Control and Robustness for Quantum Linear Systems

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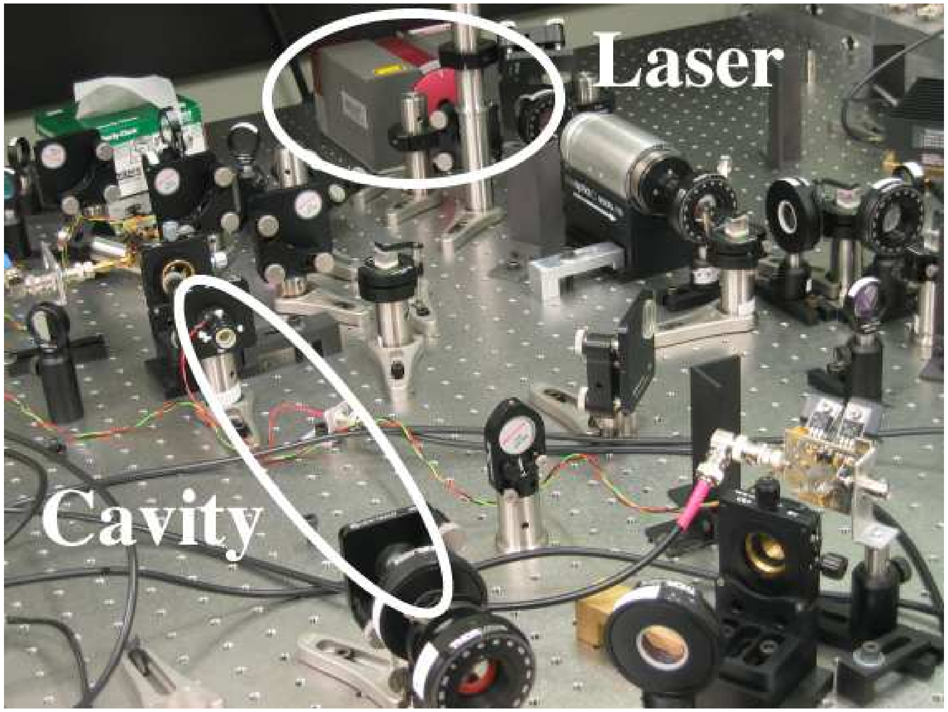
Introduction

- Developments in quantum technology and quantum information provide an important motivation for research in the area of quantum feedback control systems.

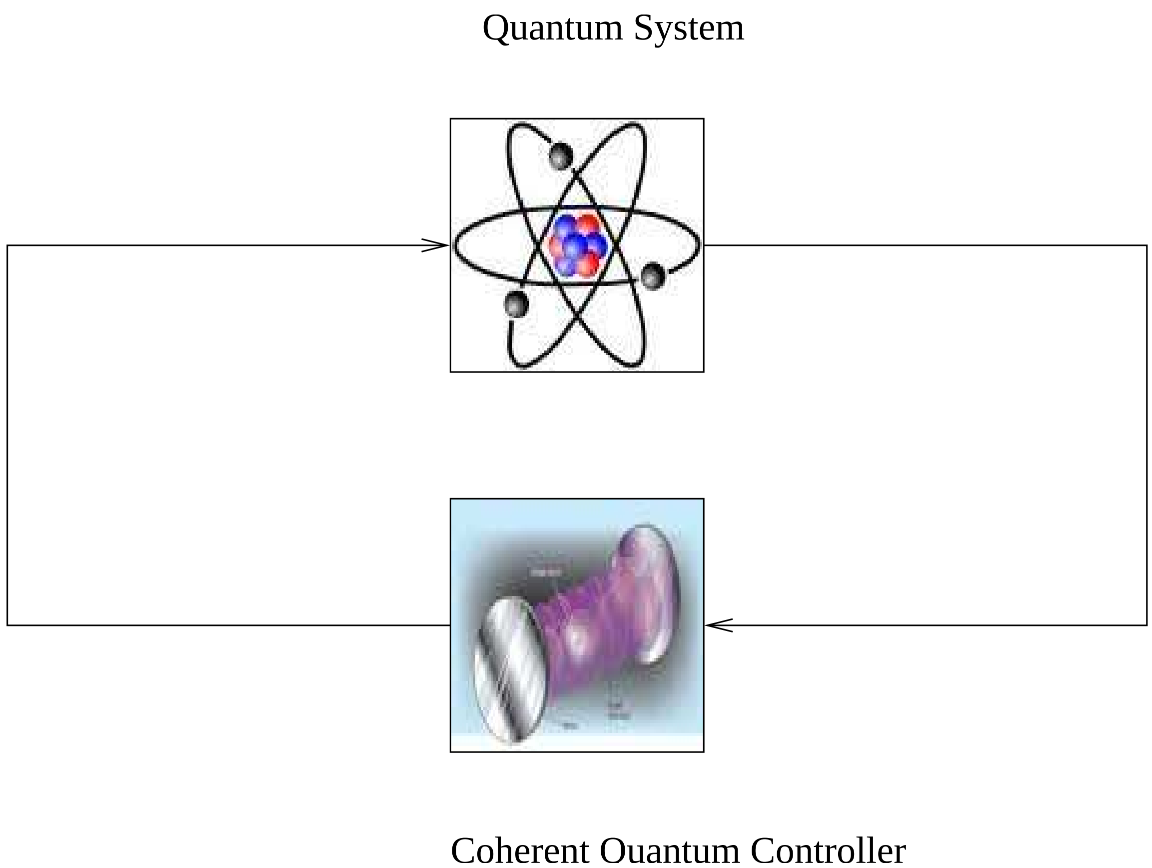


A linear quantum optics experiment at UNSW Canberra Photo courtesy of Elanor Huntington.

- The most recent Nobel prize for Physics was awarded for work in experimental (open loop) quantum control:
  Serge Haroche and David J. Wineland
  ”for ground-breaking experimental methods that enable measuring and manipulation of individual quantum systems”
- The stage will soon be reached where the development of quantum technologies will require advances in engineering rather than physics and quantum control theory is expected to play an important role here.
- Quantum physics places fundamental limits on accuracy in estimation and control. These will be the dominant issues in quantum technology.
- New control theories will be needed to deal with models of systems described by the laws of quantum physics rather than classical physics.

