Distributed Robust Consensus of Heterogeneous Uncertain Multi-agent Systems

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Abstract: This paper considers the distributed robust consensus problem of multi-agent systems with nominal linear dynamics but subject to different matching uncertainties. Due to the existence of nonidentical uncertainties, the multi-agent systems discussed in this paper are essentially heterogeneous. A distributed continuous static consensus protocol based on the relative state information is first designed, under which the consensus error is uniformly ultimately bounded and exponentially converges to a small adjustable residual set. A fully distributed adaptive consensus protocol is then designed, which, contrary to the static protocol, relies on neither the eigenvalues of the Laplacian matrix nor the upper bounds of the uncertainties. A sufficient condition for the existence of the proposed protocols is that each agent is stabilizable.

Key Words: Multi-agent system, consensus, adaptive control, robust control, uncertain systems.

1 Introduction

Cooperative control of a network of autonomous agents has been an emerging research direction and attracted a lot of attention from many scientific communities, for its potential applications in broad areas including spacecraft formation flying, sensor networks, and cooperative surveillance [1]. In the area of cooperative control, consensus is an important and fundamental problem, which means to develop distributed control policies using only local information to ensure that the agents reach an agreement on certain quantities of interest.

Two pioneering works on consensus are [2] and [3]. A theoretical explanation is provided in [2] for the alignment behavior observed in the Vicsek model [4] and a general framework of the consensus problem for networks of integrators is proposed in [3]. Since then, the consensus problem has been extensively studied by various scholars from different perspectives; see [5, 6, 7, 8, 9] and references therein. Existing consensus algorithms can be roughly categorized into two classes, namely, consensus without a leader (i.e., leaderless consensus) and consensus with a leader. The latter is also called leader-follower consensus or distributed tracking. In [5], a sufficient condition is derived to achieve consensus for multi-agent systems with jointly connected communication graphs. The authors in [6] design a distributed neighbor-based estimator to track an active leader. Distributed tracking algorithms are proposed in [10] and [11] for a network of agents with first-order dynamics. Consensus of networks of double- and high-order integrators is studied in [12, 13]. The consensus problem of multi-agent systems with general discrete- and continuous-time linear dynamics is studied in [7, 8, 9, 14, 15, 16]. It is worth noting that the design of the consensus protocols in [7, 8, 15, 16] requires the knowledge of the eigenvalues of the Laplacian matrix of the communication graph, which is actually global information. To overcome this limitation, distributed adaptive consensus protocols are proposed in [17, 18]. For the case where there exists a leader with possibly nonzero control input, distributed controllers are proposed in [19, 18] to solve the leader-follower consensus problem. A common assumption in [7, 8, 9, 14, 15, 16, 19, 18] is that the dynamics of the agents are identical and precisely known, which might be restrictive and not practical in many circumstances. In practical applications, the agents may be subject to certain parameter uncertainties or unknown external disturbances.

This paper considers the distributed robust consensus problem of multi-agent systems with identical nominal linear dynamics but subject to different matching uncertainties. A typical example belonging to this scenario is a network of mass-spring systems with different masses or unknown spring constants. Due to the existence of the nonidentical uncertainties which may be time-varying, nonlinear and unknown, the multi-agent systems discussed in this paper are essentially heterogeneous. The heterogeneous multi-agent systems in this paper contain the homogeneous linear multi-agent systems studied in [7, 8, 9, 14, 15, 16] as a special case where the uncertainties do not exist. Note that because of the existence of the uncertainties, the consensus problem in this case becomes quite challenging to solve

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and the consensus algorithms given in [7, 8, 9, 14, 15, 16] are not applicable any more.

In this paper, we present a systematic procedure to address the distributed robust consensus problem of multi-agent systems with matching uncertainties. A distributed continuous static consensus protocol based on the relative states of neighboring agents is designed, under which the consensus error is uniformly ultimately bounded and exponentially converges to a small residual set. Note that the design of this protocol relies on the eigenvalues of the Laplacian matrix and the upper bounds of the matching uncertainties. In order to remove these requirements, a fully distributed adaptive protocol is further designed, under which the residual set of the consensus error is also given. One desirable feature is that for both the static and adaptive protocols, the residual sets of the consensus error can be made to be reasonably small by properly selecting the design parameters of the protocols. It is pointed out that a sufficient condition of the existence of the proposed protocols is that each agent is stabilizable.

2 Problem Statement

In this paper, we consider a network of N autonomous agents with identical nominal linear dynamics but subject to heterogeneous uncertainties. The dynamics of the i-th agent are described by

$$\dot{x}_i = Ax_i + B[u_i + f_i(x_i,t)],\ i = 1,\cdots,N, \tag{1}$$

where $x_i \in \mathbf{R}^n$ is the state, $u_i \in \mathbf{R}^p$ is the control input, A and B are constant known matrices with compatible dimensions, and $f_i(x_i,t) \in \mathbf{R}^n$ represents the lumped uncertainty associated with the i-th agent, which is supposed to satisfy the following assumption.

Assumption 1. There exist continuous scalar valued functions $\rho_i(x_i,t)$, $i = 1,\cdots,N$, such that $\|f_i(x_i,t)\| \leq \rho_i(x_i,t)$, $i = 1,\cdots,N$, for all $t \geq 0$ and $x_i \in \mathbf{R}^n$.

The communication topology among the agents is represented by an undirected graph $\mathcal{G} = (\mathcal{V},\mathcal{E})$, where $\mathcal{V} = \{1,\cdots,N\}$ is the set of nodes (i.e., agents), and $\mathcal{E} \subset \mathcal{V}\times\mathcal{V}$ is the set of edges (i.e., communication links). An edge (i,j) $(i \neq j)$ means that agents i and j can obtain information from each other. A path between distinct nodes i_1 and i_l is a sequence of edges of the form (i_k, i_{k+1}), $k = 1,\cdots,l-1$. An undirected graph is connected if there exists a path between every pair of distinct nodes, otherwise is disconnected.

