Distributed Robust Consensus of Heterogeneous Uncertain Multi-agent Systems

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Abstract: This paper considers the distributed robust consensus problem of multi-agent systems with nominal linear dynamics but subject to different matching uncertainties. Due to the existence of nonidentical uncertainties, the multi-agent systems discussed in this paper are essentially heterogeneous. A distributed continuous static consensus protocol based on the relative state information is first designed, under which the consensus error is uniformly ultimately bounded and exponentially converges to a small adjustable residual set. A fully distributed adaptive consensus protocol is then designed, which, contrary to the static protocol, relies on neither the eigenvalues of the Laplacian matrix nor the upper bounds of the uncertainties. A sufficient condition for the existence of the proposed protocols is that each agent is stabilizable.

Key Words: Multi-agent system, consensus, adaptive control, robust control, uncertain systems.

1 Introduction

Cooperative control of a network of autonomous agents has been an emerging research direction and attracted a lot of attention from many scientific communities, for its potential applications in broad areas including spacecraft formation flying, sensor networks, and cooperative surveillance [1]. In the area of cooperative control, consensus is an important and fundamental problem, which means to develop distributed control policies using only local information to ensure that the agents reach an agreement on certain quantities of interest.

Two pioneering works on consensus are [2] and [3]. A theoretical explanation is provided in [2] for the alignment behavior observed in the Vicsek model [4] and a general framework of the consensus problem for networks of integrators is proposed in [3]. Since then, the consensus problem has been extensively studied by various scholars from different perspectives; see [5, 6, 7, 8, 9] and references therein. Existing consensus algorithms can be roughly categorized into two classes, namely, consensus without a leader (i.e., leaderless consensus) and consensus with a leader. The latter is also called leader-follower consensus or distributed tracking. In [5], a sufficient condition is derived to achieve consensus for multi-agent systems with jointly connected communication graphs. The authors in [6] design a distributed neighbor-based estimator to track an active leader. Distributed tracking algorithms are proposed in [10] and [11] for a network of agents with first-order dynamics. Consensus of networks of double- and high-order integrators is studied in [12, 13]. The consensus problem of multi-agent systems with general discrete- and continuous-time linear dynamics is studied in [7, 8, 9, 14, 15, 16]. It is worth noting that the design of the consensus protocols in [7, 8, 15, 16] requires the knowledge of the eigenvalues of the Laplacian matrix of the communication graph, which is actually global information. To overcome this limitation, distributed adaptive consensus protocols are proposed in [17, 18]. For the case where there exists a leader with possibly nonzero control input, distributed controllers are proposed in [19, 18] to solve the leader-follower consensus problem. A common assumption in [7, 8, 9, 14, 15, 16, 19, 18] is that the dynamics of the agents are identical and precisely known, which might be restrictive and not practical in many circumstances. In practical applications, the agents may be subject to certain parameter uncertainties or unknown external disturbances.

This paper considers the distributed robust consensus problem of multi-agent systems with identical nominal linear dynamics but subject to different matching uncertainties. A typical example belonging to this scenario is a network of mass-spring systems with different masses or unknown spring constants. Due to the existence of the nonidentical uncertainties which may be time-varying, nonlinear and unknown, the multi-agent systems discussed in this paper are essentially heterogeneous. The heterogeneous multi-agent systems in this paper contain the homogeneous linear multi-agent systems studied in [7, 8, 9, 14, 15, 16] as a special case where the uncertainties do not exist. Note that because of the existence of the uncertainties, the consensus problem in this case becomes quite challenging to solve.

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and the consensus algorithms given in [7, 8, 9, 14, 15, 16] are not applicable any more.

In this paper, we present a systematic procedure to address the distributed robust consensus problem of multi-agent systems with matching uncertainties. A distributed continuous static consensus protocol based on the relative states of neighboring agents is designed, under which the consensus error is uniformly ultimately bounded and exponentially converges to a small residual set. Note that the design of this protocol relies on the eigenvalues of the Laplacian matrix and the upper bounds of the matching uncertainties. In order to remove these requirements, a fully distributed adaptive protocol is further designed, under which the residual set of the consensus error is also given. One desirable feature is that for both the static and adaptive protocols, the residual sets of the consensus error can be made to be reasonably small by properly selecting the design parameters of the protocols. It is pointed out that a sufficient condition of the existence of the proposed protocols is that each agent is stabilizable.

