Generalized Criterions for Determining the Maximal Ellipsoidal Invariant Set of Linear Systems under Saturated Linear Feedback

Yuanlong Li[1], Zongli Lin[2,1]

1. Department of Automation, Shanghai Jiao Tong University, Shanghai 200240, China
E-mail: liyuanlong0301@163.com

2. Charles L. Brown Department of Electrical and Computer Engineering, University of Virginia, Charlottesville, VA 22904, U.S.A.
E-mail: zl5y@virginia.edu

Abstract: Ellipsoids have been extensively used as estimates of the domain of attraction of a linear system under a saturated linear feedback. For a linear system with a single input subject to actuator saturation, based on a convex hull representation of the saturated linear feedback, a necessary and sufficient condition for an ellipsoid to be contractively invariant was previously established, which, through the solution of an LMI problem, leads to the maximal ellipsoidal invariant set. For a linear system with multiple inputs subject to actuator saturation, it has also been proven that the optimal ellipsoid resulting from the optimization problem is the maximal one only under additional conditions. In this paper, we develop a complete characterization of the maximal ellipsoidal invariant set of a linear system with multiple inputs subject to actuator saturation, which is summarized as a comprehensive algorithm to determine if an invariant ellipsoid is the maximal one.

Key Words: Actuator saturation, ellipsoidal invariant set, domain of attraction.

1 Introduction

The stability and stabilization of a linear system subject to actuator saturation have been drawing continual interest from control theorists for decades now. For a linear system that has a pole in the open right-half plane and is subject to actuator saturation, linear feedback can only achieve local stabilization [12]. This presents an important problem of estimating the domain of attraction for a linear system subject to actuator saturation, which has attracted tremendous attention in resent years (see, *e.g.*, [2, 4, 6]).

As a subset of the domain of attraction, a contractively invariant set, from which all the trajectories of systems will remain in it and converge to the equilibrium point, is widely used as an estimate of the domain of attraction [3]. As a popular candidate invariant set, the ellipsoid has been widely used in estimating the domain of attraction for a linear system subject to actuator saturation, due to its simple representation as a level set of a quadratic Lypunov function (see, *e.g.*, [1, 4, 6, 13]). Global/local sector conditions [4] or convex hull representation [6] of saturation functions are used to express the derivative of the quadratic Lyapunov function in terms of a single or a set of negative definite quadratic functions, which ensure the negative definiteness of the derivative of the quadratic Lyapunov function. Conditions in the form of linear matrix inequalities (LMIs) are established that guarantee the negative definiteness of these quadratic functions and constrained optimization problems are formulated that result in a large contractively invariant ellipsoid.

We restrict our attention to the use of the convex hull representation, which is less conservative than the sector conditions in dealing with saturation functions. Since the conditions for the negative definiteness of quadratic Lyapunov functions are sufficient, the ellipsoid obtained through the optimization problem cannot be guaranteed to be the maximal possible contractively invariant ellipsoid. As established in [7], however, for a linear system with a single input subject to actuator saturation, the ellipsoid such obtained is the maximal one, since the conditions for the negative definiteness of those quadratic functions are proven to be necessary as well. For a linear system with multiple inputs subject to actuator saturation, it is pointed out in [8] that the contractively invariant ellipsoid obtained from the optimization problem is also the maximal possible contractively invariant ellipsoid only when certain additional conditions are satisfied. However, it is worth noting that the conditions identified in [8] are the closest to those for the maximal possible contractively invariant ellipsoid.

The main contribution of this paper is to propose a comprehensive algorithm for the complete determination of the maximal possible contractively invariant ellipsoid for a linear system with multiple inputs subject to actuator saturation. This algorithm summarizes several criterions pertaining to conditions under which an optimal ellipsoid obtained from certain optimization problems with constraints in the form of LMIs or nonlinear equations is the maximal contractively invariant one. Because of the space limitation, the proofs of all new results, including Lemmas 3-5 and Theorems 3-6, are omitted in this conference version of the paper, and can be found in the journal version [11].