The objective of this section is to solve the consensus problem for the agents in (1), i.e., to design distributed consensus protocols such that $\lim_{t\to\infty}\|x_i(t) - x_j(t)\| = 0$, $\forall\, i,j = 1,\cdots,N$.

3 Distributed Static Consensus Protocol

Based on the relative states of neighboring agents, the following distributed static consensus protocol is proposed:

$$\begin{aligned} u_i = {} & cK\sum_{j=0}^{N} a_{ij}(x_i - x_j) + \rho_i(x_i,t) \\ & \times g(K\sum_{j=0}^{N} a_{ij}(x_i - x_j)),\ i = 1,\cdots,N, \end{aligned} \tag{2}$$

where $c > 0$ is the constant coupling gain, $K \in \mathbf{R}^{p\times n}$ is the feedback gain matrix, a_{ij} is the (i,j)-th entry of the adjacency matrix associated with $\mathcal{G}$, defined as $a_{ii} = 0$, $a_{ij} = a_{ji} = 1$ if $(j,i) \in \mathcal{E}$ and $a_{ij} = a_{ji} = 0$ otherwise, and the nonlinear function $g(\cdot)$ is defined as follows: for $w \in \mathbf{R}^n$,

$$g(w) = \begin{cases} \frac{w}{\|w\|}, & \text{if } \rho_i(x_i,t)\|w\| > \kappa \\ \frac{w}{\kappa}, & \text{if } \rho_i(x_i,t)\|w\| \leq \kappa \end{cases}, \tag{3}$$

where κ is a small positive value.

Let $x = [x_1^T,\cdots,x_N^T]^T$ and $\rho = \mathrm{diag}(\rho_1,\cdots,\rho_N)$. Using (2) for (1), we can obtain the closed-loop network dynamics as

$$\begin{aligned} \dot{x} = {} & (I_N \otimes A + c\mathcal{L}\otimes BK)x + (I_N \otimes B)F(x,t) \\ & + (\rho\otimes B)G(x), \end{aligned} \tag{4}$$

where $\mathcal{L}$ denotes the Laplacian matrix of $\mathcal{G}$, defined by $\mathcal{L}_{ii} = \sum_{j=1,j\neq i}^{N} a_{ij}$ and $\mathcal{L}_{ij} = -a_{ij}$ for $i \neq j$, and

$$F(x,t) \triangleq \begin{bmatrix} f_1(x_1,t) \\ \vdots \\ f_N(x_N,t) \end{bmatrix}, G(x) \triangleq \begin{bmatrix} g(K\sum_{j=1}^{N}\mathcal{L}_{1j}x_j) \\ \vdots \\ g(K\sum_{j=1}^{N}\mathcal{L}_{Nj}x_j) \end{bmatrix}. \tag{5}$$

Regarding the algebraic properties of $\mathcal{L}$, we have

Lemma 1 [5]. (i) Zero is a simple eigenvalue of $\mathcal{L}$ with $\mathbf{1}$ as an eigenvector and all the other eigenvalues are positive if and only if $\mathcal{G}$ is connected; (ii) The smallest nonzero eigenvalue λ_2 of $\mathcal{L}$ satisfies $\lambda_2 = \min\limits_{x\neq 0, \mathbf{1}^T x = 0} \frac{x^T\mathcal{L}x}{x^T x}$.

Let $\xi = (M \otimes I_n)x$, where $M = I_N - \frac{1}{N}\mathbf{1}\mathbf{1}^T$, and $\xi = [\xi_1^T,\cdots,\xi_N^T]^T$. It is easy to see that 0 is a simple eigenvalue of M with $\mathbf{1}$ as a corresponding right eigenvector and 1 is the other eigenvalue with multiplicity $N-1$. Then, it follows that $\xi = 0$ if and only if $x_1 = \cdots = x_N$. Therefore, the consensus problem under the protocol (2) is solved if and only if ξ asymptotically converges to zero. Hereafter, we refer to ξ as the consensus error. By noting that $\mathcal{L}M = \mathcal{L}$, it is not difficult to obtain from (4) that the consensus error ξ satisfies

$$\begin{aligned} \dot{\xi} = {} & (I_N \otimes A + c\mathcal{L}\otimes BK)\xi + (M\otimes B)F(x,t) \\ & + (M\rho\otimes B)G(\xi). \end{aligned} \tag{6}$$

The following result provides a sufficient condition to design the consensus protocol (2).

Theorem 1. Suppose that the communication graph $\mathcal{G}$ is undirected and connected and Assumption 1 holds. The parameters in the distributed protocol (2) are designed as $c \geq \frac{1}{\lambda_2}$ and $K = -B^T P^{-1}$, where λ_2 is the smallest nonzero eigenvalue of $\mathcal{L}$ and $P > 0$ is a solution to the following linear matrix inequality (LMI):

$$AP + PA^T - 2BB^T < 0, \tag{7}$$

Then, the consensus error ξ of (6) is uniformly ultimately bounded and exponentially converges to the residual set

$$\mathcal{D}_1 \triangleq \{\xi : \|\xi\|^2 \leq \frac{2\lambda_{\max}(P)N\kappa}{\alpha\lambda_2}\}, \tag{8}$$

with a rate faster than $\exp(-\alpha t)$, where

$$\alpha = -\lambda_{\max}(AP + PA^T - 2BB^T)/\lambda_{\max}(P). \quad (9)$$

Proof. Consider the following Lyapunov function candidate:

$$V_1 = \frac{1}{2}\xi^T(\mathcal{L} \otimes P^{-1})\xi. \quad (10)$$

By the definition of ξ, it is easy to see that $(\mathbf{1}^T \otimes I)\xi = 0$. For a connected graph $\mathcal{G}$, it then follows from Lemma 1 that

$$V_1(\xi) \geq \frac{1}{2}\lambda_2\xi^T(I_N \otimes P^{-1})\xi \geq \frac{\lambda_2}{2\lambda_{\max}(P)}\|\xi\|^2. \quad (11)$$

