ILC for Networked Discrete Systems with Random Data Dropouts: A Switched System Approach

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Abstract: A novel approach, switched system approach, is proposed for iterative learning control problem of networked control systems with random data dropouts. The random data dropout is described as three different forms, namely, a random sequence, a binary Bernoulli random variable, and a Markov chain, respectively. The tracking error is strictly proved to converge to zero in expectation sense, mean square sense, and almost sure sense.

Key Words: Iterative Learning Control, Networked Control System, Random Data Dropout, Switched System Approach

1 Introduction

Along with the developments of telecommunication and network technologies, the structure of control system has changed greatly. In many practical applications, the conventional centralized control structure, where the controller and system are integrated together, is replaced by the networked control system (NCS), where the system and the controller are separated in the local and remote. Due to relaxation of position restriction, the networked control system possesses more flexibility for design, and thus has been an important topic of control research field [1–3]. However, because of network congestion, linkage interrupt, transmission error, and etc., the data packets are probably lost during the transmission, which could further reduce the system performance. Since the data dropout happens randomly, i.e., the data dropout is hard to predict. Thus the research on data dropout for networked control system is one of the research hot-spots [1–3].

On the other hand, many practical systems execute the operation procedure in a limited interval, termed as iteration in this paper, and then repeat. For instance, batches reactors, hard disk drives, and robotics are such kind of typical systems. For these systems, iterative learning control (ILC) could achieve precise tracking performance for a given trajectory. The key idea of this control strategy is using the input and output information generated in the previous iterations when designing the control signal for the current iteration, and thus the tracking performance can be improved from iteration to iteration. Since proposed, ILC has attracted much attention from both scholars and engineers [4–7].

Although random packet losses have been discussed numerously in conventional networked control systems, the publications related to iterative learning control are very rare. We have tried our best to search the literature, but the outcomes are limited, which is a side-reflection that this topic is on the initial step. In most related papers, the packet loss is modeled as a Bernoulli random variable, whose value is 1 when the packet is successfully transmitted and 0 otherwise.

Bu and his co-workers considered iterative learning control for networked control systems from the statistics point of view [8–13]. In [8], the linear time-invariant discrete system was lifted into the so-called super-vector form, consequently the iteration equation of tracking errors was directly obtained. Based on the equation and exponential stability for asynchronous dynamical systems [14], the stability analysis was given in [8]. Another stability result was given in [9] for SISO linear time-invariant system under data dropouts. Unlike [8], the latter took mathematical expectations to both sides of the iteration equation of tracking errors directly, and then gave the stability condition according to the mathematical expectation of tracking error. Further results were given in [10], where the relationship between data loss rate and convergence speed was studied according to the expectations of tracking errors similar to the one in [9]. Both [11] and [12] considered the nonlinear system case. Because of the nonlinearity, the techniques used in [9] are no longer suitable. Instead, the regression equation was first established, and then regression inequality was given by taking expectations, whence the effect of stochastic data dropout was eliminated so that the convergence condition was provided based on contract mapping method. Moreover, [13] provided an H_∞ iterative learning controller for a class of discrete-time systems with data dropouts. With help of the super-vector formulation, the original system is formulated as a linear discrete-time stochastic system in the iteration domain and the H_∞ performance problem in the iteration domain is defined and discussed. In short, the main idea of Bu's series publications is transforming the stochastic equation/equality into deterministic equation/equality by taking mathematical expectations and then showing the stability/convergence conditions.

Ahn and his co-workers studied this topic for MIMO time-invariant systems and proved the mean-square stability under data dropouts for iterative learning control [15–17]. The major differences among these papers are the packet loss locations. In particular, in [15] only the measurement output was assumed to be randomly lost when transmitting back to control center. Besides, the output vector was assumed to be completely lost if packet loss occurs. In practical system, maybe only part of the multi-dimensional output is lost but the other part is transmitted back. The analysis of this case was considered in [16]. [17] further discussed the case that packet loss happened to the control signals as well as output signals. The main technique used in these papers is the

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Kalman filtering based technique proposed in [18], thus all the convergence results are in mean-square sense.

Shen and Wang also investigated the ILC problem for NC-S in the random packet loss environment [19, 20]. Unlike the above reported studies, the data dropout is modeled as an arbitrary stochastic sequence with a bounded length requirement, and the almost sure convergence is achieved for stochastic NCS in these two papers. In [19], a simple P-type ILC algorithm was proposed for SISO linear time-varying stochastic system and proved that almost sure convergence based on stochastic approximation algorithm. Then the system is extended to affine nonlinear case. The unknown control direction issue was also taken into account under the random packet loss environment in [20], and an algorithm with a novel regulating approach was introduced and analyzed.

To sum up, the existing publications mainly consider data dropout in the Bernoulli random variable form, and prove convergence in mean-square sense or expectation sense. This motivates this study. Specifically, the contributions of this study are listed as follows.

- Taking the practical engineering into consideration, three different mathematical models describing random data dropouts are handled in this paper, namely, random sequence model with length requirement, Bernoulli distributed random variable model, and Markov chain model.
- A unified framework, the so-called switched system approach, is proposed to deal with the three kinds of data dropouts. The spectral radius condition for the learning gain matrix is obtained to ensure the convergence of the proposed ILC algorithm.
- The convergence in expectation sense, mean square sense, and almost sure sense is strictly proved. To this end, the first and second moments of random variables and random matrices are carefully calculated.