2 Problem Statement

In this paper, we consider a network of $N$ autonomous agents with identical nominal linear dynamics but subject to heterogeneous uncertainties. The dynamics of the $i$-th agent are described by

$$
\dot{x}_i = Ax_i + Bu_i + f_i(x_i, t), \quad i = 1, \cdots, N, \tag{1}
$$

where $x_i \in \mathbb{R}^n$ is the state, $u_i \in \mathbb{R}^p$ is the control input, $A$ and $B$ are constant matrices with compatible dimensions, and $f_i(x_i, t) \in \mathbb{R}^n$ represents the lumped uncertainty associated with the $i$-th agent, which is supposed to satisfy the following assumption.

Assumption 1. There exist continuous scalar valued functions $\rho_i(x_i, t), \ i = 1, \cdots, N$, such that $\|f_i(x_i, t)\| \leq \rho_i(x_i, t)$, for $i = 1, \cdots, N$, for all $t \geq 0$ and $x_i \in \mathbb{R}^n$.

The communication topology among the agents is represented by an undirected graph $\mathcal{G} = (\mathcal{V}, E)$, where $\mathcal{V} = \{1, \cdots, N\}$ is the set of nodes (i.e., agents), and $E \subset \mathcal{V} \times \mathcal{V}$ is the set of edges (i.e., communication links). An edge $(i, j) \ (i \neq j)$ means that agents $i$ and $j$ can obtain information from each other. A path between distinct nodes $i_k$ and $i_l$ is a sequence of edges of the form $(i_k, i_{k+1}), \ k = 1, \cdots, l-1$. An undirected graph is connected if there exists a path between every pair of distinct nodes, otherwise it is disconnected.

The objective of this section is to solve the consensus problem for the agents in (1), i.e., to design distributed consensus protocols such that

$$\lim_{t \to \infty} \|x_i(t) - x_j(t)\| = 0, \quad \forall i, j = 1, \cdots, N. \tag{2}
$$

3 Distributed Static Consensus Protocol

Based on the relative states of neighboring agents, the following distributed static consensus protocol is proposed:

$$u_i = cK \sum_{j=0}^{N} a_{ij}(x_i - x_j) + \rho_i(x_i, t) \times g(K \sum_{j=0}^{N} a_{ij}(x_i - x_j)), \quad i = 1, \cdots, N, \tag{3}
$$

where $c > 0$ is the constant coupling gain, $K \in \mathbb{R}^{p \times n}$ is the feedback gain matrix, $a_{ij}$ is the $(i, j)$-th entry of the adjacency matrix associated with $\mathcal{G}$, defined as $a_{ii} = 0$, $a_{ij} = a_{ji} = 1$ if $(j, i) \in E$ and $a_{ij} = a_{ji} = 0$ otherwise, and the nonlinear function $g(.)$ is designed as follows: for $w \in \mathbb{R}^n$,

$$g(w) = \begin{cases} \frac{w}{\|w\|^2}, & \text{if } \rho_i(x_i, t)\|w\| > \kappa, \\ \frac{w}{\|w\|}, & \text{if } \rho_i(x_i, t)\|w\| \leq \kappa, \end{cases} \tag{4}
$$

where $\kappa$ is a small positive value.