The time derivative of V_1 along the trajectory of (4) is given by

$$\begin{aligned}\dot{V}_1 &= \xi^T(\mathcal{L} \otimes P^{-1})\dot{\xi} \\ &= \xi^T[(\mathcal{L} \otimes P^{-1}A + c\mathcal{L}^2 \otimes P^{-1}BK)\xi \\ &\quad + (\mathcal{L} \otimes P^{-1}B)F(x,t) + (\mathcal{L}\rho \otimes P^{-1}B)G(\xi)].\end{aligned} \quad (12)$$

By using Assumption 2, we can obtain that

$$\begin{aligned}\xi^T(\mathcal{L} \otimes P^{-1}B)F(x,t) &= \sum_{i=1}^N\sum_{j=1}^N \mathcal{L}_{ij}\xi_j^T P^{-1}Bf_i(x_i,t) \\ &\leq \sum_{i=1}^N \|B^TP^{-1}\sum_{j=1}^N \mathcal{L}_{ij}\xi_j\|\|f_i(x_i,t)\| \\ &\leq \sum_{i=1}^N \rho_i(x_i,t)\|B^TP^{-1}\sum_{j=1}^N \mathcal{L}_{ij}\xi_j\|.\end{aligned} \quad (13)$$

Next, consider the following three cases.
i) $\rho_i(x_i,t)\|K\sum_{j=1}^N \mathcal{L}_{ij}\xi_j\| > \kappa$, $i = 1, \cdots, N$.
In this case, it follows from (3) and (5) that

$$\begin{aligned}&\xi^T(\mathcal{L}\rho \otimes P^{-1}B)G(\xi) \\ &\quad = -\sum_{i=1}^N \rho_i(x_i,t)\|B^TP^{-1}\sum_{j=1}^N \mathcal{L}_{ij}\xi_j\|.\end{aligned} \quad (14)$$

Substituting (14) and and (13) into (12) yields $\dot{V}_1 \leq \frac{1}{2}\xi^T\mathcal{X}\xi$, where

$$\mathcal{X} = \mathcal{L} \otimes (P^{-1}A + A^TP^{-1}) - 2c\mathcal{L}^2 \otimes P^{-1}BB^TP^{-1}. \quad (15)$$

ii) $\rho_i(x_i,t)\|K\sum_{j=1}^N \mathcal{L}_{ij}\xi_j\| \leq \kappa$, $i = 1, \cdots, N$.
In this case, we can get from (3) and (5) that

$$\begin{aligned}&\xi^T(\mathcal{L}\rho \otimes P^{-1}B)G(\xi) \\ &\quad = -\sum_{i=1}^N \frac{\rho_i(x_i,t)}{\kappa}\|B^TP^{-1}\sum_{j=1}^N \mathcal{L}_{ij}\xi_j\|^2 \leq 0.\end{aligned} \quad (16)$$

Substituting (14), (13), and (16) into (12) gives

$$\dot{V}_1 \leq \frac{1}{2}\xi^T\mathcal{X}\xi + N\kappa. \quad (17)$$

iii) ξ satisfies neither case i) nor case ii).

Without loss of generality, assume that $\rho_i(x_i,t)\|K\sum_{j=1}^N \mathcal{L}_{ij}\xi_j\| > \kappa$, $i = 1, \cdots, l$, and $\rho_i(x_i,t)\|K\sum_{j=1}^N \mathcal{L}_{ij}\xi_j\| \leq \kappa$, $i = l+1, \cdots, N$, where $2 \leq l \leq N-1$. In this case, we can get that

$$\begin{aligned}&\xi^T(\mathcal{L}\rho \otimes P^{-1}B)G(\xi) \\ &\quad \leq -\sum_{i=1}^l \rho_i(x_i,t)\|B^TP^{-1}\sum_{j=1}^N \mathcal{L}_{ij}\xi_j\|.\end{aligned} \quad (18)$$

Then, it follows from (12), (14), (18), and (13) that $\dot{V}_1 \leq \frac{1}{2}\xi^T\mathcal{X}\xi + (N-l)\kappa$.
Therefore, by analyzing the above three cases, we get that $\dot{V}_1$ satisfies (17) for all $\xi \in \mathbf{R}^{Nn}$. Note that (17) can be rewritten as

$$\begin{aligned}\dot{V}_1 &\leq -\alpha V_1 + \alpha V_1 + \frac{1}{2}\xi^T\mathcal{X}\xi + N\kappa \\ &= -\alpha V_1 + \frac{1}{2}\xi^T(\mathcal{X} + \alpha\mathcal{L} \otimes P^{-1})\xi + N\kappa,\end{aligned} \quad (19)$$