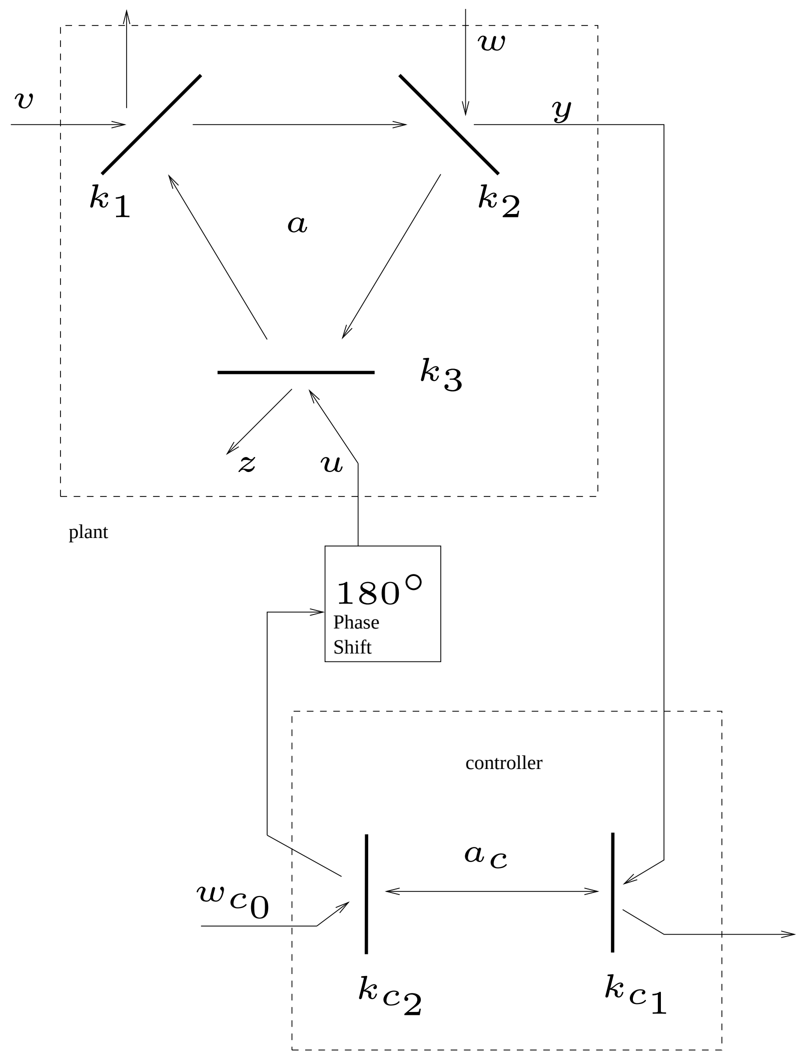
- Feedback control of quantum optical systems has potential applications in areas such as quantum communications, quantum teleportation, quantum computing, quantum error correction and gravity wave detection.
- Feedback control of quantum systems aims to achieve closed loop properties such as stability, robustness and entanglement.
- We consider models of quantum systems as quantum stochastic differential equations (QSDEs)
- These stochastic models can be used to describe quantum optical devices such as optical cavities, linear quantum amplifiers, and finite bandwidth squeezers.

- Recent papers on the feedback control of linear quantum systems have considered the case in which the feedback controller itself is also a quantum system. Such feedback control is often referred to as *coherent quantum control.*



Coherent quantum feedback control.

- One motivation for considering such coherent quantum control problems is that coherent controllers have the potential to achieve improved performance since quantum measurements inherently involve the destruction of quantum information.
- Also, coherent optical controllers may be much more practical to implement than measurement feedback based controllers.
- In a paper (James, Nurdin, Petersen, 2008), the coherent quantum H^∞ control problem was addressed.
- This paper obtained a solution to this problem in terms of a pair of algebraic Riccati equations.
- Also, in a paper (Nurdin, James, Petersen, 2009) the coherent quantum LQG problem was addressed.
- An example of a coherent quantum H^∞ system considered in (Nurdin, James, Petersen, 2008), (Maalouf Petersen 2011) is described by the following diagram:


v
w
y
k1
k2
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k3
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plant
180°
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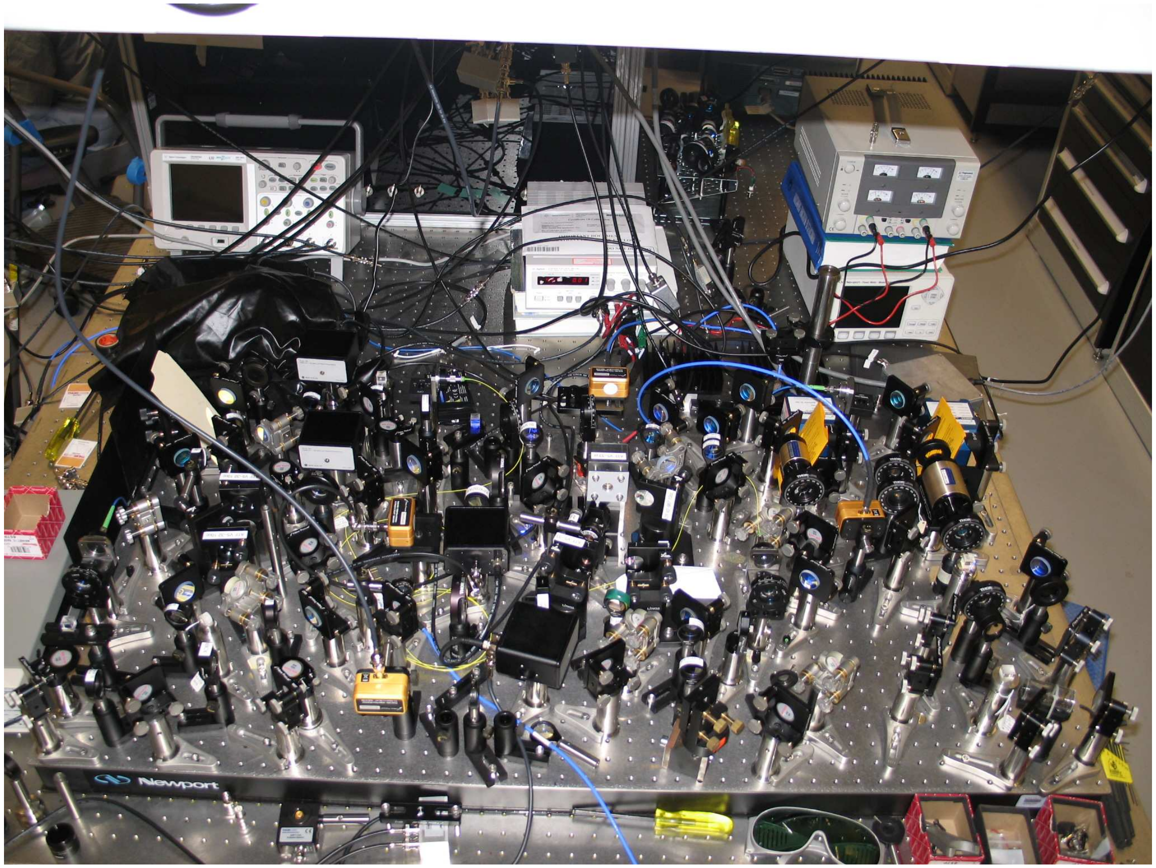
- The coherent quantum H^{∞} control approach of James Nurdin and Petersen (2008) was subsequently implemented experimentally by Hideo Mabuchi of Stanford University:



- We formulate quantum system models in the **Heisenberg Picture** of quantum mechanics which describes the time evolution of operators representing system variables such as position and momentum.
- This is as opposed to the Schrödinger picture which describes quantum systems in terms of the time evolution of the quantum state.

Werner Heisenberg

Linear Quantum System Models

- We formulate a class of linear quantum system models described by quantum stochastic differential equations (QSDEs) derived from the quantum harmonic oscillator (an infinite level quantum system).
- We begin by considering a collection of n independent quantum harmonic oscillators which are defined on a Hilbert space $\mathcal{H}$.
- Corresponding to this is a vector of *annihilation operators* a:

$$a = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}.$$

- Each annihilation operator a_i is an unbounded linear operator on $\mathcal{H}$.

- The adjoint of the operator a_i is denoted a_i^* and is referred to as a *creation operator*. We use $a^\#$ to denote the vector of a_i^*s:

$$a^\# = \begin{bmatrix} a_1^* \\ a_2^* \\ \vdots \\ a_n^* \end{bmatrix}.$$

- Physically, these operators correspond to the annihilation and creation of a photon respectively.
- Also, we use the notation $a^T = \begin{bmatrix} a_1 & a_2 & \ldots & a_n \end{bmatrix}$, and $a^\dagger = \left(a^\#\right)^T = \begin{bmatrix} a_1^* & a_2^* & \ldots & a_n^* \end{bmatrix}$.