The rest parts of the paper are arranged as follows: Section 2 provides the system formulation and the novel switched system approach; Section 3 presents the detailed convergence analysis in expectation sense, mean square sense, and almost sure senses, respectively; and some concluding remarks are summarized in Section 4.

Notations: $\mathbb{E}$ denotes the mathematical expectation. $\lambda(A)$ is the eigenvalue of a matrix A, and $\rho(A)$ is the spectral radius. $\|A\|$ is an induced norm of a matrix A. The subscript T denotes the transpose. $\otimes$ denotes the Kronecker product.

2 Problem Formulation

2.1 System Description

Consider the following SISO lifted linear system

$$y_k = Hu_k + y_0 \quad (1)$$

where k denotes the iteration number, y_k and u_k are the output and input vector, respectively. H is the Toeplitz transfer matrix defined as

$$H = \begin{bmatrix} h_1 & 0 & \cdots & 0 \\ h_2 & h_1 & \cdots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ h_n & h_{n-1} & \cdots & h_1 \end{bmatrix}$$

where $h_1, \cdots, h_n$ are the Markov parameters. Here n denotes the time interval length in an iteration. y_0 denotes the initial value. For expression clarity and without loss of any generality, in this paper it is simply assumed that $y_0 = 0$.

Remark 1. *It is worth pointing out that the model* (1) *is assumed time-invariant only to make our idea clearly elaborated, and all the following derivations are valid for the time-varying case.*

The reference trajectory is y_d, such that there exists a unique u_d satisfying the following relationship

$$y_d = Hu_d \quad (2)$$

Denote $e_k \triangleq y_d - y_k$ as the tracking error of the k-th iteration. Then the control input for the $(k+1)$-th iteration is designed by

$$u_{k+1} = u_k + Qe_k \quad (3)$$

The setup of the control system is illustrated in Fig. 1. For expressions clear, only measurement data dropouts are considered in this short paper. That is, only the measurement of the output may be lost during the transmission. The results can be directly extended to the more general case, such as data dropout happening on both the input and the output sides.

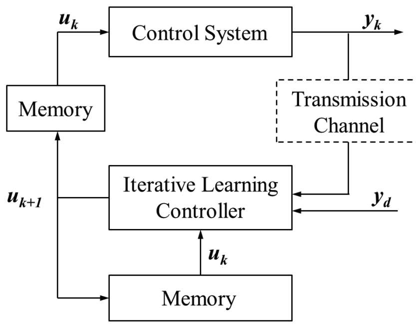

Fig. 1: Block diagram of networked control system with measurement data dropouts

Then the update law (3) under random data dropouts is rewritten as

$$u_{k+1} = u_k + \gamma Qe_k \quad (4)$$

where γ is a random variable to denote random data dropout.

2.2 Data Dropouts Models

In this paper, the outputs of an iteration are assumed to be dropped out or not together. Roughly speaking, this assumption seems restrictive. However, the assumption here is only to make our approach easy to understand. That is, the data dropouts do not have to be lost entirely. The techniques and results of this paper could be extended to the asynchronous multiple data dropouts, for example, see [21].

Considering the data dropouts happening in the practical engineering, it is observed that the data dropout is random and unpredictable in advance. Thus in order to model the randomness of data dropout, a random variable γ, which is valued as 1 or 0, is introduced, as shown in (4). Specifically, $\gamma = 1$ denotes that data dropout does not happen, while $\gamma = 0$ means data dropout happens. Here it is assumed that the controller center could detect whether the data dropout happens or not. If data dropout happens, the control signal would not update itself.

There are three principle mathematical models of γ are considered in the research of NCS, which are all studied in this paper as follows.

- RSM: Random sequence model. The measurement data dropout is random without obeying any certain probability distribution, but there is a number K such that during successive K iterations, at least in one iteration the measurement is successfully sent back [22].
- BVM: Bernoulli variable model. The random variable γ satisfies that
$$P\{\gamma=1\}=\overline{\gamma},\quad P\{\gamma=0\}=1-\overline{\gamma}, \tag{5}$$
where $\overline{\gamma}=\mathbb{E}\gamma$ with $0<\overline{\gamma}<1$ [23].
- MCM: Markov chain model. The random variable γ is valued 1 or 0 according to a two-state Markov chain. The probability transition matrix P is defined as
$$P=\begin{bmatrix} p_{00} & p_{01} \\ p_{10} & p_{11} \end{bmatrix}=\begin{bmatrix} \mu & 1-\mu \\ 1-\nu & \nu \end{bmatrix} \tag{6}$$
where $0<\mu<1$ and $0<\nu<1$ [24].

Remark 2. *In the RSM case, The number K is not necessary to be known prior, in other words, only the existence of such number is required. Thus this condition means that the measurements should not be lost too much to guarantee the convergence in almost sure sense.*

Remark 3. *The difference between the BVM case and the M-CM case lies in that the data dropout happens independently along iteration axis when it is modeled by BVM, while happens dependently along iteration axis when it is modeled by MCM.*

2.3 Switched System Approach

Define $\delta u_k \triangleq u_d-u_k$, then from (1), (2), and (4) one has

$$\begin{aligned}\delta u_{k+1}=&\delta u_k-\gamma Qe_k\\ =&\delta u_k-\gamma QH\delta u_k\\ =&(I-\gamma QH)\delta u_k\end{aligned}$$

Notice that if $\gamma=1$, i.e., the data is not lost, then the update law actually is

$$\delta u_{k+1}=(I-QH)\delta u_k \tag{7}$$

while if $\gamma=0$, i.e., the data is lost, then the update law actually is

$$\delta u_{k+1}=\delta u_k \tag{8}$$

Thus one could reformulate the update law into a switched system form

$$\delta u_{k+1}=\left[\gamma(I-QH)+(1-\gamma)I\right]\delta u_k \tag{9}$$

Therefore, the convergence of the original update law (4) could be achieved by analyzing the stability and convergence of the switched system (9). This is the novel approach introduced in this paper. For the sake of expression concise, stochastic matrix Γ_γ is introduced, which is valued as $\Gamma_1=I-QH$ and $\Gamma_0=I$. Then (9) could be formulated as

$$\delta u_{k+1}=\Gamma_{\gamma_k}\delta u_k \tag{10}$$

where γ_k indicates the value of γ in the k-th iteration.