Let $x = [x_1^T, \cdots, x_N^T]^T$ and $\rho = \text{diag}(\rho_1, \cdots, \rho_N)$. Using (2) for (1), we can obtain the closed-loop network dynamics as

$$\dot{x} = (I_N \otimes A + cL \otimes BK)x + (I_N \otimes B)F(x, t) + (\rho \otimes B)G(x), \quad \text{where } L = \sum_{i=1}^{N} L_{ii} \neq a_{ij} \text{ for } i \neq j, \text{ and } \tag{5}
$$

$$F(x, t) \triangleq \begin{bmatrix} f_1(x_1, t) \\ \vdots \\ f_N(x_N, t) \end{bmatrix}, \quad G(x) \triangleq \begin{bmatrix} g(K \sum_{j=1}^{N} L_{ij} x_j) \\ \vdots \\ g(K \sum_{j=1}^{N} L_{nj} x_j) \end{bmatrix}. \tag{6}
$$

Regarding the algebraic properties of $L$, we have

Lemma 1 [5]. (i) Zero is a simple eigenvalue of $L$ with $1$ as an eigenvector and all the other eigenvalues are positive if and only if $\mathcal{G}$ is connected; (ii) The smallest nonzero eigenvalue $\lambda_2$ of $L$ satisfies $\lambda_2 = \min_{x \neq 0, \|x\| = 1} \frac{x^TLx}{x^Tx}.$

Let $\xi = (M \otimes I_n)x$, where $M = I_N - \frac{1}{N}11^T$, and $\xi = [\xi_1^T, \cdots, \xi_N^T]^T$. It is easy to see that $0$ is a simple eigenvalue of $M$ with $1$ as a corresponding right eigenvector and $1$ is the only eigenvalue with multiplicity $N - 1$. Then, it follows that $\xi = 0$ if and only if $x_1 = \cdots = x_N$. Therefore, the consensus problem under the protocol (2) is solved if and only if $\xi$ asymptotically converges to zero. Hereafter, we refer to $\xi$ as the consensus error. By noting that $LM = L$, it is not difficult to obtain from (4) that the consensus error $\xi$ satisfies

$$\dot{\xi} = (I_N \otimes A + cL \otimes BK)\xi + (M \otimes B)F(x, t) + (M \rho \otimes B)G(\xi). \tag{7}
$$

The following result provides a sufficient condition to design the consensus protocol (2).

Theorem 1. Suppose that the communication graph $\mathcal{G}$ is undirected and connected and Assumption 1 holds. The parameters in the distributed protocol (2) are designed as $c \geq \frac{1}{\lambda_2}$ and $K = -BB^TP^{-1}$, where $\lambda_2$ is the smallest nonzero eigenvalue of $L$ and $P > 0$ is a solution to the following linear matrix inequality (LMI):

$$AP + PA^T - 2BB^T < 0. \tag{8}
$$

Then, the consensus error $\xi$ of (6) is uniformly ultimately bounded and exponentially converges to the residual set

$$\mathcal{D}_1 \triangleq \{ \xi : \|\xi\|^2 \leq \frac{2\lambda_{max}(P)NK}{\alpha\lambda_2} \}. \tag{9}$$
with a rate faster than $\exp(-\alpha t)$, where
\[
\alpha = -\lambda_{\text{max}}(AP + PA^T - 2BB^T)/\lambda_{\text{max}}(P).
\] (9)

**Proof.** Consider the following Lyapunov function candidate:
\[
V_1 = \frac{1}{2} \xi^T (\mathcal{L} \otimes P^{-1}) \xi.
\] (10)

By the definition of $\xi$, it is easy to see that $(I^T \otimes I) \xi = 0$. For a connected graph $G$, it then follows from Lemma 1 that
\[
V_1(\xi) \geq \frac{1}{2} \lambda_{\text{max}} \xi^T (I_N \otimes P^{-1}) \xi \geq \frac{\lambda_2}{2\lambda_{\text{max}}(P)} \|\xi\|^2.
\] (11)

The time derivative of $V_1$ along the trajectory of (4) is given by
\[
\dot{V}_1 = \xi^T (\mathcal{L} \otimes P^{-1}) \dot{\xi} = \xi^T [(\mathcal{L} \otimes P^{-1} A) + cL^2 \otimes P^{-1} BK] \xi + (\mathcal{L} \otimes P^{-1} B)F(x,t) + (\mathcal{L} \rho \otimes P^{-1} B)G(\xi).
\] (12)

