ &= O\left(\log^6(i-1)\log^2 r_{i-2}(\vartheta)\frac{\phi_{i-1}^2(\vartheta)}{r_{i-2}(\vartheta)} + \log^2 i\right) \\ &= O\left(\log^6(i-1)\log^2 r_{i-2}(\vartheta)\frac{r_{i-1}(\vartheta)}{r_{i-2}(\vartheta)}\right), \text{a.s..} \end{aligned} \tag{38}$$

Then, according to (35) and (36), one has

$$y_i^2 = O\left(t_{k+1}^3 \log^2 \bar{r}_k(\vartheta)\frac{\bar{r}_k(\vartheta)}{\bar{r}_{k-1}(\vartheta)}\right), \quad \text{a.s..} \tag{39}$$

Observe that by the definition of $\bar{r}_k(\vartheta)$ and Assumption A3, one has

$$\begin{aligned} \bar{r}_{k+1}(\vartheta) &= \sum_{i=0}^{k+1} \phi_{t_i}^2(\vartheta) \le C_2^2 \sum_{i=0}^{k+1} \|\varphi_{t_i}\|^{2b} \\ &= O\left(\sum_{i=0}^{k+1} y_{t_i}^{2b}\right) = O\left(\sum_{i=0}^{t_{k+1}} y_i^{2b}\right). \end{aligned} \tag{40}$$

Hence, with (39), one has

$$\begin{aligned} \bar{r}_{k+1}(\vartheta) &= O\left(t_{k+1}^{3b+1}\left(\log^2 \bar{r}_k(\vartheta)\frac{\bar{r}_k(\vartheta)}{\bar{r}_{k-1}(\vartheta)}\right)^b + 1\right) \\ &= O\left(t_{k+1}^{3b+1}\left(\log^2 \bar{r}_k(\vartheta)\frac{\bar{r}_k(\vartheta)}{\bar{r}_{k-1}(\vartheta)}\right)^b\right), \end{aligned} \tag{41}$$

where the last equality follows from Lemma 4.2 that $\lim_{k\to\infty} \bar{r}_k(\vartheta) = \infty$.

As a result, by taking logarithm on both sides of (41), it yields for $k \to \infty$,

$$\begin{aligned} & (1-o(1))\log \bar{r}_{k+1}(\vartheta) \\ \leq \; & b\left(\log \bar{r}_k(\vartheta) - \log \bar{r}_{k-1}(\vartheta)\right) \\ & +(3b+1)\log t_{k+1} + O(1), \quad \text{a.s..} \end{aligned} \tag{42}$$

Suppose that there is a set $E \subset \Omega$ with $P(E) > 0$ and a positive random variable k' such that for all $k \geq k'$ that are sufficiently large,

$$\bar{r}_{k+1}(\vartheta) \geq \left(2^{\log^2 t_{k+1}} t_{k+1}^{-4}\right)^{\frac{1}{2b}} \qquad \text{on } E.$$

Then, $\log t_{k+1} = o(\log \bar{r}_{k+1}(\vartheta))$ as $k \to \infty$, and hence by (42), for any $k \geq k'$,

$$(1-o(1))\log \bar{r}_{k+1}(\vartheta) \leq b\left(\log \bar{r}_k(\vartheta) - \log \bar{r}_{k-1}(\vartheta)\right) \tag{43}$$

holds almost surely on the set E.

Now, let $\zeta_k = \log \bar{r}_k(\vartheta)$. Then, on set E, (43) becomes

$$(1-o(1))\zeta_{k+1} \leq b(\zeta_k - \zeta_{k-1}), \quad \text{a.s.,} \tag{44}$$

where $\zeta_k \nearrow \infty$ almost surely. Define $z_k \triangleq \dfrac{\zeta_k}{\zeta_{k-1}}$ and denote random $z \triangleq \limsup_{k\to\infty} z_k$, which is clearly positive (including positive infinity). Dividing (44) by a_{k+1}, then for a sufficiently large k,

$$(1-o(1)) + \frac{b}{z_{k+1}z_k} \leq \frac{b}{z_{k+1}}.$$

Taking limit inferior on both sides of the above inequality, one obtains that on set E,

$$z^2 - bz + b \leq 0, \quad \text{a.s.,} \tag{45}$$

which is impossible since the supremum limit z is finite by (45), and $z^2 - bz + b > 0$ for all $z \in \mathbb{R}$ whenever $b < 4$. Thus, (43) cannot be true and the lemma is proved. $\square$

One can arrive at the following conclusion based on Lemma 4.3. The proof is omitted here for brevity.