The following Borel-Cantelli lemma [25] is needed in the convergence analysis for system (9).

Lemma 1 (Borel-Cantelli Lemma)**.** *If $\{K_n, n\ge 1\}$ is a sequence of events with $\sum_{n=1}^{\infty}P\{K_n\}<\infty$, then $P\{K_n, i.o.\}=0$, where i.o. is the abbreviation of "infinitely often".*

3 Convergence Analysis of the Algorithm

Noticing (9), one can observe that if data dropout does not happen, then the input signal would be updated, while if data dropout happens, then the input signal remains unchanged. Thus the convergence would depends on the case that data dropout does not happen, and then on the design of Q. In order to ensure the convergence of the iterative learning control algorithm, the following design condition is required.

Design condition: Q is designed to satisfy that all eigenvalues of $I-QH$ lie interior to the unit circle.

Note that this condition implies that $\rho(I-QH)<1$, where $\rho(A)$ denotes the spectral radius of a matrix A. Besides, there is a large degree of freedom on the design of learning gain Q. Sometimes one can directly design Q such that

$$0<I-QH<I \tag{11}$$

Remark 4. *The above condition* (11) *on Q seems restrictive for the designing. On one hand,* (11) *is a linear matrix inequality of Q, thus the solution could be obtained simply by using LMI toolbox. For example, taking the non-causal learning gain, i.e., letting $Q=sH^T$ with s being a small coefficient, the condition* (11) *is satisfied. On the other hand, the simple form of Q in* (11) *would help us to make a neat proof of the convergence analysis. Actually, the design condition of Q is sufficient enough.*

3.1 The RSM Case

The convergence analysis for the RSM case could be done in a deterministic way as there is a length requirement on the consecutive data dropouts. Thus the tracking errors generated by random mode switching could be bounded by a convergent sequence generated by an associated deterministic algorithm, as shown in the following.

For this purpose, let us divide the iteration axis into segments with length K, where K is defined in the RSM. In other words, the iteration indices $k=0,1,2,\cdots$ are split into $[0,1,2,\cdots,K-1]$, $[K,K+1,\cdots,2K-1]$, $\cdots,[jK,\cdots,(j+1)K-1],\cdots$. In the sequel of this subsection, the j-th segment means $[(j-1)K,(j-1)K+1,\cdots,jK-1]$. According to the length requirement of RSM, one has that during each segment, say the i-th segment, there exist at least one iteration, say $(i-1)K+j_0$, such that $\Gamma_{\gamma_{(i-1)K+j_0}}=I-QH$.

Let us first express the input error δu_k at the first iteration of every segment, i.e., $\delta u_{(j-1)K}$, $j=1,2,\cdots$. Denote θ_j as the cumulative total of successful transmission from the plant to the controller during the j-th segment. In other words, $\theta_j = \sum_{i=0}^{K-1} \gamma_{(j-1)K+i}$, then it is clear that θ_j is a random variable. According to the formulation of RSM, one has that $1 \leq \theta_j \leq K$, $j = 1, 2, \cdots$. Then

$$\begin{aligned} \delta u_{jK} &= \Gamma_{\gamma_{jK-1}} \cdots \Gamma_{\gamma_{(j-1)K}} \delta u_{(j-1)K} \\ &= \Gamma_1^{\theta_j} \delta u_{(j-1)K} \end{aligned} \tag{12}$$

Then the following convergence theorems could be constructed based on (12)

Theorem 1. *System* (10) *with data dropouts variable* γ *obeying RSM converges to zero in the sense of* $\mathbb{E}\delta u_k \xrightarrow[k\to\infty]{} 0$*, if* $\rho(I - QH) < 1$.

Proof. By (12) it follows that

$$\begin{aligned} \mathbb{E}\delta u_{jK} =& \mathbb{E}\left(\Gamma_1^{\theta_j}\Gamma_1^{\theta_{j-1}} \cdots \Gamma_1^{\theta_1} \delta u_0\right) \\ =& \mathbb{E}\left(\Gamma_1^{\sum_{i=1}^{j}\theta_i} \delta u_0\right) \end{aligned}$$

Since $1 \leq \theta_j \leq K$, $j = 1, 2, \cdots$, it is obvious that $\sum_{i=1}^{j} \theta_i \xrightarrow[j\to\infty]{} \infty$. Noticing that $\rho(I - QH) = \rho(\Gamma_1) < 1$, it is obtained that

$$\Gamma_1^{\sum_{i=1}^{j}\theta_i} \xrightarrow[j\to\infty]{} 0$$

Thus $\mathbb{E}\delta u_{jK} \xrightarrow[j\to\infty]{} 0$.