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Canonical Commutation Relations

- The operators a_i and a_i^* are such that the following *canonical commutation relations* are satisfied

$$[a_i, a_j^*] := a_i a_j^* - a_j^* a_i = \delta_{ij}$$

where δ_{ij} denotes the Kronecker delta multiplied by the identity operator on the Hilbert space $\mathcal{H}$.

- We also have the commutation relations

$$[a_i, a_j] = 0, \ [a_i^*, a_j^*] = 0.$$

- These relations encapsulate Heisenberg's uncertainty relation.

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- Using the above operator vector notation, the commutation relations can be written as

$$\begin{aligned}\left[\begin{bmatrix} a \\ a^{\#} \end{bmatrix}, \begin{bmatrix} a \\ a^{\#} \end{bmatrix}^{\dagger}\right] &= \begin{bmatrix} a \\ a^{\#} \end{bmatrix}\begin{bmatrix} a \\ a^{\#} \end{bmatrix}^{\dagger} - \left(\begin{bmatrix} a \\ a^{\#} \end{bmatrix}^{\#}\begin{bmatrix} a \\ a^{\#} \end{bmatrix}^{T}\right)^{T} \\ &= \Theta\end{aligned}$$

where $\Theta = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}$ is the commutation matrix.

Quantum Wiener Processes

- The quantum harmonic oscillators described above are assumed to be coupled to m external independent quantum fields modeled by a vector of bosonic annihilation field operators $\mathcal{A}(t)$.
- For each annihilation field operator $\mathcal{A}_j(t)$, there is a corresponding creation field operator $\mathcal{A}_j^*(t)$.
- These quantum fields may be electromagnetic fields such as a light beam produced by a laser.

Hamiltonian, Coupling and Scattering Operators

- In order to describe the joint evolution of the quantum harmonic oscillators and quantum fields, we first specify the *Hamiltonian operator* for the quantum system which is a self adjoint operator on $\mathcal{H}$ of the form

$$H = \frac{1}{2}\left[\, a^{\dagger} \;\; a^{T} \,\right] M \begin{bmatrix} a \\ a^{\#} \end{bmatrix}$$

where $M \in \mathbb{C}^{2n\times 2n}$ is a Hermitian matrix of the form

$$M = \begin{bmatrix} M_1 & M_2 \\ M_2^{\#} & M_1^{\#} \end{bmatrix}$$

and $M_1 = M_1^{\dagger}$, $M_2 = M_2^{T}$.

- Here, $M^{\dagger}$ denotes the complex conjugate transpose of the complex matrix M, M^{T} denotes the transpose of the complex matrix M, and $M^{\#}$ denotes the complex conjugate of the complex matrix M.

- Also, we specify the *coupling operator* for the quantum system to be a vector of operators of the form

$$L = \begin{bmatrix} N_1 & N_2 \end{bmatrix} \begin{bmatrix} a \\ a^\# \end{bmatrix}$$

where $N_1 \in \mathbb{C}^{m \times n}$ and $N_2 \in \mathbb{C}^{m \times n}$. This describes the interaction between the quantum fields and the quantum system.

- Also, we write

$$\begin{bmatrix} L \\ L^\# \end{bmatrix} = N \begin{bmatrix} a \\ a^\# \end{bmatrix} = \begin{bmatrix} N_1 & N_2 \\ N_2^\# & N_1^\# \end{bmatrix} \begin{bmatrix} a \\ a^\# \end{bmatrix}.$$

- In addition, we define a *scattering matrix* which is a unitary matrix $S \in \mathbb{C}^{m \times m}$. This describes the interactions between the different quantum fields.

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Quantum Stochastic Differential Equations

- The quantities (S, L, H) define the joint evolution of the quantum harmonic oscillators and the quantum fields.
- A set of QSDEs describing the quantum system can be obtained.
- The QSDEs for the linear quantum system can be written as

$$\begin{bmatrix} da(t) \\ da(t)^{\#} \end{bmatrix} = F \begin{bmatrix} a(t) \\ a(t)^{\#} \end{bmatrix} dt + G \begin{bmatrix} d\mathcal{A}(t) \\ d\mathcal{A}(t)^{\#} \end{bmatrix};$$

$$\begin{bmatrix} d\mathcal{A}^{out}(t) \\ d\mathcal{A}^{out}(t)^{\#} \end{bmatrix} = H \begin{bmatrix} a(t) \\ a(t)^{\#} \end{bmatrix} dt + K \begin{bmatrix} d\mathcal{A}(t) \\ d\mathcal{A}(t)^{\#} \end{bmatrix}.$$

- In many ways these systems behave like classical (complex) linear stochastic systems except that the variables $a_i(t)$ and $a_i^*(t)$ are non-commutative.
- Also, there are restrictions on the structure of the matrices F, G, H, K (physical realizability).

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- We can derive the following formulas for the matrices in the QSDE model, in terms of the Hamiltonian and coupling matrices.

$$
\begin{aligned}
F &= -i\Theta M - \frac{1}{2}\Theta N^\dagger J N; \\
G &= -\Theta N^\dagger \begin{bmatrix} S & 0 \\ 0 & -S^\# \end{bmatrix}; \\
H &= N; \\
K &= \begin{bmatrix} S & 0 \\ 0 & S^\# \end{bmatrix}.
\end{aligned}
$$

- Here

$$
J = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}.
$$

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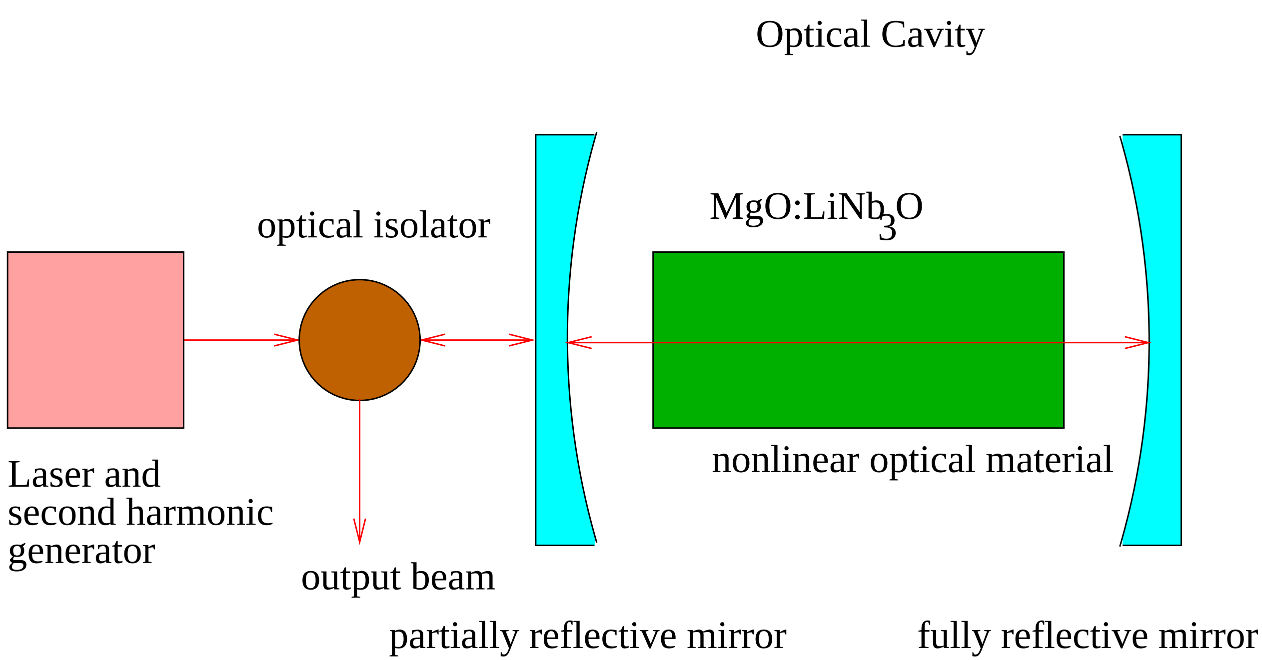
Example

Optical Parametric Amplifier: Squeezer

- An optical parametric amplifier (OPA) can be used to produce squeezed light in which the quantum noise in one quadrature is squeezed relative to the noise in the other quadrature and yet Heisenberg's uncertainty relation still holds.