where $\alpha > 0$.
Because $\mathcal{G}$ is connected, it follows from lemma 1 that zero is a simple eigenvalue of $\mathcal{L}$ and all the other eigenvalues are positive. Let $U = [\,\frac{\mathbf{1}}{\sqrt{N}}\ Y_1\,]$ and $U^T = \begin{bmatrix}\frac{\mathbf{1}^T}{\sqrt{N}} \\ Y_2\end{bmatrix}$, with $Y_1 \in \mathbf{R}^{N\times(N-1)}$, $Y_2 \in \mathbf{R}^{(N-1)\times N}$, be such unitary matrices that $U^T\mathcal{L}U = \Lambda \triangleq \operatorname{diag}(0, \lambda_2, \cdots, \lambda_N)$, where $\lambda_2 \leq \cdots \leq \lambda_N$ are the nonzero eigenvalues of $\mathcal{L}$. Let $\bar{\xi} \triangleq [\bar{\xi}_1^T, \cdots, \bar{\xi}_N^T]^T = (U^T \otimes P^{-1})\xi$. By the definitions of ξ and $\bar{\xi}$, it is easy to see that $\bar{\xi}_1 = (\frac{\mathbf{1}^T}{\sqrt{N}} \otimes P^{-1})\xi = (\frac{\mathbf{1}^T}{\sqrt{N}}M \otimes P^{-1})x = 0$. Then, it follows that

$$\begin{aligned}&\xi^T(\mathcal{X} + \alpha\mathcal{L} \otimes P^{-1})\xi \\ &= \bar{\xi}^T[\Lambda \otimes (AP + PA^T + \alpha P) - 2c\Lambda^2 \otimes BB^T]\bar{\xi} \\ &\leq \sum_{i=2}^N \lambda_i\bar{\xi}_i^T(AP + PA^T + \alpha P - 2BB^T)\bar{\xi}_i.\end{aligned} \quad (20)$$

Because $\alpha = -\lambda_{\max}(AP + PA^T - 2BB^T)/\lambda_{\max}(P)$, we can see from (20) that $\xi^T(\mathcal{X} + \alpha\mathcal{L} \otimes P^{-1})\xi \leq 0$. Then, we can get from (19) that

$$\dot{V}_1 \leq -\alpha V_1 + N\kappa. \quad (21)$$

By using the well-known Comparison lemma (Lemma 3.4 in [20]), we can obtain from (21) that

$$V_1(\xi) \leq [V_1(\xi(0)) - \frac{N\kappa}{\alpha}]\exp(-\alpha t) + \frac{N\kappa}{\alpha}, \quad (22)$$

which, by (11), implies that ξ exponentially converges to the residual set $\mathcal{D}_1$ in (8) with a rate not less than $\exp(-\alpha t)$. ■

Remark 1. The distributed consensus protocol (2) consists of a linear part and a nonlinear part, where the term $\rho_i(x_i,t)g(K\sum_{j=1}^N a_{ij}(x_i - x_j))$ is used to suppress the effect of the uncertainties $f_i(x_i,t)$. For the case where $f_i(x_i,t) = 0$, we can accordingly remove $\rho_i(x_i,t)g(K\sum_{j=1}^N a_{ij}(x_i - x_j))$ from (2), which can recover the static consensus protocols as in [7, 8]. As shown in Proposition 2 of [7], a necessary and sufficient condition for the existence of a $P > 0$ to the LMI (7) is that (A, B) is stabilizable. Therefore, a sufficient condition for the existence of (2) satisfying Theorem 1 is that (A, B) is stabilizable.

Remark 2. Note that the residual set $\mathcal{D}_1$ depends on the smallest nonzero eigenvalue of $\mathcal{L}$, the number of agents, the largest eigenvalue of P, and the size κ of the boundary layer. By choosing a sufficiently small κ, the consensus error ξ under the protocol (2) can converge to an arbitrarily small neighborhood of zero, which is acceptable in most applications.

4 Distributed Adaptive Consensus Protocol

In the last section, the design of the distributed protocol (2) relies on the minimal nonzero eigenvalue λ_2 of $\mathcal{L}$ and the upper bounds $\rho_i(x_i, t)$ of the matching uncertainties $f_i(x_i, t)$. However, λ_2 is global information in the sense that each agent has to know the entire communication graph to compute it. Besides, the bounds $\rho_i(x_i, t)$ of the uncertainties $f_i(x_i, t)$ might not be easily obtained in some cases. In this section, we will design fully distributed protocols without requring either λ_2 or $\rho_i(x_i, t)$.

Before moving forward, we introduce a modified assumption regarding the bounds of the lumped uncertainties $f_i(x_i, t), i = 1, \cdots, N$.

Assumption 2. There are positive constants d_i and e_i such that $\|f_i(x_i, t)\| \le d_i + e_i\|x_i\|, i = 1, \cdots, N$.

Based on the local state information of neighboring agents, we propose the following distributed adaptive protocol to each agent:

$$
\begin{aligned}
u_i &= \bar{d}_i K \sum_{j=1}^N a_{ij}(x_i - x_j) + r(K \sum_{j=1}^N a_{ij}(x_i - x_j)),\\
\dot{\bar{d}}_i &= \tau_i[-\varphi_i \bar{d}_i + \sum_{j=1}^N a_{ij}(x_i - x_j)^T \Gamma \sum_{j=1}^N a_{ij}(x_i - x_j)\\
&\quad + \|K \sum_{j=1}^N a_{ij}(x_i - x_j)\|],\\
\dot{\bar{e}}_i &= \epsilon_i[-\psi_i \bar{e}_i + \|K \sum_{j=1}^N a_{ij}(x_i - x_j)\|\|x_i\|],
\end{aligned}
\tag{23}
$$

where $\bar{d}_i(t)$ and $\bar{e}_i(t)$ are the adaptive gains associated with the i-th agent, $\Gamma \in \mathbf{R}^{n\times n}$ is the feedback gain matrix, τ_i and ϵ_i are positive scalars, φ_i and ψ_i are small positive constants chosen by the designer, the nonlinear function $r(\cdot)$ is defined as follows: for $w \in \mathbf{R}^n$,

$$
r(w) = \begin{cases} \frac{w(\bar{d}_i + \bar{e}_i\|x_i\|)}{\|w\|}, & \text{if } (\bar{d}_i + \bar{e}_i\|x_i\|)\|w\| > \kappa \\ \frac{w(\bar{d}_i + \bar{e}_i\|x_i\|)^2}{\kappa}, & \text{if } (\bar{d}_i + \bar{e}_i\|x_i\|)\|w\| \le \kappa \end{cases},
\tag{24}
$$

and the rest of the variables are defined as in (2).