By using Assumption 2, we can obtain that
\[
\xi^T (\mathcal{L} \otimes P^{-1} B)F(x,t) = \sum_{i=1}^N \sum_{j=1}^N L_{ij} \xi^T P^{-1} B f_i(x,t) \\
\leq \sum_{i=1}^N \|B^T P^{-1} \| \sum_{j=1}^N L_{ij} \|f_i(x,t)\| \\
\leq \sum_{i=1}^N \rho_i(x_i,t) \|B^T P^{-1} \| \sum_{j=1}^N L_{ij} \xi_j.
\]
(13)

Next, consider the following three cases.

i) $\rho_i(x_i,t) \|K \sum_{j=1}^N L_{ij} \xi_j\| > \kappa$, $i = 1, \cdots, N$.

In this case, it follows from (3) and (5) that
\[
\xi^T (\mathcal{L} \rho \otimes P^{-1} B)G(\xi) = -\sum_{i=1}^N \rho_i(x_i,t) \|B^T P^{-1} \| \sum_{j=1}^N L_{ij} \xi_j.
\]
(14)

Substituting (14) and (13) into (12) yields $\dot{V}_1 \leq \frac{1}{2} \xi^T \mathcal{X} \xi$, where
\[
\mathcal{X} = \mathcal{L} \otimes (P^{-1} A + A^T P^{-1}) - 2cL^2 \otimes P^{-1} BB^T P^{-1}.
\] (15)

ii) $\rho_i(x_i,t) \|K \sum_{j=1}^N L_{ij} \xi_j\| \leq \kappa$, $i = 1, \cdots, N$.

In this case, we can get from (3) and (5) that
\[
\xi^T (\mathcal{L} \rho \otimes P^{-1} B)G(\xi) = -\sum_{i=1}^N \rho_i(x_i,t) \kappa \|B^T P^{-1} \| \sum_{j=1}^N L_{ij} \xi_j^2 \leq 0.
\]
(16)

Substituting (14), (13), and (16) into (12) gives
\[
\dot{V}_1 \leq \frac{1}{2} \xi^T \mathcal{X} \xi + N\kappa.
\] (17)

iii) $\xi$ satisfies neither case i) nor case ii).

Without loss of generality, assume that $\rho_i(x_i,t) \|K \sum_{j=1}^N a_{ij}(x_i - x_j)\| > \kappa$, $i = 1, \cdots, l$, and $\rho_i(x_i,t) \|K \sum_{j=1}^N a_{ij}(x_i - x_j)\| \leq \kappa$, $i = l + 1, \cdots, N$, where $2 \leq l \leq N - 1$. In this case, we can get that
\[
\xi^T (\mathcal{L} \rho \otimes P^{-1} B)G(\xi) \leq -\sum_{i=1}^l \rho_i(x_i,t) \|B^T P^{-1} \| \sum_{j=1}^N L_{ij} \xi_j. 
\]
(18)

Then, it follows from (12), (14), (18), and (13) that $\dot{V}_1 \leq \frac{1}{2} \xi^T \mathcal{X} \xi + (N - l)\kappa$.

Therefore, by analyzing the above three cases, we get that $\dot{V}_1$ satisfies (17) for all $\xi \in \mathbb{R}^{Nn}$. Note that (17) can be rewritten as
\[
\dot{V}_1 \leq -\alpha V_1 + \frac{1}{2} \xi^T \mathcal{X} \xi + N\kappa
\]
(19)

where $\alpha > 0$.