Schematic diagram of a squeezer.



- A simplified model of such a squeezer has (S, L, H) parameters
  - $S = I$;
  - $L = \sqrt{2\kappa}a$;
  - $H = \frac{i}{2}\chi\left(a^2 - {a^\dagger}^2\right)$.
- Here κ is a parameter depending on the reflectivity of the partially reflecting mirror and χ is a complex parameter depending on the strength of the nonlinear optical material.
- Using the above formulas, this leads to an approximate linearized QSDE model of a squeezer as follows:

$$\begin{aligned} da &= -\left(\kappa a + \chi a^*\right) dt + \sqrt{2\kappa} d\mathcal{A}; \\ da^* &= -\left(\kappa a^* + \chi^* a\right) dt + \sqrt{2\kappa} d\mathcal{A}^*. \end{aligned}$$

- This model is a QSDE quantum linear system model of the form considered above.

Physical Realizability

- Not all QSDEs of the form considered above correspond to physical quantum systems which satisfy all of the laws of quantum mechanics.
- For physical systems, the laws of quantum mechanics require that the commutation relations be satisfied for all times.
- This motivates a notion of physical realizability.
- This notion is of particular importance in the problem of coherent quantum feedback control in which the controller itself is a quantum system.
- In this case, if a controller is synthesized using a method such as quantum H^∞ control or quantum LQG control, it important that the controller can be implemented as a physical quantum system.

Definition. *QSDEs of the form considered above are* physically realizable *if there exist suitably structured complex matrices* $\Theta = \Theta^\dagger$, $M = M^\dagger$, N, S *such that* $S^\dagger S = I$, *and*

$$F = -i\Theta M - \frac{1}{2}\Theta N^\dagger J N; \;\; G = -\Theta N^\dagger \begin{bmatrix} S & 0 \\ 0 & -S^\# \end{bmatrix};$$

$$H = N; \;\; K = \begin{bmatrix} S & 0 \\ 0 & S^\# \end{bmatrix};$$

- The conditions in the above definition require that the QSDEs correspond to a collection of quantum harmonic oscillators with dynamics defined by the operators

$$\left(S, L = N \begin{bmatrix} a \\ a^\# \end{bmatrix}, H = \frac{1}{2} \begin{bmatrix} a^\dagger & a^T \end{bmatrix} M \begin{bmatrix} a \\ a^\# \end{bmatrix} \right).$$

Theorem. *The above QSDEs are physically realizable if and only if there exist complex matrices* $\Theta = \Theta^\dagger$ *and* S *such that* $S^\dagger S = I$*, and*

$$F\Theta + \Theta F^\dagger + GJG^\dagger = 0;$$

$$G = -\Theta H^\dagger \begin{bmatrix} S & 0 \\ 0 & -S^\# \end{bmatrix};$$

$$K = \begin{bmatrix} S & 0 \\ 0 & S^\# \end{bmatrix}; \text{ where } J = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}$$

- Note that the first of these conditions is equivalent to the preservation of the commutation relations for all times.

- We can also characterize physical realizability in terms of the transfer function matrix $\Gamma(s) = H\left(sI - F\right)^{-1} G + K$ of a linear quantum system.

Theorem. *(See Shaiju and Petersen, 2012) The linear quantum system defined by the above QSDEs is physically realizable if and only if the system transfer function matrix $\Gamma(s)$ satisfies*

$$\Gamma(-s^*)^\dagger J \Gamma(s) = J$$

(i.e., the system is (J, J)-unitary) and the matrix K is of the form $K = \begin{bmatrix} S & 0 \\ 0 & S^\# \end{bmatrix}$ where $S^\dagger S = I$.

Recap

- Coherent feedback controllers for linear quantum systems can be synthesized in a variety of ways to ensure that the controller is physically realizable.
- These include quantum H^∞ control

M. R. James, H. I. Nurdin, and I. R. Petersen, "H^∞ control of linear quantum stochastic systems," *IEEE Transactions on Automatic Control*, vol. 53, no. 8, pp. 1787–1803, 2008.

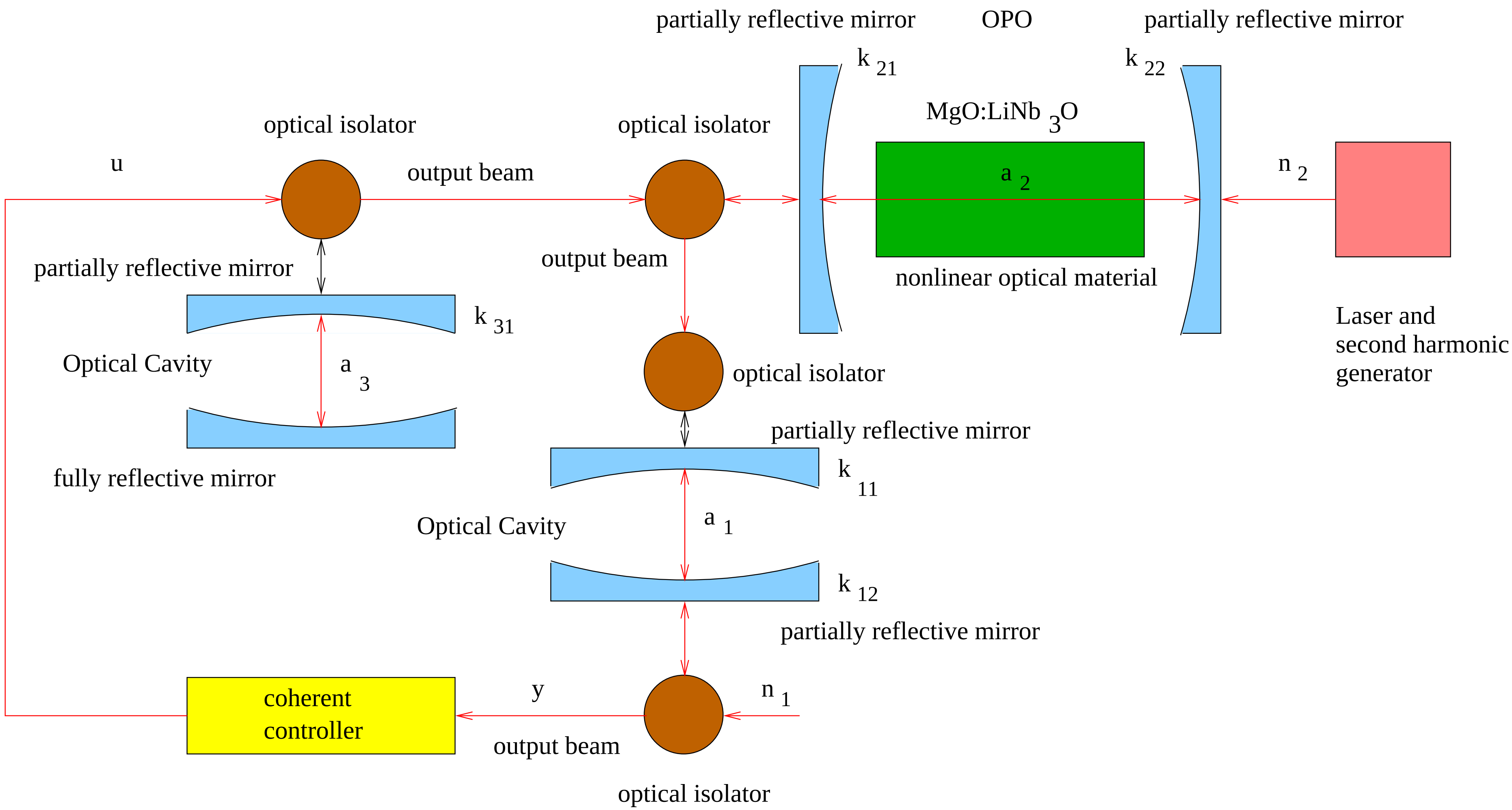
- Another approach is quantum LQG control