Let the consensus error ξ be defined as in (6) and $\overline{D} = \mathrm{diag}(\bar{d}_1, \cdots, \bar{d}_N)$. Then, it is not difficult to get from (1) and (23) that the closed-loop network dynamics can be written as

$$
\begin{aligned}
\dot{\xi} &= (I_N \otimes A + M\overline{D}\mathcal{L} \otimes BK)\xi + (M \otimes B)F(x, t)\\
&\quad + (M \otimes B)R(\xi),\\
\dot{\bar{d}}_i &= \tau_i[-\varphi_i \bar{d}_i + \sum_{j=1}^N \mathcal{L}_{ij}\xi_j^T \Gamma \sum_{j=1}^N \mathcal{L}_{ij}\xi_j + \|K \sum_{j=1}^N \mathcal{L}_{ij}\xi_j\|],\\
\dot{\bar{e}}_i &= \epsilon_i[-\psi_i \bar{e}_i + \|K \sum_{j=1}^N \mathcal{L}_{ij}\xi_j\|\|x_i\|],\ i = 1, \cdots, N,
\end{aligned}
\tag{25}
$$

where

$$
R(\xi) \triangleq \begin{bmatrix} r(K\sum_{j=1}^N \mathcal{L}_{1j}\xi_j) \\ \vdots \\ r(K\sum_{j=1}^N \mathcal{L}_{Nj}\xi_j) \end{bmatrix},
\tag{26}
$$

and the rest of the variables are defined as in (4).

To establish the ultimate boundedness of the states ξ, $\bar{d}_i$, and $\bar{e}_i$ of (25), we use the following Lyapunov function candidate

$$
V_2 = \frac{1}{2}\xi^T(\mathcal{L} \otimes P^{-1})\xi + \sum_{i=1}^N \frac{\tilde{d}_i^2}{2\tau_i} + \sum_{i=1}^N \frac{\tilde{e}_i^2}{2\epsilon_i},
\tag{27}
$$

where $\tilde{e}_i = \bar{e}_i - e_i$, $\tilde{d}_i = \bar{d}_i - \beta$, $i = 1, \cdots, N$, and $\beta \ge \max_{i=1,\cdots,N}\{d_i, \frac{1}{\lambda_2}\}$.

Theorem 2. Suppose that $\mathcal{G}$ is connected and Assumptions 2 holds. The feedback gain matrices of the distributed adaptive protocol (23) are designed as $K = -B^T P^{-1}$ and $\Gamma = P^{-1}BB^T P^{-1}$, where $P > 0$ is a solution to the LMI (7). Then, both the consensus error ξ and the adaptive gains $\bar{d}_i$ and $\bar{e}_i$, $i = 1, \cdots, N$, in (25) are uniformly ultimately bounded and the following statements hold.

i) For any φ_i and ψ_i, ξ, $\tilde{d}_i$, and $\tilde{e}_i$ exponentially converge to the residual set

$$
\mathcal{D}_2 \triangleq \{\xi, \tilde{d}_i, \tilde{e}_i\ :\ V_2 < \frac{1}{2\delta}\sum_{i=1}^N(\beta^2\varphi_i + e_i^2\psi_i) + \frac{N\kappa}{4\delta}\},
\tag{28}
$$

with a convergence rate faster than $\exp(-\delta t)$, where $\delta \triangleq \min_{i=1,\cdots,N}\{\alpha, \varphi_i\tau_i, \psi_i\epsilon_i\}$ and α is defined as in (9).

ii) If small φ_i and ψ_i satisfy $\varrho \triangleq \max_{i=1,\cdots,N}\{\varphi_i\tau_i, \psi_i\epsilon_i\} < \alpha$, then in addition to i), ξ exponentially converges to the residual set

$$
\begin{aligned}
\mathcal{D}_3 \triangleq \{\xi\ :\ \|\xi\|^2 \le \frac{\lambda_{\max}(P)}{\lambda_2\rho}[\sum_{i=1}^N(\beta^2\varphi_i + e_i^2\psi_i)\\
+ \frac{1}{2}N\kappa]\}.
\end{aligned}
\tag{29}
$$

with a rate faster than $\exp(-\varrho t)$.