Next consider the other iterations of the j-th segment except the first one, i.e., $(j-1)K + i$, $i = 1, 2, \cdots, K-1$. One could find that

$$\delta u_{(j-1)K+i} = \Gamma_{\gamma_{(j-1)K+i-1}} \cdots \Gamma_{\gamma_{(j-1)K}} \delta u_{(j-1)K}$$

which further implies that

$$\|\mathbb{E}\delta u_{(j-1)K+i}\| \leq \|\mathbb{E}\delta u_{(j-1)K}\|, \ \forall 1 \leq i \leq K-1 \tag{13}$$

Thus $\mathbb{E}\delta u_k \xrightarrow[k\to\infty]{} 0$ and the proof is completed. □

Theorem 2. *System* (10) *with data dropouts variable* γ *obeying RSM converges to zero almost surely, if* $\rho(I - QH) < 1$.

Proof. Based on similar steps in the proof of Theorem 1, one can have

$$\mathbb{E}\|\delta u_{jK}\| \leq \mathbb{E}\|\Gamma_1^{\sum_{i=1}^{j}\theta_i}\delta u_0\| \leq \|\Gamma_1\|^j \mathbb{E}\|\delta u_0\|$$

and

$$\mathbb{E}\|\delta u_{(j-1)K+i}\| \leq \mathbb{E}\|\delta u_{(j-1)K}\|, \ \forall 1 \leq i \leq K-1$$

Then it follows

$$\begin{aligned} \sum_{k=1}^{\infty} \mathbb{E}\|\delta u_k\| =& \sum_{j=1}^{\infty}\sum_{i=0}^{K-1} \mathbb{E}\|\delta u_{(j-1)K+i}\| \\ \leq& \sum_{j=1}^{\infty}\sum_{i=0}^{K-1} \mathbb{E}\|\delta u_{(j-1)K}\| \\ \leq& K\sum_{j=1}^{\infty} \mathbb{E}\|\delta u_{(j-1)K}\| < \infty \end{aligned}$$

Then using Markov inequality, for any $\epsilon > 0$ one has

$$\sum_{k=1}^{\infty} P\{\|\delta u_k\| > \epsilon\} \leq \sum_{k=1}^{\infty} \frac{\mathbb{E}\|\delta u_k\|}{\epsilon} < \infty$$

Therefore, $\delta u_k \to 0$ almost surely according to Lemma 1. The proof thus is completed. □

To show the mean square convergence of δu_k, it only needs to show that $\mathbb{E}\delta u_k \delta u_k^T \to 0$. The following theorem shows the convergence.

Theorem 3. *The system* (10) *with data dropouts variable* γ *obeying RSM converges to zero in the mean square sense, if* $\rho(I - QH) < 1$.

Proof. We first consider the initial iteration of every segment. Following similar steps of Theorem 1, there is a suitable constant $0 < \sigma < 1$ such that $\|I - QH\| < \sigma$. Then by (12)

$$\begin{aligned} \left\|\mathbb{E}\left(\delta u_{jK}\delta u_{jK}^T\right)\right\| =& \left\|\mathbb{E}\left(\Gamma_1^{\theta_j}\delta u_{(j-1)K}\delta u_{(j-1)K}^T(\Gamma_1^{\theta_j})^T\right)\right\| \\ \leq& \sigma^2 \left\|\mathbb{E}\left(\delta u_{(j-1)K}\delta u_{(j-1)K}^T\right)\right\| \end{aligned}$$

Thus $\mathbb{E}\left(\delta u_{jK}\delta u_{jK}^T\right) \xrightarrow[j\to\infty]{} 0$.

Next consider the other iterations of the j-th segment except the first one, i.e., $(j-1)K + i$, $i = 1, 2, \cdots, K-1$. Using similar steps in the proof of Theorem 1, it is clear that

$$\left\|\mathbb{E}\left(\delta u_{(j-1)K+i}\delta u_{(j-1)K+i}^T\right)\right\| \leq \left\|\mathbb{E}\left(\delta u_{(j-1)K}\delta u_{(j-1)K}^T\right)\right\|$$

which completes the proof. □

3.2 The BVM Case

Notice that in the BVM case, the data dropout is mutually independent along the iteration axis. Thus there may be a quite long length of consecutive data dropout iterations. Therefore, deterministic analysis techniques in last subsection is no longer workable. As a matter of fact, the convergence analysis for the BVM case could be done by calculating the first and second moments of the switching path, which actually is product of random matrices.

Taking (10) into account, let

$$Z_k = \Gamma_{\gamma_k}\Gamma_{\gamma_{k-1}} \cdots \Gamma_{\gamma_1} \tag{14}$$

where Γ_{γ_k} is a random matrix taking values in $\{\Gamma_0, \Gamma_1\}$ with the following probabilities

$$\begin{aligned} P\{\Gamma_{\gamma_k} = \Gamma_1\} = P\{\gamma_k = 1\} = \overline{\gamma} \\ P\{\Gamma_{\gamma_k} = \Gamma_0\} = P\{\gamma_k = 0\} = 1 - \overline{\gamma} \end{aligned}$$

for all iteration index k.

Then the switched system (10) leads

$$\delta u_{k+1} = Z_k \delta u_0 \tag{15}$$

Remind that $\{\gamma_k\}$ is mutually independent. We first calculate the mean and covariance of the sample path Z_k.