Because $G$ is connected, it follows from Lemma 1 that zero is a simple eigenvalue of $\mathcal{L}$ and all the other eigenvalues are positive. Let $U = \left[ \frac{1}{\sqrt{N}} Y_1 \right]$ and $U^T = \left[ \frac{1}{\sqrt{N}} Y_2 \right]$, with $Y_1 \in \mathbb{R}^{N \times (N-1)}$, $Y_2 \in \mathbb{R}^{N \times N}$, be such unitary matrices that $U^T \mathcal{L} U = \Lambda \triangleq \text{diag}(0, \lambda_2, \cdots, \lambda_N)$, where $\lambda_2 \leq \cdots \leq \lambda_N$ are the nonzero eigenvalues of $\mathcal{L}$. Let $\xi \triangleq \begin{bmatrix} \xi_1^T, \cdots, \xi_N^T \end{bmatrix}^T = (U^T \otimes P^{-1}) \xi$. By the definitions of $\xi$ and $\xi$, it is easy to see that $\xi_1 = (\frac{1}{\sqrt{N}} \otimes P^{-1}) \xi = \frac{1}{\sqrt{N}} \mathcal{M} \otimes P^{-1} x = 0$. Then, it follows that
\[
\xi^T (\mathcal{X} + \alpha \mathcal{L} \otimes P^{-1} \xi)
\]
\[
= \xi^T [\Lambda \otimes (AP + PA^T + \alpha P) - 2cL^2 \otimes BB^T] \xi
\]
\[
\leq \sum_{i=2}^N \lambda_i \xi_i^T (AP + PA^T + \alpha P - 2BB^T) \xi_i.
\]
(20)

Because $\alpha = -\lambda_{\text{max}}(AP + PA^T - 2BB^T)/\lambda_{\text{max}}(P)$, we can see from (20) that $\xi^T (\mathcal{X} + \alpha \mathcal{L} \otimes P^{-1} \xi) \leq 0$. Then, we can get from (19) that
\[
\dot{V}_1 \leq -\alpha V_1 + N\kappa.
\]
(21)

By using the well-known Comparison lemma (Lemma 3.4 in [20]), we can obtain from (21) that
\[
V_1(\xi) \leq [V(\xi(0)) - \frac{N\kappa}{\alpha}\exp(-\alpha t)] + \frac{N\kappa}{\alpha},
\]
(22)

which, by (11), implies that $\xi$ exponentially converges to the residual set $D_1$ in (8) with a rate not less than $\exp(-\alpha t)$.

**Remark 1.** The distributed consensus protocol (2) consists of a linear part and a nonlinear part, where the term $\rho_i(x_i,t) g(\sum_{j=1}^N a_{ij}(x_i - x_j))$ is used to suppress the effect of the uncertainties $f_i(x_i,t)$. For the case where $f_i(x_i,t) = 0$, we can accordingly remove $\rho_i(x_i,t) g(\sum_{j=1}^N a_{ij}(x_i - x_j))$ from (2), which can recover the static consensus protocols as in [7, 8]. As shown
in Proposition 2 of [7], a necessary and sufficient condition for the existence of a $P > 0$ to the LMI (7) is that $(A, B)$ is stabilizable. Therefore, a sufficient condition for the existence of (2) satisfying Theorem 1 is that $(A, B)$ is stabilizable.

**Remark 2.** Note that the residual set $D_1$ depends on the smallest nonzero eigenvalue of $L$, the number of agents, the largest eigenvalue of $P$, and the size $\kappa$ of the boundary layer. By choosing a sufficiently small $\kappa$, the consensus error $\xi$ under the protocol (2) can converge to an arbitrarily small neighborhood of zero, which is acceptable in most applications.

## 4 Distributed Adaptive Consensus Protocol

In the last section, the design of the distributed protocol (2) relies on the minimal nonzero eigenvalue $\lambda_2$ of $L$ and the upper bounds $\rho_i(x_i, t)$ of the matching uncertainties $f_i(x_i, t)$. However, $\lambda_2$ is global information in the sense that each agent has to know the entire communication graph to compute it. Besides, the bounds $\rho_i(x_i, t)$ of the uncertainties $f_i(x_i, t)$ might not be easily obtained in some cases. In this section, we will design fully distributed protocols without requiring either $\lambda_2$ or $\rho_i(x_i, t)$.

Before moving forward, we introduce a modified assumption regarding the bounds of the lumped uncertainties $f_i(x_i, t)$, $i = 1, \ldots, N$.

**Assumption 2.** There are positive constants $d_i$ and $e_i$ such that $\|f_i(x_i, t)\| \leq d_i + e_i \|x_i\|$, $i = 1, \ldots, N$.