Proof. The time derivative of V_2 along (25) can be obtained as

$$
\begin{aligned}
\dot{V}_2 &= \xi^T(\mathcal{L}\otimes P^{-1})\dot{\xi} + \sum_{i=1}^{N}\frac{\tilde{d}_i}{\tau_i}\dot{\tilde{d}}_i + \sum_{i=1}^{N}\frac{\tilde{e}_i}{\epsilon_i}\dot{\tilde{e}}_i \\
&= \xi^T[(\mathcal{L}\otimes P^{-1}A + \mathcal{L}\widetilde{D}\mathcal{L}\otimes P^{-1}BK)\xi \\
&\quad + \mathcal{L}\otimes P^{-1}B)F(x,t) + (\mathcal{L}\otimes P^{-1}B)R(\xi)] \\
&\quad + \sum_{i=1}^{N}\tilde{d}_i[-\varphi_i(\tilde{d}_i+\beta) + \sum_{j=1}^{N}\mathcal{L}_{ij}\xi_j^T\Gamma\sum_{j=1}^{N}\mathcal{L}_{ij}\xi_j \\
&\quad + \|K\sum_{j=1}^{N}\mathcal{L}_{ij}\xi_j\|] + \sum_{i=1}^{N}\tilde{e}_i[-\psi_i(\tilde{e}_i+e_i) \\
&\quad + \|K\sum_{j=1}^{N}\mathcal{L}_{ij}\xi_j\|\|x_i\|],
\end{aligned}
\tag{30}
$$

where $\widetilde{D}(t) = \mathrm{diag}(\tilde{d}_1+\beta,\cdots,\tilde{d}_N+\beta)$.
By noting that $K = -BP^{-1}$, it is easy to get that

$$
\begin{aligned}
\xi^T(\mathcal{L}\widetilde{D}\mathcal{L}\otimes P^{-1}BK)\xi &= -\sum_{i=1}^{N}(\tilde{d}_i+\beta)\sum_{j=1}^{N}\mathcal{L}_{ij}\xi_j^T \\
&\quad \times P^{-1}BB^TP^{-1}\sum_{j=1}^{N}\mathcal{L}_{ij}\xi_j.
\end{aligned}
\tag{31}
$$

In light of Assumption 2, we can obtain that

$$
\begin{aligned}
&\xi^T(\mathcal{L}\otimes P^{-1}B)F(x,t) \\
&= \sum_{j=1}^{N}(d_i+e_i\|x_i\|)\|B^TP^{-1}\sum_{j=1}^{N}\mathcal{L}_{ij}\xi_j\|.
\end{aligned}
\tag{32}
$$

In what follows, we consider three cases.
i) $(\bar{d}_i+\bar{e}_i\|x_i\|)\|K\sum_{j=1}^{N}\mathcal{L}_{ij}\xi_j\| > \kappa$, $i=1,\cdots,N$.
In this case, we can get from (24) and (26) that

$$
\begin{aligned}
&\xi^T(\mathcal{L}\otimes P^{-1}B)R(\xi) \\
&= -\sum_{i=1}^{N}[\tilde{d}_i+\beta+(\tilde{e}_i+e_i)\|x_i\|]\|B^TP^{-1}\sum_{j=1}^{N}\mathcal{L}_{ij}\xi_j\|.
\end{aligned}
\tag{33}
$$

Substituting (31), (32), and (33) into (30) yields

$$
\begin{aligned}
\dot{V}_2 &\le \frac{1}{2}\xi^T\mathcal{Y}\xi - \sum_{i=1}^{N}(\beta-d_i)\|B^TP^{-1}\sum_{j=1}^{N}\mathcal{L}_{ij}\xi_j\| \\
&\quad - \frac{1}{2}\sum_{i=1}^{N}(\varphi_i\tilde{d}_i^2+\psi_i\tilde{e}_i^2) + \frac{1}{2}\sum_{i=1}^{N}(\beta^2\varphi_i+e_i^2\psi_i),
\end{aligned}
$$

where we have used the fact that $-\tilde{d}_i^2-\tilde{d}_i\beta \le -\frac{1}{2}\tilde{d}_i^2+\frac{1}{2}\beta^2$ and $\mathcal{Y} \triangleq \mathcal{L}\otimes(P^{-1}A+A^TP^{-1})-2\beta\mathcal{L}^2\otimes P^{-1}BB^TP^{-1}$.
ii) $(\bar{d}_i+\bar{e}_i\|x_i\|)\|K\sum_{j=1}^{N}\mathcal{L}_{ij}\xi_j\| \le \kappa$, $i=1,\cdots,N$.
In this case, we can get from (24) and (26) that

$$
\begin{aligned}
&\xi^T(\mathcal{L}\otimes P^{-1}B)R(\xi) \\
&= -\sum_{i=1}^{N}\frac{(\bar{d}_i+\bar{e}_i\|x_i\|)^2}{\kappa}\|B^TP^{-1}\sum_{j=1}^{N}\mathcal{L}_{ij}\xi_j\|^2.
\end{aligned}
\tag{34}
$$

Then, it follows from (31), (32), (34), and (30) that

$$
\begin{aligned}
\dot{V}_2 &\le \frac{1}{2}\xi^T\mathcal{Y}\xi - \sum_{i=1}^{N}(\beta-d_i)\|B^TP^{-1}\sum_{j=1}^{N}\mathcal{L}_{ij}\xi_j\| + \frac{1}{4}N\kappa \\
&\quad - \frac{1}{2}\sum_{i=1}^{N}(\varphi_i\tilde{d}_i^2+\psi_i\tilde{e}_i^2) + \frac{1}{2}\sum_{i=1}^{N}(\beta^2\varphi_i+e_i^2\psi_i).
\end{aligned}
\tag{35}
$$