Lemma 2. *Let* $S^k = \{Z_k$: *taken over all sample paths*$\}$*, then the mean of* S^k*, denoted as* M_k*, is given recursively by*

$$M_k = (\overline{\gamma}\Gamma_1 + (1 - \overline{\gamma})\Gamma_0)M_{k-1} \tag{16}$$

Proof. Let $S_i^k = \{Z_k \in S^k : \Gamma_{\gamma_k} = \Gamma_i\}, i = 0, 1$. It is obvious that S^k is the disjoint union of S_0^k and S_1^k. From the definition of the mean, one has

$$
\begin{aligned}
M_k = &\sum_{Z_k \in S^k} P\{Z_k\} Z_k \\
=&\sum_{j=0}^{1} \sum_{Z_{k-1}\in S^{k-1}} P\{\Gamma_{\gamma_k} = \Gamma_j\} P\{Z_{k-1}\} \Gamma_j Z_{k-1} \\
=&\sum_{j=0}^{1} P\{\Gamma_{\gamma_k} = \Gamma_j\} \Gamma_j \sum_{Z_{k-1}\in S^{k-1}} P\{Z_{k-1}\} Z_{k-1} \\
=&(\overline{\gamma}\Gamma_1 + (1-\overline{\gamma})\Gamma_0) \sum_{Z_{k-1}\in S^{k-1}} P\{Z_{k-1}\} Z_{k-1} \\
=&(\overline{\gamma}\Gamma_1 + (1-\overline{\gamma})\Gamma_0) M_{k-1}
\end{aligned}
$$

Thus the proof is completed. □

Lemma 3. *Let $S^k = \{Z_k$: taken over all sample paths$\}$, then the covariance of the S^k, denoted as V_k, is given by*

$$V_k = C_k - M_k M_k^T \tag{17}$$

where C_k is generated recursively as

$$C_k = \overline{\gamma}\Gamma_1 C_{k-1} \Gamma_1^T + (1-\overline{\gamma})\Gamma_0 C_{k-1} \Gamma_0^T \tag{18}$$

Proof. The covariance is calculated as

$$V_k = \sum_{Z_k \in S^k} P\{Z_k\}(Z_k - M_k)(Z_k - M_k)^T$$

Then by decomposition it leads to the following derivation

$$
\begin{aligned}
V_k =&\sum_{j=0}^{1} \sum_{Z_{k-1}\in S^{k-1}} \Big[P\{\Gamma_{\gamma_k} = \Gamma_j\} P\{Z_{k-1}\} \\
&\times (\Gamma_j Z_{k-1} - M_k)(\Gamma_j Z_{k-1} - M_k)^T \Big] \\
=&\sum_{j=0}^{1} P\{\Gamma_{\gamma_k} = \Gamma_j\} \\
&\times \Big[\sum_{Z_{k-1}\in S^{k-1}} P\{Z_{k-1}\} \Gamma_j Z_{k-1} Z_{k-1}^T \Gamma_j^T \\
&- \sum_{Z_{k-1}\in S^{k-1}} P\{Z_{k-1}\} M_k Z_{k-1}^T \Gamma_j^T \\
&- \sum_{Z_{k-1}\in S^{k-1}} P\{Z_{k-1}\} \Gamma_j Z_{k-1} M_k^T \\
&+ \sum_{Z_{k-1}\in S^{k-1}} P\{Z_{k-1}\} M_k M_k^T \Big] \\
=&\sum_{j=0}^{1} P\{\Gamma_{\gamma_k} = \Gamma_j\} \\
&\times \Big[\sum_{Z_{k-1}\in S^{k-1}} P\{Z_{k-1}\} \Gamma_j Z_{k-1} Z_{j-1}^T \Gamma_j^T \\
&- M_k M_{k-1}^T \Gamma_j^T - \Gamma_j M_{k-1} M_k^T + M_k M_k^T \Big] \\
=&\overline{\gamma} \sum_{Z_{k-1}\in S^{k-1}} P\{Z_{k-1}\} \Gamma_1 Z_{k-1} Z_{k-1}^T \Gamma_1^T \\
&(1-\overline{\gamma}) \sum_{Z_{k-1}\in S^{k-1}} P\{Z_{k-1}\} \Gamma_0 Z_{k-1} Z_{k-1}^T \Gamma_0^T \\
&- M_k M_k^T
\end{aligned}
$$

On the other hand

$$V_k = \sum_{Z_k \in S^k} P\{Z_k\}(Z_k Z_k^T - M_k M_k^T)$$

Let

$$C_k = \sum_{Z_k \in S^k} P\{Z_k\} Z_k Z_k^T$$

then it is obvious that

$$C_k = \overline{\gamma}\Gamma_1 C_{k-1} \Gamma_1^T + (1-\overline{\gamma})\Gamma_0 C_{k-1} \Gamma_0^T$$

by combing the last two expressions of V_k. This completes the proof. □

The following theorems prove the convergence of switched system (10) for the BVM case.