Lemma 4.4 *Under Assumptions A1–A3, if there is some $\bar{\vartheta} \in \mathbb{R}$ such that the corresponding $t_k \to \infty$ almost surely, then*

$$P\left\{\sup_{\vartheta\in\mathbb{R}} \frac{r_{t+i}(\vartheta)}{r_{t+i-1}(\vartheta)} < 2C_3^4, -m+1 \leq i \leq 0, \ i.o.\right\} = 1.$$

Lemma 4.5 *Let Assumptions A1–A3 be satisfied, then*

$$\sup_{t\geq 1}\sup_{\vartheta\in\mathbb{R}} \frac{r_t(\vartheta)}{r_{t-1}(\vartheta)} < \infty, \quad a.s.. \tag{46}$$

Proof. Take a $\vartheta \in \mathbb{R}$. Observe that for this ϑ, if $\{t_k\}$ is a finite subsequence on some set $G \subset \Omega$ with positive probability, then by (26), there is a random $k_0 \geq 0$ such that for all $t \geq t_{k_0}$ on G,

$$\frac{\phi_t^2(\vartheta)}{\bar{r}_{k_0}(\vartheta)} \leq \frac{\phi_{t_{k_0}}^2(\vartheta)}{\bar{r}_{k_0-1}(\vartheta)}.$$

This in fact leads to the boundness of $\phi_t(\vartheta)$ on G. Furthermore, since $r_t(\vartheta) \geq r_0(\vartheta) \geq 1$, by (2) and Assumption A3, for all $t \geq t_{k_0}$, one has on G that

$$\frac{r_t(\vartheta)}{r_{t-1}(\vartheta)} \leq 1 + \phi_t^2(\vartheta) \leq \frac{C_2^4\left(\sum_{i=0}^{k_0}\|\varphi_{t_i}\|^2\right)\|\varphi_{t_{k_0}}\|^2}{C_1^2\sum_{i=0}^{k_0-1}\|\varphi_{t_i}\|^2}.$$

Hence, (46) holds on G by (3).

Now, without loss of generality, we assume that $\{t_k\}$ is an infinite random sequence almost everywhere for ϑ. This is because the argument in the following can be viewed to be discussed on a restriction probability space of $(\Omega, \mathcal{F}, P)$ on G^c.

According to Lemmas 4.2 and 4.4, there is a sufficiently large random integer t_0 such that for some positive random constant M,

$$\inf_{\vartheta\in\mathbb{R}} r_t(\vartheta) \geq Mt, \quad \text{a.s.,} \quad \forall\, t \geq t_0 \tag{47}$$

and for $-m+1 \leq i \leq 0$,

$$\sup_{\vartheta\in\mathbb{R}} \frac{r_{t_0+i}(\vartheta)}{r_{t_0+i-1}(\vartheta)} < 2C_3^4, \quad \text{a.s..}$$

We will use an induction method to prove that for all $t \geq t_0$,

$$\sup_{\vartheta\in\mathbb{R}} \frac{r_t(\vartheta)}{r_{t-1}(\vartheta)} < 2C_3^4, \quad \text{a.s..} \tag{48}$$

Assume for some integer $k \geq t_0$,

$$\sup_{\vartheta\in\mathbb{R}} \frac{r_{k+i}(\vartheta)}{r_{k+i-1}(\vartheta)} < 2C_3^4, \ -m+1 \leq i \leq 0, \quad \text{a.s..}$$

We proceed to check (48) for $t = k+1$. First, similar to (38), by Proposition 3.1, for $k-m+1 \leq i \leq k$, one has

$$\begin{aligned} y_{i+1}^2 &= O\left(\log^6 i \log^2 r_{i-1} \sup_{\vartheta\in\mathbb{R}} \frac{r_i(\vartheta)}{r_{i-1}(\vartheta)}\right) \\ &= O\left(\log^6 i \log^2 r_{i-1}\right), \quad \text{a.s.,} \end{aligned} \tag{49}$$

where r_{i-1} is defined by Proposition 3.1.