Note that to get (35), we have used the following fact: $-\frac{(\bar{d}_i+\bar{e}_i\|x_i\|)^2}{\kappa}\|B^TP^{-1}\sum_{j=1}^{N}\mathcal{L}_{ij}\xi_j\|^2 + (\bar{d}_i+\bar{e}_i\|x_i\|)\|B^TP^{-1}\sum_{j=1}^{N}\mathcal{L}_{ij}\xi_j\| \le \frac{1}{4}\kappa$ for $(\bar{d}_i+\bar{e}_i\|x_i\|)\|K\sum_{j=1}^{N}\mathcal{L}_{ij}\xi_j\| \le \kappa$.
iii) $(\bar{d}_i+\bar{e}_i\|x_i\|)\|K\sum_{j=1}^{N}\mathcal{L}_{ij}\xi_j\| > \kappa$, $i=1,\cdots,l$, and $(\bar{d}_i+\bar{e}_i\|x_i\|)\|K\sum_{j=1}^{N}\mathcal{L}_{ij}\xi_j\| \le \kappa$, $i=l+1,\cdots,N$, where $2 \le l \le N-1$.
By following similar steps in the two cases above, it is not difficult to get that

$$
\begin{aligned}
\dot{V}_2 &\le \frac{1}{2}\xi^T\mathcal{Y}\xi - \sum_{i=1}^{N}(\beta-d_i)\|B^TP^{-1}\sum_{j=1}^{N}\mathcal{L}_{ij}\xi_j\| \\
&\quad - \frac{1}{2}\sum_{i=1}^{N}(\varphi_i\tilde{d}_i^2+\psi_i\tilde{e}_i^2) + \frac{1}{2}\sum_{i=1}^{N}(\beta^2\varphi_i+e_i^2\psi_i) \\
&\quad + \frac{1}{4}(N-l)\kappa.
\end{aligned}
$$

Therefore, based on the above three cases, we can get that $\dot{V}_2$ satisfies (35) for all $\xi \in \mathbf{R}^{Nn}$. Because $\beta \ge \max_{i=1,\cdots,N} d_i$ and $\beta\lambda_2 \ge 1$, it follows from (35) that

$$
\begin{aligned}
\dot{V}_2 &\le \frac{1}{2}\xi^T\mathcal{Y}\xi - \frac{1}{2}\sum_{i=1}^{N}(\varphi_i\tilde{d}_i^2+\psi_i\tilde{e}_i^2) \\
&\quad + \frac{1}{2}\sum_{i=1}^{N}(\beta^2\varphi_i+e_i^2\psi_i) + \frac{1}{4}N\kappa.
\end{aligned}
\tag{36}
$$

Note that (36) can be rewritten into

$$
\begin{aligned}
\dot{V}_2 &\le -\delta V_2 + \frac{1}{2}\xi^T(\mathcal{Y}+\delta\mathcal{L}\otimes P^{-1})\xi - \frac{1}{2}\sum_{i=1}^{N}[(\varphi_i-\frac{\delta}{\tau_i})\tilde{d}_i^2 \\
&\quad + (\psi_i-\frac{\delta}{\epsilon_i})\tilde{e}_i^2)] + \frac{1}{2}\sum_{i=1}^{N}(\beta^2\varphi_i+e_i^2\psi_i) + \frac{1}{4}N\kappa.
\end{aligned}
\tag{37}
$$

Because $\beta\lambda_2 \ge 1$ and $0 < \delta \le \alpha$, by following similar steps in the proof of Theorem 1, we can show that $\xi^T(\mathcal{Y}+\delta\mathcal{L}\otimes P^{-1})\xi \le 0$. Further, by noting that $\delta \le \min_{i=1,\cdots,N}\{\varphi_i\tau_i, \psi_i\epsilon_i\}$, it follows from (37) that

$$
\dot{V}_2 \le -\delta V_2 + \frac{1}{2}\sum_{i=1}^{N}(\beta^2\varphi_i+e_i^2\psi_i) + \frac{1}{4}N\kappa,
\tag{38}
$$

which implies that

$$
\begin{aligned}
V_2 &\le [V_2(0) - \frac{N\kappa}{4\delta} - \frac{1}{2\delta}\sum_{i=1}^{N}(\beta^2\varphi_i+e_i^2\psi_i)]\exp(-\delta t) \\
&\quad + \frac{1}{2\delta}\sum_{i=1}^{N}(\beta^2\varphi_i+e_i^2\psi_i) + \frac{N\kappa}{4\delta}.
\end{aligned}
\tag{39}
$$

Therefore, V_2 exponentially converges to the residual set $\mathcal{D}_2$ in (28) with a rate faster than $\exp(-\delta t)$, implying that ξ, $\bar{d}_i$, and $\bar{e}_i$ are uniformly ultimately bounded.

Next, if $\varrho \triangleq \max_{i=1,\cdots,N}\{\varphi_i\tau_i, \psi_i\epsilon_i\} < \alpha$, we can obtain a smaller residual set for ξ by rewriting (37) into

$$\begin{aligned}\dot{V}_2 &\leq -\varrho V_2 + \frac{1}{2}\xi^T(\mathcal{Y} + \alpha\mathcal{L}\otimes P^{-1})\xi + \frac{1}{4}N\kappa \\ &\quad - \frac{\alpha-\varrho}{2}\xi^T(\mathcal{L}\otimes P^{-1})\xi + \frac{1}{2}\sum_{i=1}^{N}(\beta^2\varphi_i + e_i^2\psi_i) \\ &\leq -\varrho V_2 - \frac{\lambda_2(\alpha-\varrho)}{2\lambda_{\max}(P)}\|\xi\|^2 + \frac{1}{4}N\kappa \\ &\quad + \frac{1}{2}\sum_{i=1}^{N}(\beta^2\varphi_i + e_i^2\psi_i).\end{aligned} \tag{40}$$

Obviously, it follows from (40) that $\dot{V}_2 \leq -\varrho V_2$ if $\|\xi\|^2 > \frac{\lambda_{\max}(P)}{\lambda_2(\alpha-\varrho)}[\sum_{i=1}^{N}(\beta^2\varphi_i + e_i^2\psi_i) + \frac{1}{2}N\kappa]$. Then, by noting $V_2 \geq \frac{\lambda_2}{2\lambda_{\max}(P)}\|\xi\|^2$, we can get that ξ exponentially converges to the residual set $\mathcal{D}_3$ in (29) with a rate faster than $\exp(-\varrho t)$. ■

Remark 3. From (28) and (29), we can observe that the residual sets $\mathcal{D}_2$ and $\mathcal{D}_3$ decrease as κ decreases. Given κ, smaller φ_i and ψ_i give a smaller bound for ξ and at the same time yield a larger bound for $\bar{d}_i$ and $\bar{e}_i$. For the case where $\varphi_i = 0$ and $\psi_i = 0$, $\bar{d}_i$ and $\bar{e}_i$ will tend to infinity. In real implementations, if large $\bar{d}_i$ and $\bar{e}_i$ are acceptable, we can choose φ_i, ψ_i, and κ to be relatively small in order to guarantee a small ξ.