H. I. Nurdin, M. R. James, and I. R. Petersen, "Coherent quantum LQG control," *Automatica*, vol. 45, no. 8, pp. 1837–1846, 2009.

- However, so far this quantum LQG problem has only been solved by brute force optimization methods and there do not as yet general methods to solve large scale quantum LQG control problems.

Example

- We consider a problem of stabilization via coherent feedback control.
- In this example, an unstable linear quantum optical system is to be controlled via a coherent quantum feedback controller in which the controller itself is a quantum system. This requires that the controller is physically realizable.
- The quantum system to be controlled consists of the cascade connection of an optical parametric oscillator (OPO) and two optical cavities.



Schematic diagram of coherent quantum control system.

- The following linear quantum system model is constructed for the quantum plant.

$$\begin{bmatrix} da_1(t) \\ da_1(t)^* \\ da_2(t) \\ da_2(t)^* \\ da_3(t) \\ da_3(t)^* \end{bmatrix} = F \begin{bmatrix} a_1(t) \\ a_1(t)^* \\ a_2(t) \\ a_2(t)^* \\ a_3(t) \\ a_3(t)^* \end{bmatrix} dt + G \begin{bmatrix} du(t) \\ du(t)^* \end{bmatrix} + D \begin{bmatrix} dn_1(t) \\ dn_1(t)^* \\ dn_2(t) \\ dn_2(t)^* \end{bmatrix};$$

$$\begin{bmatrix} dy(t) \\ dy(t)^* \end{bmatrix} = H \begin{bmatrix} a_1(t) \\ a_1(t)^* \\ a_2(t) \\ a_2(t)^* \\ a_3(t) \\ a_3(t)^* \end{bmatrix} dt + K \begin{bmatrix} dn_1(t) \\ dn_1(t)^* \\ dn_2(t) \\ dn_2(t)^* \end{bmatrix}.$$

- Here

$$F = \begin{bmatrix} -0.5007 & 0 & -0.0374 & 0 & -0.0410 & 0 \\ 0 & -0.5007 & 0 & -0.0374 & 0 & -0.0410 \\ 0 & 0 & -1.0000 & -1.0500 & -1.0954 & 0 \\ 0 & 0 & -1.0500 & -1.0000 & 0 & -1.0954 \\ 0 & 0 & 0 & 0 & -0.6000 & 0 \\ 0 & 0 & 0 & 0 & 0 & -0.6000 \end{bmatrix};$$

$$G = \begin{bmatrix} -0.0374 & 0 \\ 0 & -0.0374 \\ -1.0000 & 0 \\ 0 & -1.0000 \\ -1.0954 & 0 \\ 0 & -1.0954 \end{bmatrix};$$

$$H = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}; K = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}.$$

- The eigenvalues of the matrix F in this quantum plant model are calculated to be $\lambda = -0.5007, \ -0.5007, \ 0.05, \ -2.05, \ -0.6, \ -0.6$, and thus the plant is unstable.
- We propose to control this quantum plant using a coherent feedback controller designed using the pole-placement and reduced order observer method.

- The reduced order observer based controller is defined as follows:

$$dz = F_c z dt + G_c \begin{bmatrix} dy \\ dy^* \end{bmatrix};$$
$$\begin{bmatrix} du \\ du^* \end{bmatrix} = H_c z dt + K_c \begin{bmatrix} dy \\ dy^* \end{bmatrix}$$

where

$$\begin{aligned} F_c &= F_{22} - H_o F_{12} - (G_2 - H_0 G_1) K_2; \\ G_c &= F_{21} + F_{22} H_o - H_o F_{12} H_o - H_o F_{11} \\ &\quad + (G_2 - H_o G_1)(K_1 + K_2 H_o); \\ H_c &= K_2; \;\; K_c = K_1 + K_2 H_o. \end{aligned}$$

- The observer gain matrix H_o is chosen as

$$H_o = 10^3 \times \begin{bmatrix} 1.5928 & 3.7229 \\ 3.7229 & 1.5928 \\ -1.9274 & -3.3729 \\ -3.3729 & -1.9274 \end{bmatrix},$$

which leads to observer poles $\lambda = -11, \ -11, \ -10, \ -10$.

- Also, the state feedback controller matrix is chosen as

$$K_{sf} = \begin{bmatrix} 1.2376 & -0.0008 & 0.0201 & 0.0043 & 0.0238 & 0.0007 \\ -0.0008 & 1.2376 & 0.0043 & 0.0201 & 0.0007 & 0.0238 \end{bmatrix},$$

which leads to the state feedback closed loop poles $\lambda = -2.1498, \ -0.9618, \ -0.6882, \ -0.0882 \pm 0.0368\imath, \ -0.4102$.

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- Then, the poles of the closed loop system corresponding to the quantum plant and the controller are calculated to be $\lambda = -2.1498, \ -0.0882 \pm 0.0368\imath, \ -0.9618, \ -0.4102, \ -0.6882, \ -11.0000, \ -11.0000, \ -10.0000, \ -10.0000,$ and thus this controller stabilizes the given quantum plant.
- Now, we wish to implement this controller as a coherent quantum controller.
- This requires that the controller is physically realizable. However, the above state space realization of the quantum controller does not satisfy the canonical physical realizability conditions.
- We will apply the above result to determine if the controller transfer function matrix $\Gamma_c(s) = H_c\left(sI - F_c\right)^{-1} G_c + K_c$ is physically realizable.
- Indeed, we calculate $K_c = \Gamma_c(\infty) = I$ and

$$\Gamma_c^{\sim}(s) J \Gamma_c(s) = J,$$

and hence the controller transfer function matrix is (J, J)-unitary.

- Also, the poles of the controller transfer function matrix are calculated to be at $s = -10.5301 \pm 1.3545\imath$ and $s = -10.5251 \pm 1.3385\imath$.
- Hence, it follows that the conditions of the theorem are satisfied and therefore, the controller transfer function matrix $\Gamma_c(s)$ is physically realizable and can be implemented as a quantum system.
- We can implement the controller transfer function as in interconnection of optical devices using the algorithm in

H. Nurdin, “Synthesis of linear quantum stochastic systems via quantum feedback networks,” *IEEE Transactions on Automatic Control*, vol. 55, no. 4, pp. 1008 –1013, April 2010.