Theorem 4. *System* (10) *with data dropouts variable γ obeying BVM converges to zero in the sense of $\mathbb{E}\delta u_k \xrightarrow[k\to\infty]{} 0$, if $\rho(I - QH) < 1$.*

Proof. From (15), one has

$$\mathbb{E}\delta u_k = M_{k-1} \mathbb{E}\delta u_0$$

Then from the recurrence of M_k, i.e., (16), it is obvious to have

$$\mathbb{E}\delta u_k = (\overline{\gamma}\Gamma_1 + (1-\overline{\gamma})\Gamma_0)^{k-1} \mathbb{E}\delta u_0$$

If $\rho(I - QH) < 1$, i.e., all eigenvalues of Γ_1 lie interior to the unit circle, it is effortless to reach that all eigenvalues of $\overline{\gamma}\Gamma_1 + (1-\overline{\gamma})\Gamma_0$ lie interior to the unit circle by noticing that $\Gamma_0 = I$, whose every eigenvalue is 1. Thus the proof is completed. □

Theorem 5. *System* (10) *with data dropouts variable γ obeying BVM converges to zero almost surely, if $\rho(I - QH) < 1$.*

Proof. Based on the condition $\rho(I - QH) < 1$ and noticing that $\Gamma_0 = I$, it follows that

$$|\rho(\overline{\gamma}\Gamma_1 + (1-\overline{\gamma})\Gamma_0)| < 1$$

Further, one could have

$$\overline{\gamma}\|\Gamma_1\| + (1-\overline{\gamma})\|\Gamma_0\| < 1$$

where $\|\cdot\|$ is the induced norm.

Notice that the data dropouts happen independently along the iteration axis, thus directly calculating the expectation by similar steps of Lemma 2, one has

$$
\begin{aligned}
\mathbb{E}\|\delta u_k\| &= (\overline{\gamma}\|\Gamma_1\| + (1-\overline{\gamma})\|\Gamma_0\|)\, \mathbb{E}\|\delta u_{k-1}\| \\
&= (\overline{\gamma}\|\Gamma_1\| + (1-\overline{\gamma})\|\Gamma_0\|)^k\, \mathbb{E}\|\delta u_0\|
\end{aligned}
$$

Thus it is obvious that

$$\sum_{k=1}^{\infty} \mathbb{E}\|\delta u_k\| < \infty$$

Then by Markov inequality, for any $\epsilon > 0$ one has

$$\sum_{k=1}^{\infty} P\{\|\delta u_k\| > \epsilon\} \leq \sum_{k=1}^{\infty} \frac{\mathbb{E}\|\delta u_k\|}{\epsilon} < \infty$$

Therefore, $\delta u_k \to 0$ almost surely by Lemma 1. The proof thus is completed. □

To prove the mean square convergence of δu_k, it only needs to show that $\mathbb{E}\delta u_k \delta u_k^T \to 0$, which corresponds to C_m in Lemma 3. Let us first give the following lemma.

Lemma 4. *Matrix C_k defined by* (18) *is positive definite.*

Proof. First note that $C_0 = I$. Assume that C_{k-1} is positive definite, then one has that $x^T C_k x = \overline{\gamma} x^T \Gamma_1 C_{k-1} \Gamma_1^T x + (1-\overline{\gamma}) x^T \Gamma_0 C_{k-1} \Gamma_0^T x > 0, \forall x \neq 0$. Thus the lemma is valid by mathematical induction. □

Then by the recurrence of C_k defined in (18), we can have the following mean square convergence result.

Theorem 6. *The system* (10) *with data dropouts variable γ obeying BVM converges to zero in the mean square sense, if $\rho(I - QH) < 1$.*

Proof. Our target is to show $\mathbb{E}\delta u_k \delta u_k^T \to 0$ as $k \to \infty$, which is sufficient to prove the exponentially stable of C_k.

Following similar steps of the proof of Theorem 5, there is a suitable constant $0 < \eta < 1$ such that

$$\overline{\gamma}\|\Gamma_1\|^2 + (1-\overline{\gamma})\|\Gamma_0\|^2 < \eta$$

Then from the recurrence of C_k one has

$$\begin{aligned}
\|C_k\| =& \|\overline{\gamma}\Gamma_1 C_{k-1}\Gamma_1^T + (1-\overline{\gamma})\Gamma_0 C_{k-1}\Gamma_0^T\| \\
\leq& \|\overline{\gamma}\Gamma_1 C_{k-1}\Gamma_1^T\| + \|(1-\overline{\gamma})\Gamma_0 C_{k-1}\Gamma_0^T\| \\
\leq& \overline{\gamma}\|\Gamma_1\|^2\|C_{k-1}\| + (1-\overline{\gamma})\|\Gamma_0\|^2\|C_{k-1}\| \\
=& \left(\overline{\gamma}\|\Gamma_1\|^2 + (1-\overline{\gamma})\|\Gamma_0\|^2\right)\|C_{k-1}\| \\
<& \eta\|C_{k-1}\|
\end{aligned}$$

Thus the exponential stability of C_k is established and hence $C_k \to 0$. The proof is completed. □

3.3 The MCM Case

Now revisit the switch regression (10). It is easily seen that $\{\delta u_k, k = 0, 1, 2, \cdots\}$ is not a Markov process when γ_k is modeled by a Markov chain. Thus the techniques used for the BVM case in last subsection can not be applied in the MCM case. However, the joint process $\{\delta u_k, \gamma_k\}$ is a Markov process, thus we could handle this case by considering δu_k and γ_k simultaneously. To this end, define the indicator function $I_{\{\text{event}\}}$ as

$$I_{\{\text{event}\}} = \begin{cases} 1, & \text{if the event indicated is fulfilled} \\ 0, & \text{otherwise} \end{cases}$$

With the help of this indicator function, we could work with $\delta u_k I_{\{\gamma_k = i\}}$ and $\delta u_k \delta u_k^T I_{\{\gamma_k = i\}}$ instead of δu_k and $\delta u_k \delta u_k^T$ themselves, respectively, $i = 0, 1$. Using the techniques in [28], one can to obtain the difference equation like M_k and C_k in the BVM case.