Based on the local state information of neighboring agents, we propose the following distributed adaptive protocol to each agent:

$$u_i = \bar{d}_i \sum_{j=1}^{N} a_{ij}(x_i - x_j) + r(K \sum_{j=1}^{N} a_{ij}(x_i - x_j)),$$

$$\dot{\bar{d}}_i = \tau_i [\varphi_i \bar{d}_i + \sum_{j=1}^{N} a_{ij}(x_i - x_j)^T \Gamma \sum_{j=1}^{N} a_{ij}(x_i - x_j) + \|K \sum_{j=1}^{N} a_{ij}(x_i - x_j)\|],$$

$$\dot{e}_i = e_i [\varphi_i \bar{e}_i + \|K \sum_{j=1}^{N} a_{ij}(x_i - x_j)\|],$$

where $\bar{d}_i(t)$ and $\bar{e}_i(t)$ are the adaptive gains associated with the $i$-th agent, $\bar{d}_i$, $\bar{e}_i$, $\varphi_i$, and $\psi_i$ are positive scalars, $\bar{d}_i$ and $\bar{e}_i$ are positive scalars chosen by the designer, the nonlinear function $r(\cdot)$ is defined as follows: for $w \in \mathbb{R}^n$,

$$r(w) = \begin{cases} \frac{w(d_i + e_i \|x_i\|)}{\|w\|}, & \text{if } (d_i + e_i \|x_i\|) \|w\| > \kappa \\ \frac{w(d_i + e_i \|x_i\|)^2}{\kappa^2}, & \text{if } (d_i + e_i \|x_i\|) \|w\| \leq \kappa \end{cases},$$

and the rest of the variables are defined as in (2).

Let the consensus error $\xi$ be defined as in (6) and $\mathcal{T} = \text{diag}(d_1, \ldots, d_N)$. Then, it is not difficult to get from (1) and (23) that the closed-loop network dynamics can be written as

$$\dot{\xi} = (I_N \otimes A + M \mathcal{T} \otimes B K) \xi + (M \otimes B) F(x, t) + (M \otimes B) R(\xi),$$

$$\dot{\bar{d}}_i = \tau_i [\varphi_i \bar{d}_i + \sum_{j=1}^{N} L_{ij} \xi_j \Gamma \sum_{j=1}^{N} L_{ij} \xi_j + \|K \sum_{j=1}^{N} L_{ij} \xi_j\|],$$

$$\dot{e}_i = e_i [\varphi_i \bar{e}_i + \|K \sum_{j=1}^{N} L_{ij} \xi_j\|],$$

where

$$R(\xi) = \begin{bmatrix} r(K \sum_{j=1}^{N} L_{ij} \xi_j) \\ \vdots \\ r(K \sum_{j=1}^{N} L_{Nj} \xi_j) \end{bmatrix},$$

and the rest of the variables are defined as in (4).

To establish the ultimate boundedness of the states $\xi, \bar{d}_i$, and $\bar{e}_i$ of (25), we use the following Lyapunov function candidate

$$V_2 = \frac{1}{2} \xi^T (L \otimes P^{-1}) \xi + \sum_{i=1}^{N} \bar{d}^2_i + \sum_{i=1}^{N} \bar{e}^2_i,$$

where $\bar{d}_i = \bar{d}_i - \bar{d}_i$, $\bar{d}_i = \bar{d}_i - \beta$, $i = 1, \ldots, N$, and $\beta \geq \max_{i=1, \ldots, N} \{d_i, \frac{1}{2} \}$.