Robust Stability of Nonlinear Quantum Systems

- Most interesting quantum phenomena such as entanglement and squeezing for continuous variable (infinite level) quantum systems occur when we have a nonlinear quantum system rather than a linear one.
- We would like to extend our existing linear quantum systems theory to the nonlinear case.
- Also, the issue of robustness to nonlinear perturbations in the dynamics is important in any linear feedback control system.
- We now consider the problem of robust stability for a quantum system defined in terms of a triple (S, L, H) in which the quantum system Hamiltonian is decomposed as $H = H_1 + H_2$ where H_1 is a known nominal (quadratic) Hamiltonian and H_2 is a perturbation (non-quadratic) Hamiltonian, which is contained in a specified set of Hamiltonians $\mathcal{W}$.

- Our solution to this problem is a quantum version of the **small gain theorem** for quantum systems which are nominally linear but are subject to sector bounded nonlinearities.
- This result can be applied to the stability analysis of perturbed quantum linear systems.
- Alternatively, it can be applied to closed loop quantum systems obtained when we apply a coherent quantum H^∞ controller to a linear quantum system subject to nonlinear sector bounded perturbations.

- Consider an open quantum system defined by parameters (S, L, H) where $H = H_1 + H_2$.
- We let
$$H_2 = f(\zeta, \zeta^*)$$
- Here ζ is a scalar operator on the underlying Hilbert space.

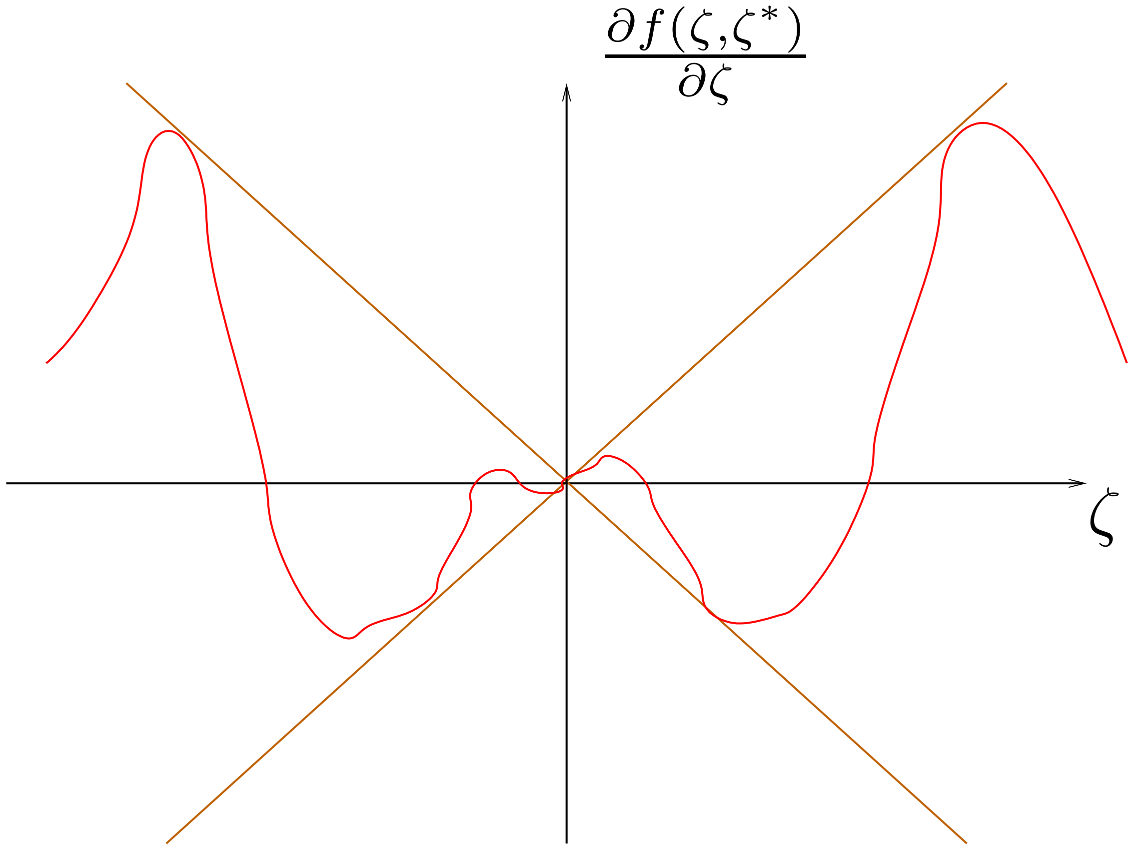
- Also, we consider the sector bound condition

$$\frac{\partial f(\zeta,\zeta^*)}{\partial \zeta}^* \frac{\partial f(\zeta,\zeta^*)}{\partial \zeta} \leq \frac{1}{\gamma^2}\zeta\zeta^* + \delta_1$$

and the smoothness condition

$$\frac{\partial^2 f(\zeta,\zeta^*)}{\partial \zeta^2}^* \frac{\partial^2 f(\zeta,\zeta^*)}{\partial \zeta^2} \leq \delta_2.$$

- Representation of the sector bound condition:



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- Then we define the set of allowable perturbations $\mathcal{W}$ as follows:

Definition.

$$\mathcal{W} = \left\{ \begin{array}{l} H_2 = f(\zeta, \zeta^*) \textit{ such that} \\ \frac{\partial f(\zeta,\zeta^*)}{\partial \zeta}^* \frac{\partial f(\zeta,\zeta^*)}{\partial \zeta} \leq \frac{1}{\gamma^2}\zeta\zeta^* + \delta_1 \textit{ and} \\ \frac{\partial^2 f(\zeta,\zeta^*)}{\partial \zeta^2} \frac{\partial^2 f(\zeta,\zeta^*)}{\partial \zeta^2}^* \leq \delta_2 \end{array} \right\}.$$

- Also as previously in the case of linear quantum systems, H_1 is of the form

$$H_1 = \frac{1}{2} \begin{bmatrix} a^\dagger & a^T \end{bmatrix} M \begin{bmatrix} a \\ a^\# \end{bmatrix}$$

where $M \in \mathbb{C}^{2n \times 2n}$ is a Hermitian matrix of the form

$$M = \begin{bmatrix} M_1 & M_2 \\ M_2^\# & M_1^\# \end{bmatrix}$$

and $M_1 = M_1^\dagger$, $M_2 = M_2^T$.

- In addition, we assume L is of the form

$$L = \begin{bmatrix} N_1 & N_2 \end{bmatrix} \begin{bmatrix} a \\ a^\# \end{bmatrix}$$

where $N_1 \in \mathbb{C}^{m \times n}$ and $N_2 \in \mathbb{C}^{m \times n}$. Also, $S = I$.

Definition. *An uncertain open quantum system defined by* (S, L, H) *where* $H = H_1 + H_2$ *with quadratic* H_1 *as above,* $H_2 \in \mathcal{W}$*, and linear* L *as above, is said to be* **robustly mean square stable** *if there exist constants* $c_1 > 0$*,* $c_2 > 0$ *and* $c_3 \geq 0$ *such that for any* $H_2 \in \mathcal{W}$*,*

$$\left\langle \begin{bmatrix} a(t) \\ a(t)^{\#} \end{bmatrix}^{\dagger} \begin{bmatrix} a(t) \\ a(t)^{\#} \end{bmatrix} \right\rangle$$

$$\leq c_1 e^{-c_2 t} \left\langle \begin{bmatrix} a \\ a^{\#} \end{bmatrix}^{\dagger} \begin{bmatrix} a \\ a^{\#} \end{bmatrix} \right\rangle + c_3 \ \forall t \geq 0.$$

- We define

$$\begin{aligned}\zeta &= E_1 a + E_2 a^{\#} \\ &= \begin{bmatrix} E_1 & E_2 \end{bmatrix} \begin{bmatrix} a \\ a^{\#} \end{bmatrix} = \tilde{E} \begin{bmatrix} a \\ a^{\#} \end{bmatrix}\end{aligned}$$

  where ζ is assumed to be a scalar operator.