We will use standard notation. For a vector $u = [u_1 \ \ u_2 \ \ \cdots \ \ u_m]^{\mathrm{T}}$, $|u|_\infty := \max_i |u_i|$. For two integers $k_1, k_2, k_1 < k_2$, $I[k_1, k_2] := \{k_1, k_1+1, \cdots, k_2\}$. For a positive definite $P \in \mathbf{R}^{n\times n}$ and a positive scalar ρ, $\mathcal{E}(P,\rho) := \{x \in \mathbf{R}^n : x^{\mathrm{T}}Px \le \rho\}$, $\mathcal{E}^o(P,\rho) := \{x \in \mathbf{R}^n : x^{\mathrm{T}}Px < \rho\}$ and $\partial\mathcal{E}(P,\rho) := \{x \in \mathbf{R}^n : x^{\mathrm{T}}Px = \rho\}$. For a matrix $H \in \mathbf{R}^{m\times n}$, $\mathcal{L}(H) := \{x \in \mathbf{R}^n : |Hx|_\infty \le 1\}$. For a matrix A, $\lambda_{\max}(A)$ denotes the maximal eigenvalue of A, and $\mathrm{He}(A) = A^{\mathrm{T}} + A$.

2 Preliminaries

Consider a linear system

$$\dot{x} = Ax + Bu, \quad x \in \mathbf{R}^n, u \in \mathbf{R}^m. \tag{1}$$

This work is supported by National Natural Science Foundation of China under Grant No. 61273105.

Under a saturated linear feedback $u = \mathrm{sat}(Fx)$, the closed-loop system is

$$\dot{x} = Ax + B\mathrm{sat}(Fx), \tag{2}$$

where $\mathrm{sat} : \mathbf{R}^m \to \mathbf{R}^m$ denotes the vector valued standard saturation function, which is defined as $\mathrm{sat}(u) = [\mathrm{sat}(u_1), \mathrm{sat}(u_2), \cdots, \mathrm{sat}(u_m)]^{\mathrm{T}}$, $\mathrm{sat}(u_i) = \mathrm{sgn}(u_i)\min\{1, |u_i|\}$. A signal u_i is said to saturate if $|u_i| > 1$ and it is said to saturate critically if $|u_i| = 1$. Given a positive definite matrix $P \in \mathbf{R}^{n\times n}$, let $V(x) = x^{\mathrm{T}}Px$. The ellipsoid $\mathcal{E}(P, \rho)$ is said to contractively invariant if

$$\dot{V}(x) = 2x^{\mathrm{T}}P(Ax + B\mathrm{sat}(Fx)) < 0,$$

for all $x \in \mathcal{E}(P, \rho) \setminus \{0\}$. It is said to be contractively invariant on a set $\Omega \subset \mathbf{R}^n$ if $\dot{V}(x) < 0$ for all $x \in (\mathcal{E}(P, \rho) \cap \Omega) \setminus \{0\}$. The following fact is clear.

Fact 1 *Let $\rho_c := \sup\{\rho > 0 : \ \mathcal{E}(P, \rho) \ \text{is contractively invariant}\}$. Then, a $\rho^* > 0$ is such that $\rho^* = \rho_c$ if and only if $\dot{V}(x) < 0, \forall\, x \in \mathcal{E}^o(P, \rho^*) \setminus \{0\}$, and $\dot{V}(x_0) = 0$ for some $x_0 \in \partial\mathcal{E}(P, \rho^*)$.*

For use later in the paper, we denote the maximal contractively invariant ellipsoid as $\mathcal{E}(P, \rho_c)$, and refer to $x_0 \in \partial\mathcal{E}(P, \rho_c)$ such that $\dot{V}(x_0) = 0$ as an extreme state.

We next recall the convex hull representation of a saturated linear feedback from [6]. Let $\mathcal{D}$ denote the set of $m\times m$ diagonal matrices whose diagonal elements are either 1 or 0. There are 2^m such matrices in $\mathcal{D}$, and we label them as $D_i, i \in I[1, 2^m]$. Denote $D_i^- = I - D_i$. Clearly, $D_i^- \in \mathcal{D}$.