**Theorem 2.** Suppose that $G$ is connected and Assumptions 2 holds. The feedback gain matrices of the distributed adaptive protocol (23) are designed as $K = -B^T P^{-1}$ and $\Gamma = -P^{-1} B B^T P^{-1}$, where $P > 0$ is a solution to the LMI (7). Then, both the consensus error $\xi$ and the adaptive gains $\bar{d}_i$ and $\bar{e}_i$, $i = 1, \ldots, N$, in (25) are uniformly ultimately bounded and the following statements hold.

i) For any $\varphi_i$ and $\psi_i$, $\xi, \bar{d}_i$, and $\bar{e}_i$ exponentially converge to the residual set

$$D_2 \triangleq \{ \xi, \bar{d}_i, \bar{e}_i : V_2 < \frac{N}{2\delta} \sum_{i=1}^{N} (\beta^2 \varphi_i + e_i^2 \psi_i) + \frac{N \kappa}{4\delta} \},$$

with a convergence rate faster than $\exp(-\delta t)$, where $\delta \triangleq \min_{i=1, \ldots, N} \{\varphi_i, \psi, \psi_i \psi_i \} < \alpha$, in addition to i), $\xi$ exponentially converges to the residual set

$$D_3 \triangleq \{ \xi : \|\xi\|^2 \leq \frac{\lambda_{\max}(P)}{\lambda_\alpha^2 P} \sum_{i=1}^{N} (\beta^2 \varphi_i + e_i^2 \psi_i) + \frac{1}{2} N \kappa^2 \}.$$

with a rate faster than $\exp(-\alpha t)$.

**Proof.** The time derivative of $V_2$ along (25) can be obtained
as

\[ V_2 = \xi^T(\mathcal{L} \otimes P^{-1})\xi + \sum_{i=1}^N \frac{d_i}{\tau_i} + \sum_{i=1}^N \beta_i \xi_i. \]

Therefore, based on the above three cases, we can get

\[ V_2 \leq \frac{1}{2} \xi^T \mathbb{Y} \xi - \sum_{i=1}^N (\beta - d_i) ||BTP^{-1} \sum_{j=1}^N L_{ij} \xi_j|| + \frac{1}{4} N\kappa. \]

By following similar steps in the two cases above, it is not difficult to get that

\[ V_2 \leq \frac{1}{2} \xi^T \mathbb{Y} \xi - \frac{1}{2} \sum_{i=1}^N (\beta - d_i) ||BTP^{-1} \sum_{j=1}^N L_{ij} \xi_j|| \]

\[ + \frac{1}{2} \sum_{i=1}^N (\beta^2 \phi_i + \psi_i^2) + \frac{1}{4} \sum_{i=1}^N (\beta^2 \phi_i + \psi_i^2) + \frac{1}{4} \sum_{i=1}^N (\beta^2 \phi_i + \psi_i^2) + \frac{1}{4} N\kappa. \]

Therefore, based on the above three cases, we can get that \( V_2 \) satisfies (35) for all \( \xi \in \mathbb{R}^{Nn} \). Because \( \beta \geq \max_{i=1, \ldots, N} \beta_i \) and \( \beta \lambda_2 \geq 1 \), it follows from (35) that

\[ V_2 \leq \frac{1}{2} \xi^T \mathbb{Y} \xi - \frac{1}{2} \sum_{i=1}^N (\beta - d_i) ||BTP^{-1} \sum_{j=1}^N L_{ij} \xi_j|| \]

\[ + \frac{1}{2} \sum_{i=1}^N (\beta^2 \phi_i + \psi_i^2) + \frac{1}{4} \sum_{i=1}^N (\beta^2 \phi_i + \psi_i^2) + \frac{1}{4} N\kappa. \]

Note that (36) can be rewritten into

\[ V_2 \leq \frac{1}{2} \xi^T (\mathcal{Y} + \mathcal{L} \otimes P^{-1}) \xi - \frac{1}{2} \sum_{i=1}^N (\beta - \frac{\delta}{\tau_i}) d_i^2 \]

\[ + (\beta - \frac{\delta}{\tau_i}) e_i^2 \] + \frac{1}{2} \sum_{i=1}^N \left( \phi_i \right)^2 \phi_i + \frac{1}{2} \sum_{i=1}^N \left( \psi_i \right)^2 \psi_i + \frac{1}{4} N\kappa. \]