Notice that

$$\mathbb{E}\delta u_k = \sum_{i=0}^{1} \mathbb{E}\left(\delta u_k I_{\{\gamma_k = i\}}\right) \tag{19}$$

and

$$\mathbb{E}\delta u_k \delta u_k^T = \sum_{i=0}^{1} \mathbb{E}\left(\delta u_k \delta u_k^T I_{\{\gamma_k = i\}}\right) \tag{20}$$

The following notations are introduced for further analysis, $i = 0, 1$,

$$\begin{aligned}
\phi_k(i) &= \mathbb{E}\left(\delta u_k I_{\{\gamma_k = i\}}\right) \\
\Phi_k &= \begin{pmatrix} \phi_k(0) \\ \phi_k(1) \end{pmatrix} \\
\psi_k(i) &= \mathbb{E}\left(\delta u_k \delta u_k^T I_{\{\gamma_k = i\}}\right) \\
\Psi_k &= (\psi_k(0), \psi_k(1))
\end{aligned}$$

Then it follows that $\mathbb{E}\delta u_k = \phi_k(0) + \phi_k(1)$ and $\mathbb{E}\delta u_k \delta u_k^T = \psi_k(0) + \psi_k(1)$. Now we give the recursive difference equations for $\phi_k(i)$ and $\psi_k(i)$.

Lemma 5. *Consider system* (10) *where γ_k is modeled by MCM, then*

1) $\phi_{k+1}(j) = \sum_{i=0}^{1} p_{ij}\Gamma_i \phi_k(i)$;

2) $\psi_{k+1}(j) = \sum_{i=0}^{1} p_{ij}\Gamma_i \psi_k(i)\Gamma_i^T$.

where p_{ij}, $i, j = 0, 1$ is defined in (6).

Proof. For 1), it comes from

$$\begin{aligned}
\phi_{k+1}(j) &= \sum_{i=0}^{1} \mathbb{E}\left(\Gamma_i \delta u_k I_{\{\gamma_{k+1} = j\}} I_{\{\gamma_k = i\}}\right) \\
&= \sum_{i=0}^{1} \Gamma_i \mathbb{E}\left(\delta u_k I_{\{\gamma_k = i\}} P\{\gamma_{k+1} = j | \gamma_k = i\}\right) \\
&= \sum_{i=0}^{1} p_{ij}\Gamma_i \phi_k(i)
\end{aligned}$$

For 2), notice that

$$\begin{aligned}
\psi_{k+1}(j) &= \sum_{i=0}^{1} \mathbb{E}\left(\Gamma_i \delta u_k (\Gamma_i \delta u_k)^T I_{\{\gamma_{k+1} = j\}} I_{\{\gamma_k = i\}}\right) \\
&= \sum_{i=0}^{1} \Gamma_i \mathbb{E}\left(\delta u_k \delta u_k^T I_{\{\gamma_k = i\}} P\{\gamma_{k+1} = j | \gamma_k = i\}\right)\Gamma_i^T \\
&= \sum_{i=0}^{1} p_{ij}\Gamma_i \psi_k(i)\Gamma_i^T
\end{aligned}$$

The proof is thus completed. □

I_n denotes an $n \times n$ identity matrix, where n is the length of an iteration defined in (1) or H matrix. Then define

$$\mathcal{M} = (P^T \otimes I_n) \begin{bmatrix} \Gamma_0 & 0 \\ 0 & \Gamma_1 \end{bmatrix}$$

where P is the probability transition matrix defined in MCM.

From Lemma 5, one obtains the following relationship

$$\Phi_{k+1} = \mathcal{M}\Phi_k \tag{21}$$

Thus it is easy to declare that if the spectral radius of $\mathcal{M}$ is less than 1, then $\Phi_k \xrightarrow[k\to\infty]{} 0$, which means that $\mathbb{E}\delta u_k \xrightarrow[k\to\infty]{} 0$. We formulate it in the following theorem without a proof.

Theorem 7. *System* (10) *with data dropouts variable* γ *obeying MCM converges to zero in the sense of* $\mathbb{E}\delta u_k \xrightarrow[k\to\infty]{} 0$*, if* $\rho(\mathcal{M}) < 1$.

For Ψ_k, it is unable to construct a difference equation similar to (21). Instead, let us introduce the operator $\mathcal{T}$ according to Lemma 5 in order to give a compact formulation of Ψ_k,

$$\mathcal{T}(\cdot) = (\mathcal{T}_0(\cdot), \mathcal{T}_1(\cdot))$$

where

$$\mathcal{T}_j(\Psi_k) = \sum_{i=0}^{1} p_{ij}\Gamma_i\psi_k(i)\Gamma_i^T, \quad j = 0, 1$$

Then by Lemma 5 one has

$$\Psi_{k+1} = \mathcal{T}(\Psi_k) \tag{22}$$

Then it is seen that the convergence analysis should be given on the operator $\mathcal{T}$ rather than some specific difference equation, which involves additional difficulties. To make this paper neat, some complicated proofs are omitted here and will be provided in another paper.

It is evident that if $\rho(\mathcal{T}) < 1$, then Ψ_k tends to zero as $k \to \infty$. This is formulated in the following theorem.

Theorem 8. *System* (10) *with data dropouts variable* γ *obeying BVM converges to zero in the mean square sense, if* $\rho(\mathcal{T}) < 1$.