Furthermore, in virtue of (3) and (47),

$$\inf_{\vartheta\in\mathbb{R}} r_k(\vartheta) \geq \max\left\{Mk, \frac{r_{i-1}}{C_3^2}, k-m+1 \leq i \leq k\right\}, \tag{50}$$

which together with (49) implies that as $t \to \infty$,

$$\begin{aligned} & \sup_{\vartheta\in\mathbb{R}} \frac{r_{k+1}(\vartheta)}{r_k(\vartheta)} \\ = \; & 1 + \sup_{\vartheta\in\mathbb{R}} \frac{\phi_{k+1}^2(\vartheta)}{r_k(\vartheta)} \\ = \; & 1 + O\left(\frac{\sum_{i=k-m+1}^{k}\left(\log^{6b} i \log^{2b} r_{i-1}\right)}{\inf_{\vartheta\in\mathbb{R}} r_k(\vartheta)}\right) \\ = \; & 1 + O\left(\frac{1}{\sqrt{k}}\right) \\ < \; & 2C_3^4, \quad \text{a.s.,} \end{aligned} \tag{51}$$

which shows that (48) holds for all $t \geq t_0$ by induction. The lemma is thus proved. $\square$

Lemma 4.6 *For any $\vartheta \in \mathbb{R}$, define $a_t(\vartheta) \triangleq \dfrac{r_{t-1}(\vartheta)}{r_t(\vartheta)}$ and*

$$\rho_t(\vartheta) \triangleq \frac{|\phi_t(\vartheta)| \left(\sum_{i=1}^t |\phi_{i-1}(\vartheta) w_i|\right)}{r_{t-1}(\vartheta)}.$$

Then, under Assumption A1, as $t \to \infty$,

$$\sum_{i=1}^t \sup_{\vartheta \in \mathbb{R}} a_i(\vartheta) \rho_i^2(\vartheta) = O\left(\log^3 r_t + t\right), \quad a.s..$$

Proof: The proof is omitted. □

Proof of Theorem 2.1: First, we provide an estimation of $\tilde{f}_t(\theta) = f(\theta, \varphi_t) - f(\hat{\theta}_t, \varphi_t)$. Similar to (19), for any $t \geq 1$,

$$\begin{aligned} \tilde{f}_t(\theta) &= f(g_t(W_t), \varphi_t) - f(g_t(0), \varphi_t) \\ &= \frac{\partial f(\theta, \varphi_t)}{\partial \theta} \frac{\partial g_t(W_t)}{\partial W_t}\Big|_{W_t = \bar{W}_t^*} W_t \\ &= -\frac{\phi_t(\bar{\theta}_t^*) \left(\sum_{i=1}^t \phi_{i-1}(\bar{\theta}_t^*) w_i\right)}{r_{t-1}(\bar{\theta}_t^*)}, \end{aligned} \tag{52}$$

where $\bar{W}_t^*$ is some random variable and $\bar{\theta}_t^* = g_t(\bar{W}_t^*)$.

Clearly, for any $t \geq 1$, $|\tilde{f}_t(\theta)| \leq \rho_t(\bar{\theta}_t^*)$, where $\rho_t(\cdot)$ is define by Lemma 4.6. Therefore, by (7) and Lemma 4.6, it can be derived that

$$\begin{aligned} \sum_{i=1}^t y_{i+1}^2 &= O\left(\sum_{i=1}^t \left(\tilde{f}_i^2(\theta) + w_{i+1}^2\right)\right) \\ &= O\left(\sum_{i=1}^t a_i(\bar{\theta}_i^*) \tilde{f}_i^2(\theta) \sup_{k \geq 1} \sup_{\vartheta \in \mathbb{R}} \frac{r_k(\vartheta)}{r_{k-1}(\vartheta)} + t\right) \\ &= O\left(\sum_{i=1}^t \sup_{\vartheta \in \mathbb{R}} a_i(\vartheta) \rho_i^2(\vartheta) + t\right) \\ &= O\left(\log^3 r_t + t\right), \quad \text{a.s..} \end{aligned} \tag{53}$$

Now, similar to (41), by using the Jensen inequality, one has

$$r_t = O\left(\sum_{i=1}^t y_i^{2b}\right) = O\left(\left(\sum_{i=1}^t y_i^2\right)^b\right),$$

which, with (53), leads to

$$\sum_{i=1}^t y_{i+1}^2 = O\left(\log^3\left(\sum_{i=1}^t y_{i+1}^2\right) + t\right), \quad \text{a.s.,}$$

which can be rewritten as

$$(1 + o(1)) \sum_{i=1}^t y_{i+1}^2 = O\left(t\right), \quad \text{a.s..}$$

The proof of Theorem 2.1 is completed. □

5 Concluding Remarks

In this paper, we have studied a class of nonlinearly parameterized uncertain systems in discrete time. The strong consistency of the NLS estimator for a general regression model in the presence of nonlinear paramterization was established under a certain excitation condition. Based on the NLS estimator, the adaptive stabilization was also achieved for the nonlinearly parameterized uncertain systems with a scalar unknown parameter.