Because \( \beta \lambda_2 \geq 1 \) and \( 0 < \delta \leq \alpha \), by following similar steps in the proof of Theorem 1, we can show that

\[ \xi^T (\mathcal{Y} + \mathcal{L} \otimes P^{-1}) \xi \leq 0. \]

Furthermore, by noting that \( \delta \leq \min_{i=1, \ldots, N} \{ \frac{\beta^2}{\tau_i}, \beta e_i \} \), it follows from (37) that

\[ V_2 \leq \frac{1}{2} \xi^T (\mathcal{Y} + \mathcal{L} \otimes P^{-1}) \xi - \frac{1}{2} \sum_{i=1}^N (\beta^2 \phi_i + \psi_i^2) + \frac{1}{4} N\kappa, \]

which implies that

\[ V_2 \leq \frac{1}{2} \xi^T (\mathcal{Y} + \mathcal{L} \otimes P^{-1}) \xi - \frac{1}{2} \sum_{i=1}^N (\beta^2 \phi_i + \psi_i^2) + \frac{1}{4} N\kappa. \]
Therefore, $V_2$ exponentially converges to the residual set $D_2$ in (28) with a rate faster than $\exp(-\delta t)$, implying that $\xi, \tilde{d}_i$, and $\tilde{e}_i$ are uniformly ultimately bounded.

Next, if $\phi \triangleq \max_{i=1,\ldots,N} \{\phi_1, \psi, \psi_i \} < \alpha$, we can obtain a smaller residual set for $\xi$ by rewriting (37) into

$$
\dot{V}_2 \leq -\theta V_2 + \frac{1}{2} \xi^T (3'[\alpha \mathcal{L} \otimes \mathbf{P}^{-1}] \xi + \frac{1}{4} N \kappa
- \frac{\alpha - \theta}{2} \xi^T (\mathcal{L} \otimes \mathbf{P}^{-1}) \xi + \frac{1}{2} \sum_{i=1}^{N} (\beta^2 \phi_i + e_i^2 \psi_i)
\leq -\theta V_2 - \frac{\lambda_2 (\alpha - \theta)}{2 \lambda_{\text{max}}(\mathbf{P})} \| \xi \|^2 + \frac{1}{4} N \kappa
+ \frac{1}{2} \sum_{i=1}^{N} (\beta^2 \phi_i + e_i^2 \psi_i).
$$

(40)

Obviously, it follows from (40) that $\dot{V}_2 \leq -\theta V_2$ if $\| \xi \|^2 > \frac{\lambda_2 (\alpha - \theta)}{2 \lambda_{\text{max}}(\mathbf{P})} (\sum_{i=1}^{N} (\beta^2 \phi_i + e_i^2 \psi_i) + \frac{1}{2} N \kappa)$. Then, by noting $V_2 \geq \frac{\lambda_1}{2 \lambda_{\text{max}}(\mathbf{P})} \| \xi \|^2$, we can get that $\xi$ exponentially converges to the residual set $D_3$ in (29) with a rate faster than $\exp(-\delta t)$. $\blacksquare$

**Remark 3.** From (28) and (29), we can observe that the residual sets $D_2$ and $D_3$ decrease as $\kappa$ decreases. Given $\kappa$, smaller $\phi_i$ and $\psi_i$ give a smaller bound for $\xi$ and at the same time yield a larger bound for $d_i$ and $e_i$. For the case where $\phi_i = 0$ and $\psi_i = 0$, $d_i$ and $e_i$ will tend to infinity. In real implementations, if large $d_i$ and $e_i$ are acceptable, we can choose $\phi_i$, $\psi_i$, and $\kappa$ to be relatively small in order to guarantee a small $\xi$.

5 Conclusion

This paper has addressed the robust consensus problem for multi-agent systems with heterogeneous matching uncertainties. Distributed static and adaptive consensus protocols have been designed, under which the consensus error has been shown to be ultimately bounded and exponentially converges to small adjustable residual sets. An interesting future topic is to consider more general uncertainties which do not necessarily satisfy the matching condition. Another direction is to discuss the case with general directed and switching communication graphs.

REFERENCES