5 Conclusion

This paper has addressed the robust consensus problem for multi-agent systems with heterogeneous matching uncertainties. Distributed static and adaptive consensus protocols have been designed, under which the consensus error has been shown to be ultimately bounded and exponentially converges to small adjustable residual sets. An interesting future topic is to consider more general uncertainties which do not necessarily satisfy the matching condition. Another direction is to discuss the case with general directed and switching communication graphs.

REFERENCES

[1] R. Olfati-Saber, J. Fax, and R. Murray, "Consensus and cooperation in networked multi-agent systems," *Proceedings of the IEEE*, vol. 95, no. 1, pp. 215–233, 2007.

[2] A. Jadbabaie, J. Lin, and A. Morse, "Coordination of groups of mobile autonomous agents using nearest neighbor rules," *IEEE Transactions on Automatic Control*, vol. 48, no. 6, pp. 988–1001, 2003.

[3] R. Olfati-Saber and R. Murray, "Consensus problems in networks of agents with switching topology and time-delays," *IEEE Transactions on Automatic Control*, vol. 49, no. 9, pp. 1520–1533, 2004.

[4] T. Vicsek, A. Czirók, E. Ben-Jacob, I. Cohen, and O. Shochet, "Novel type of phase transition in a system of self-driven particles," *Physical Review Letters*, vol. 75, no. 6, pp. 1226–1229, 1995.

[5] W. Ren and R. Beard, "Consensus seeking in multiagent systems under dynamically changing interaction topologies," *IEEE Transactions on Automatic Control*, vol. 50, no. 5, pp. 655–661, 2005.

[6] Y. Hong, G. Chen, and L. Bushnell, "Distributed observers design for leader-following control of multi-agent networks," *Automatica*, vol. 44, no. 3, pp. 846–850, 2008.

[7] Z. Li, Z. Duan, G. Chen, and L. Huang, "Consensus of multiagent systems and synchronization of complex networks: A unified viewpoint," *IEEE Transactions on Circuits and Systems I: Regular Papers*, vol. 57, no. 1, pp. 213–224, 2010.

[8] H. Zhang, F. Lewis, and A. Das, "Optimal design for synchronization of cooperative systems: State feedback, observer, and output feedback," *IEEE Transactions on Automatic Control*, vol. 56, no. 8, pp. 1948–1952, 2011.

[9] K. You and L. Xie, "Network topology and communication data rate for consensusability of discrete-time multi-agent systems," *IEEE Transactions on Automatic Control*, vol. 56, no. 10, pp. 2262–2275, 2011.

[10] W. Ren, "Consensus tracking under directed interaction topologies: Algorithms and experiments," *IEEE Transactions on Control Systems Technology*, vol. 18, no. 1, pp. 230–237, 2010.

[11] Y. Cao, W. Ren, and Y. Li, "Distributed discrete-time coordinated tracking with a time-varying reference state and limited communication," *Automatica*, vol. 45, no. 5, pp. 1299–1305, 2009.

[12] W. Ren and E. Atkins, "Distributed multi-vehicle coordinated control via local information exchange," *International Journal of Robust and Nonlinear Control*, vol. 17, no. 10-11, pp. 1002–1033, 2007.

[13] F. Jiang and L. Wang, "Consensus seeking of high-order dynamic multi-agent systems with fixed and switching topologies," *International Journal of Control*, vol. 85, no. 2, pp. 404–420, 2010.

[14] S. Tuna, "Conditions for synchronizability in arrays of coupled linear systems," *IEEE Transactions on Automatic Control*, vol. 54, no. 10, pp. 2416–2420, 2009.

[15] J. Seo, H. Shim, and J. Back, "Consensus of high-order linear systems using dynamic output feedback compensator: Low gain approach," *Automatica*, vol. 45, no. 11, pp. 2659–2664, 2009.

[16] C. Ma and J. Zhang, "Necessary and sufficient conditions for consensusability of linear multi-sgent systems," *IEEE Transactions on Automatic Control*, vol. 55, no. 5, pp. 1263–1268, 2010.

[17] Z. Li, W. Ren, X. Liu, and M. Fu, "Consensus of multiagent systems with general linear and Lipschitz nonlinear dynamics using distributed adaptive protocols," *IEEE Transactions on Automatic Control*, in press, 2012.

[18] Z. Li, W. Ren, X. Liu, and L. Xie, "Distributed consensus of linear multi-agent systems with adaptive dynamic protocols," *Automatica*, in press, 2013.

[19] Z. Li, X. Liu, W. Ren, and L. Xie, "Distributed tracking control for linear multiagent systems with a leader of bounded unknown input," *IEEE Transactions on Automatic Control*, vol. 58, no. 2, pp. 518–523, 2013.

[20] H. Khalil, *Nonlinear Systems*. Englewood Cliffs, NJ: Prentice Hall, 2002.