References

[1] H. Chen and L. Guo, *Identification and Stochastic Adaptive Control*, Birkhauser: Boston, MA, 1991.

[2] L. Guo, "Convergence and logarithm laws of self-tuning regulators", *Automatica*, vol. 31, pp. 435–450, 1995.

[3] L. Guo, "On critical stability of discrete-time adaptive nonlinear control", *IEEE Trans. Autom. Control*, vol. 42, pp. 1488–1499, 1997.

[4] D. Huang and L. Guo, "Estimation of nonstationary Armax models based on the Hannan-Rissanen method," *The Annals of Statistics*, vol. 18, 1729–1756, 1990.

[5] P. V. Kokotović and M. Arcak, "Constructive nonlinear control: Progress in the 90s", in *Proc. 14th IFAC World Congr.*, Beijing, China, pp. 49–77, July 1999.

[6] M. Krstić, I. Kanellakopoulos, and P. V. Kokotović, *Non-linear and Adaptive Control Design*, Wiley: New York, 1995.

[7] H. B. Ma, "Finite-model adaptive control using WLS-like algorithm", *Automatica*, vol. 43, pp. 677–684, 2007.

[8] H. B. Ma, "An 'impossibility' theorem on a class of high-order discrete-time nonlinear control systems", *Syst. Control Lett.*, vol. 57, pp. 497–504, 2008.

[9] C. Li and L.-L. Xie, "On robust stability of discrete-time adaptive nonlinear control", *Syst. Control Lett.*, vol. 55, pp. 452–458, 2006.

[10] C. Li and L. Guo, "On adaptive stabilization of nonlinearly parameterized discrete-time systems", in *Proc. 27th Chinese Control Conference*, Kunming, pp. 484–487, July, 2008.

[11] C. Li and L. Guo, "A new critical theorem for adaptive nonlinear stabilization", *Automatica*, vol. 46, pp. 999–1007, 2010.

[12] C. Li and L. Guo, "On feedback capability in a class of nonlinearly parameterized uncertain systems", *IEEE Trans. Autom. Control*, vol. 56, pp. 2946–2951, 2011.

[13] C. Li and L. Guo, "A dynamical inequality for the output of uncertain nonlinear systems", *Science China*, vol. 56, pp. 1–9, 2013.

[14] C. Li and J. Lam, "Stabilization of discrete-time nonlinear uncertain systems by feedback based on LS algorithm", *SIAM J. Control and Optimization*, vol. 51, pp. 1128–1151, 2013.

[15] C. Li, L.-L. Xie, and L. Guo, "A polynomial criterion for adaptive stabilizability of discrete-time nonlinear systems", *Communications in Information and Systems*, vol. 6, pp. 273–298, 2006.

[16] D. W. Marquardt, "An algorithm for least-squares estimation of nonlinear parameters", *SIAM Journal on Applied Mathematics*, vol. 11, pp. 431–441, 1963.

[17] F. P. Skantze, A. Kojić, A. -P. Loh and A. M. Annaswamy. "Adaptive estimation of discrete-time systems with nonlinear parameterization", *Automatica*, vol.36, pp. 1879–1887, 2000.

[18] T. L. Lai, "Asymptotic properties of nonlinear least squares estimates in stochastic regression models", *The Annals of Statistics*, vol. 22, pp. 1917–1930, 1994.

[19] L.-L. Xie and L. Guo, "Fundamental limitations of discrete-time adaptive nonlinear control", *IEEE Trans. Autom. Control*, vol. 44, 1777–1782, 1999.

[20] L.-L. Xie and L. Guo, "Adaptive control of discrete-time nonlinear systems with structural uncertainties", In: Lectures on Systems, Control, and Information, AMS/IP, pp. 49–90, 2000.

[21] L.-L. Xie and L. Guo, "How much uncertainty can be dealt with by feedback?", *IEEE Trans. Autom. Control*, vol. 45, pp. 2203–2217, 2000.

[22] F. Xue, L. Guo, and M. Y. Huang, "Towards understanding the capability of adaptation for time-varying systems", *Automatica*, vol. 37, pp. 1551–1560, 2001.