Lemma 1 *[6] Let $F, H \in \mathbf{R}^{m\times n}$. Then, for any $x \in \mathcal{L}(H)$,*

$$\mathrm{sat}(Fx) \in \mathrm{co}\left\{D_iFx + D_i^-Hx : i \in I[1, 2^m]\right\},$$

where co stands for the convex hull.

From Lemma 1, the m dimensional nonlinear function $\mathrm{sat}(Fx)$ is expressed as a linear combination of the 2^m auxiliary linear feedbacks. Under this expression the conditions under which the ellipsoid $\mathcal{E}(P, \rho)$ is a contractively invariant set of the closed-loop system (2) were established in [6, 9] as follows.

Theorem 1 *Given an ellipsoid $\mathcal{E}(P, \rho)$, if there exists an $H \in \mathbf{R}^{m\times n}$ such that*

$$\mathrm{He}(P(A + B(D_iF + D_i^-H))) < 0, \quad \forall\, i \in I[1, 2^m], \tag{3}$$

and $\mathcal{E}(P, \rho) \subset \mathcal{L}(H)$, then $\mathcal{E}(P, \rho)$ is contractively invariant under the feedback $u = \mathrm{sat}(Fx)$.

Clearly, the condition

$$(A + BF)^{\mathrm{T}}P + P(A + BF) < 0 \tag{4}$$

in (3) is necessary. Theorem 1 provides a set of sufficient conditions under which $\mathcal{E}(P, \rho)$ is contractively invariant. These conditions are presented in terms of linear matrix inequalities. Moreover, as established in [6], for a single input system, that is, $m = 1$, the resulting linear matrix inequalities that characterize the invariance of an ellipsoid $\mathcal{E}(P, \rho)$ are also necessary.

Next, we will focus on the largest invariant ellipsoid $\mathcal{E}(P, \rho)$ that satisfies the conditions of Theorem 1 for a given positive definite matrix P. Based on Theorem 1, the following optimization problem can be formulated:

$$\begin{aligned} &\sup_{H} \ \rho \\ &\text{s.t.} \ \text{a)} \ \mathrm{He}\left(P(A + B(D_iF + D_i^-H)\right) \le 0, \ i \in I[1, 2^m], \\ &\qquad \text{b)} \ \rho h_jP^{-1}h_j^{\mathrm{T}} \le 1, j \in I[1, m]. \end{aligned} \tag{5}$$

Define $\rho^* = \sup_H \ \rho$. For a single input system, $\rho_c = \rho^*$, since Conditions a) and b) are necessary for $\mathcal{E}(P, \rho)$ to be contractively invariant. For multiple input systems, that is, $m \ge 2$, it is pointed out in [8] that $\rho^* = \rho_c$ is conditional. For completeness and convenience of presentation, we transcribe the main result in [8] that guarantees $\rho^* = \rho_c$.

Theorem 2 *Let (ρ^*, H^*) be an optimal solution to (5). Suppose that*

1) *there is only one j such that $\rho^*h_j^*P^{-1}h_j^{*\mathrm{T}} = 1$ (i.e., the boundary of $\mathcal{E}(P, \rho)$ only touches one pair of the hyperplanes $h_j^*x = \pm 1$);*
2) *there is only one i satisfying $\lambda_{\max}(T_i) = 0$, where $T_i = \mathrm{He}(P(A + B(D_iF + D_i^-H^*)))$. The matrix T_i has a single eigenvalue at 0 and the only nonzero element in D_i^- is the jth diagonal one ($D_i^-H^*$ choose only h_j^*).*

Let $x_0 = \rho^*P^{-1}h_j^{*\mathrm{T}}$, then x_0 is the unique intersection of $\mathcal{E}(P, \rho^*)$ with $h_j^*x = 1$.