Proof. Recalling from (22), one has $\Psi_k = \mathcal{T}^k(\Psi_0)$ and

$$\mathbb{E}\delta u_k \delta u_k^T = \psi_k(0) + \psi_k(1) = \mathcal{T}_0^k(\Psi_0) + \mathcal{T}_1^k(\Psi_0)$$

Therefore, it follows from $\rho(\mathcal{T}) < 1$ that $\mathbb{E}\delta u_k \delta u_k^T \xrightarrow[k\to\infty]{} 0$. □

However, the condition $\rho(\mathcal{T}) < 1$ is not convenient to check for application. As have been studied in many publications related to Markov jump systems, the above condition is equivalent to the following one

$$\rho\left((P^T \otimes I_{n^2})\text{diag}(\Gamma_i \otimes \Gamma_i)\right) < 1, \tag{23}$$

Or, one can check the following Riccati-like conditions: for any positive matrices $E_0 >$ and $E_1 > 0$, there exist positive matrices $F_0 > 0$ and $F_1 > 0$ such that

$$\begin{aligned} p_{00}\Gamma_0 F_0 \Gamma_0^T + p_{10}\Gamma_1 F_1 \Gamma_1^T - F_0 = -E_0 \\ p_{01}\Gamma_0 F_0 \Gamma_0^T + p_{11}\Gamma_1 F_1 \Gamma_1^T - F_1 = -E_1 \end{aligned}$$

If the Markov model degenerates to the BVM case, i.e., $p_{ij} = p_j$ for every i, j. Then the condition reduces to that every eigenvalue of $p_0\Gamma_0 \otimes \Gamma_0 + p_1\Gamma_1 \otimes \Gamma_1$ lies interior to the unit circle. Noticing that $\rho(A \otimes B) = \rho(A)\rho(B)$, it follows that this condition coincides with the BVM case, which actually is a special case of MCM.

Theorem 9. *System* (10) *with data dropouts variable* γ *obeying MCM converges to zero almost surely, if* $\rho(\mathcal{T}) < 1$.

Proof. If $\rho(\mathcal{T}) < 1$, then for some $\alpha > 0$ and $0 < \beta < 1$, one has $\|\mathcal{T}^k\| \leq \alpha\beta^k$. Therefore, recalling (22), it follows

$$\begin{aligned} \mathbb{E}\|\delta u_k\|^2 =& \mathbb{E}\left(\text{tr}(\delta u_k \delta u_k^T)\right) \\ =& \text{tr}\mathbb{E}\left(\delta u_k \delta u_k^T I_{\{\gamma_k=0\}}\right) + \text{tr}\mathbb{E}\left(\delta u_k \delta u_k^T I_{\{\gamma_k=0\}}\right) \\ =& \text{tr}(\psi_k(0)) + \text{tr}(\psi_k(1)) \leq n(\|\psi_k(0)\| + \|\psi_k(1)\|) \\ =& n\|\Psi_k\| \leq n\|\mathcal{T}^k\|\|\Psi_0\| \leq n\alpha\beta^k\|\delta u_0\|^2 \end{aligned}$$

Therefore,

$$\sum_{k=0}^{\infty} \mathbb{E}\|\delta u_k\|^2 < \infty$$

The proof is completed by using Lemma 1. □

Remark 5. *How to link the design condition and convergence condition in the MCM case is somewhat a difficult problem, which needs further efforts. There have been quite a large amount of studies on stability of Markov jump/switch system that would help us a lot to deal this issue. Besides, the design condition would be enough if some practical situation is considered, such as the stochastic* τ*-stability in [29].*

3.4 Further Remarks

Remark 6. *Generally speaking, convergence in mean square sense and in almost sure sense cannot be implicit by each other. The underling reason that we could established the two convergence properties meanwhile is that the convergence in mean square sense is stronger than the native one. As we could see, the convergence in mean square sense actually is exponential, which hits that* $\sum_{k=1}^{\infty} Var(\delta u_k) < \infty$*, whence by Chebyshev's inequality and Borel-Cantelli lemma, the almost sure convergence of* δu_k *to zero is also established.*

Remark 7. *In [15] it is remarked that as long as the measurements are not totally lost, the convergence is always guaranteed. As one could see above, it is only required that* $0 < \overline{\gamma} < 1$*, for the data dropout rate, to show the convergence in both almost sure and mean square senses. Thus our study also confirms this conclusion.*

Remark 8. *The more general case of* (9) *with* n *alternative models has been studied in some previous publications [26, 27], where the distribution of random switch parameters are assumed as uniformly distribution [26] and poisson distribution [27]. But the techniques used there help us to deal with the convergence analysis for the BVM case.*

4 Concluding Remarks

In this paper, the iterative learning control for networked discrete-time systems with random data dropouts is considered. The data dropout is described by three different kinds of mathematical models, namely, an arbitrary random sequence, a binary Bernoulli random variable, and a Markov chain, respectively. The Bernoulli random variable model has been studied a lot in many previous publications, while the other two have attracted little attention. In this paper, not only the three models are considered, but also we provide a novel approach, called switched system approach, to guarantee the convergence of the iterative learning control algorithm under random data dropouts. To this end, the original iterative learning control model is first reformulated into a switched system form. Then the convergence analysis for different data dropouts models is detailed in the expectation sense, mean square sense, and almost sure sense, respectively. For the random sequence model, the deterministic analysis techniques are used; for the Bernoulli random variable model, the first and second moments of the random path are calculated to show the convergence; while for the Markov chain model, some operators are introduced to give a convergence condition. Due to limited space, the illustrative simulations are omitted in this paper.

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