- The following frequency domain small gain condition provides a sufficient condition for robust mean square stability when $H_2 \in \mathcal{W}$:

1. The matrix

$$F = -iJM - \frac{1}{2}JN^\dagger JN \text{ is Hurwitz;}$$

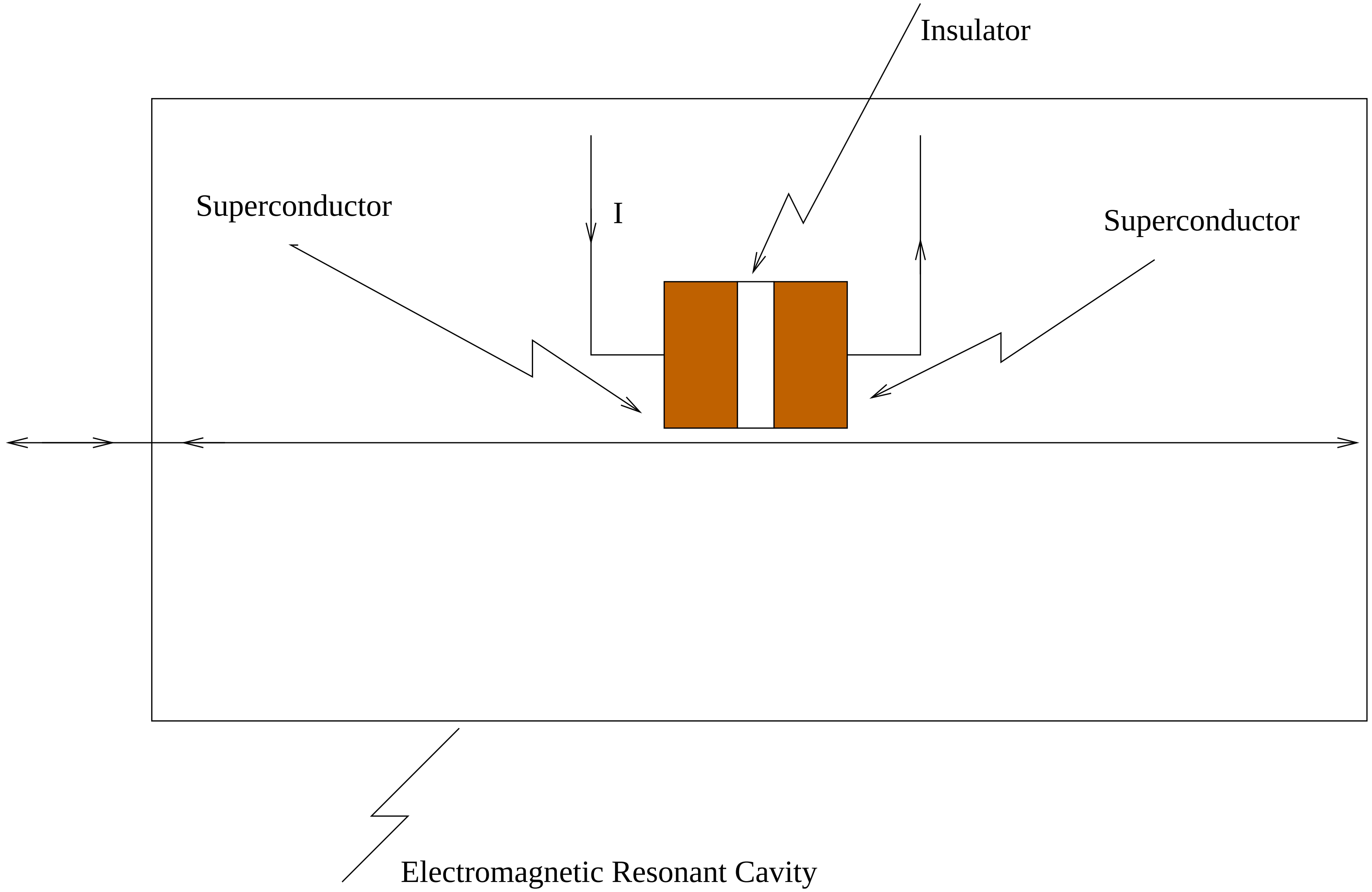
2.

$$\left\| \tilde{E}\left(sI - F\right)^{-1} J\tilde{E}^\dagger \right\|_\infty < \frac{\gamma}{2}.$$

Theorem. *Consider an uncertain open quantum system defined by (S, L, H) such that $H = H_1 + H_2$ where H_1 is quadratic as above, L is linear as above and $H_2 \in \mathcal{W}$. Furthermore, assume that the above frequency domain small conditions are satisfied. Then the uncertain quantum system is robustly mean square stable.*

A Josephson Junction in a Resonant Cavity System

- A Josephson junction consists of a thin insulating material between two superconducting layers as illustrated below:



- The following Hamiltonian can be obtained for this quantum system

$$H = \frac{1}{2} \left[\, a^\dagger \;\; a^T \, \right] M \begin{bmatrix} a \\ a^\# \end{bmatrix} - \mu \cos(\frac{a_2 + a_2^*}{\sqrt{2}})$$

where $a = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$, M is a Hermitian matrix and $\mu > 0$.

- This leads to the perturbation Hamiltonian

$$H_2 = f(\zeta, \zeta^*) = -\mu \cos(\frac{\zeta + \zeta^*}{\sqrt{2}})$$

where $\zeta = a_2$.

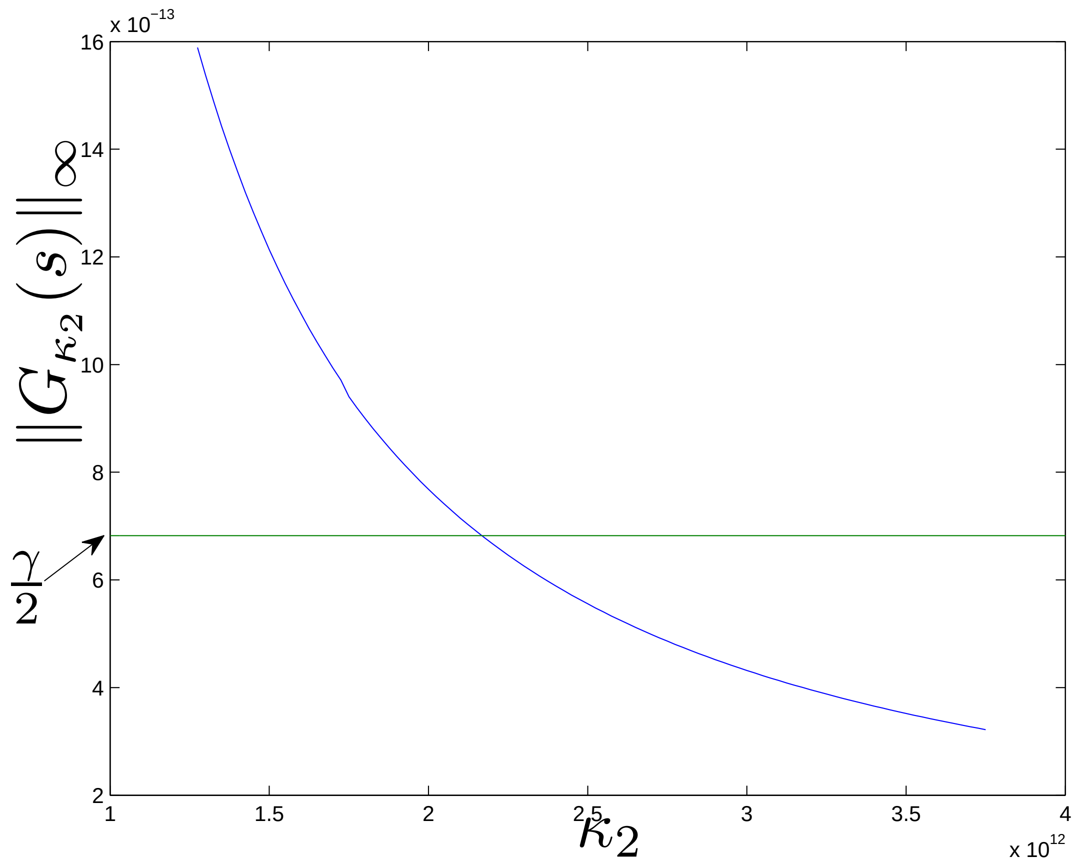
- The derivative of a cosine function is a sine function which is sector bounded with $\gamma = \frac{\mu}{\sqrt{2}}$.

- We assume that the cavity and Josephson modes are coupled to fields corresponding to coupling operators of the form

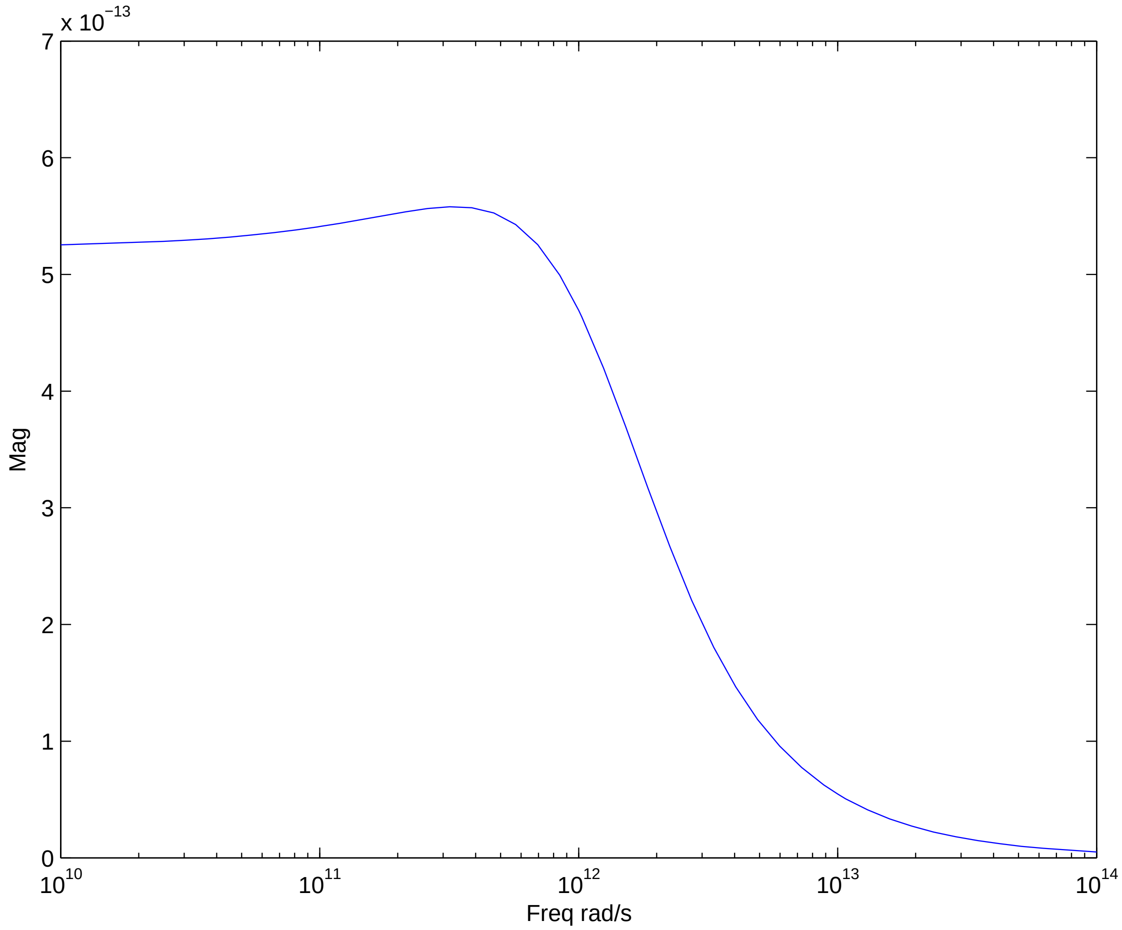
$$L = \begin{bmatrix} \sqrt{\kappa_1} a_1 \\ \sqrt{\kappa_2} a_2 \end{bmatrix}.$$

- We choose physically reasonable values for the parameters in this system except for the parameter κ_2 which we allow to vary.
- For various values of κ_2 we form the transfer function $G_{\kappa_2}(s) = \tilde{E}\left(sI - F\right)^{-1} J\tilde{E}^{\dagger}$ and calculate its H_{∞} norm.

- A plot of $\|G_{\kappa_2}(s)\|_\infty$ versus κ_2 is shown below:



- From this plot we can see that stability can be guaranteed for $\kappa_2 > 2.2 \times 10^{12}$.
- Hence, choosing a value of $\kappa_2 = 2.5 \times 10^{12}$, it follows that stability of Josephson junction system can be guaranteed.
- Indeed, with this value of κ_2, we calculate the matrix $F = -iJM - \frac{1}{2}JN^\dagger JN$ and find its eigenvalues to be $-5.0000 \times 10^{10} \pm 3.3507 \times 10^3 i$ and $-1.2500 \times 10^{12} \pm 1.4842 \times 10^3 i$ which implies that the matrix F is Hurwitz.
- Also, a magnitude Bode plot of the corresponding transfer function $G_{\kappa_2}(s)$ is shown below which implies that $\|G_{\kappa_1,\kappa_2}(s)\|_\infty = 5.5554 \times 10^{-13} < \gamma/2 = 6.8209 \times 10^{-13}$ and hence, we conclude that the quantum system is robustly mean square stable.


x 10^-13
7
6
5
4
3
2
1
0
Mag
10^10
10^11
10^12
10^13
10^14
Freq rad/s

Conclusions

- The control of linear quantum systems is an emerging field with important applications in quantum optics.
- The theory of linear quantum systems has close connections to standard linear systems theory but with the important distinction that the variables of interest are non-commutative.
- Central to the theory of quantum linear systems is the notion of physical realizability which characterizes when a linear system model is really quantum.
- It is also possible to extend the theory to allow for nonlinear quantum systems using classical ideas from robust control theory.
- The study of nonlinear quantum systems is needed to capture truly quantum phenomenon such as entanglement, squeezing and quantum superpositions.

- We are also extending our theory to the theory of finite level quantum systems (e.g. atoms) interacting with quantum fields:

  L. A. D. Espinosa, Z. Miao, I. R. Petersen, V. Ugrinovskii, and M. R. James, “On the preservation of commutation and anticommutation relations of n-level quantum systems,” in *Proceedings of the 2013 American Control Conference*, Washington, DC, June 2013.

- In this case, the QSDEs are bilinear systems rather than linear systems.
- All of these areas are rich with theoretical challenges which seem very ameniable to the tools of control theory.

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