3) *If $|f_kx_0| \le 1$ for all $k \ne j$,*

then $\rho^* = \rho_c$.

Theorem 2 presents a criterion for determining if the conditions in the optimization problem (5) result in the maximal possible contractively invariant ellipsoid for a linear system with multiple inputs subject to actuator saturation. As shown by an example in [8], the optimal solution to (5) does not always satisfy the conditions in Theorem 2. This implies that the conditions in Theorem 2 are of conservativeness. On the one hand, the result of Theorem 2 is established on the optimization problem (5), which results from Lemma 1. An alternative convex hull representation of saturated linear feedback [10] with less conservativeness than Lemma 1 is recalled as follows:

Lemma 2 *Let $F, H_i \in \mathbf{R}^{m\times n}, \ i \in I[1, m]$. Then, for any $x \in \mathcal{L}(H_i)$,*

$$\mathrm{sat}(Fx) \in \mathrm{co}\left\{D_iFx + D_i^-H_ix : i \in I[1, 2^m]\right\}. \tag{6}$$

A result similar to Lemma 2 can be found in [1, 13]. As shown in Lemma 2, an m-dimensional saturated linear feedback can be expressed as a linear combination of a set of 2^m auxiliary linear feedbacks, each of which associates with an H_i. On the other hand, the limitation of Theorem 2 also lies on that the extreme state only causes one input to saturate.

Motivated by the limitations of Theorem 2, we will start our presentation of the main results in the next sections with a result that generalizes Theorem 2. Several improved criterions will be proposed to determine the maximal possible contractively invariant ellipsoid, and, as a result, a comprehensive algorithm for the determination of the maximal contractively ellipsoidal invariant set will be developed for a linear system with multiple inputs subject to actuator saturation.

3 The Maximal Contractively Invariant Ellipsoid: LMIs Approach

3.1 An alternative criterion

Based on the observation of the limitations of Theorem 2, we apply Lemma 2 to obtain the optimization problem:

$$\begin{aligned} &\sup_{H_i,\, i\in I[1,2^m]} \ \rho \\ &\text{s.t.} \ \text{a)} \ \mathrm{He}\left(P(A + B(D_iF + D_i^-H_i)\right) \le 0, \ i \in I[1, 2^m], \\ &\qquad \text{b)} \ \rho h_{ij}P^{-1}h_{ij}^{\mathrm{T}} \le 1, \ \ i \in I[1, 2^m], \ j \in I[1, m], \end{aligned} \tag{7}$$

where $h_{ij} \in \mathbf{R}^{1\times n}$ is the jth row of H_i. Note that there exists no coupling of the auxiliary feedback matrices H_i's between the 2^m LMIs in a). Thus, the optimization problem (7) can be decoupled as a set of optimization problems:

$$\begin{aligned} &\sup_{H_i} \ \rho_i, \quad i \in I[1, 2^m], \\ &\text{s.t. a) } \mathrm{He}\left(P(A + B(D_iF + D_i^- H_i)\right) \le 0, \\ &\quad\ \ \text{b) } \rho_i h_{ij} P^{-1} h_{ij}^{\mathrm{T}} \le 1, \ \ j \in I[1, m]. \end{aligned} \tag{8}$$

Let $(\rho^*, H_{01}^*, H_{02}^*, \cdots, H_{02^m}^*)$ and (ρ_i^*, H_i^*), $i \in I[1, 2^m]$, be the optimal solutions to optimization problems (7) and (8), respectively. Clearly, $\rho^* = \min_{i\in I[1,2^m]}\{\rho_i^*\}$. For every optimization problem in (8), except the one associated with $D_i = I$, there exists a j such that $\rho_i^* h_{ij}^* P^{-1} {h^*}_{ij}^{\mathrm{T}} = 1$ and $\lambda_{\max}(S_i) = 0$, where $S_i = \mathrm{He}\left(P(A + B(D_iF + D_i^- H_i^*))\right)$. Hence, for the optimization problem (7), there must be a j such that $\rho^* h_{0ij}^* P^{-1} {h_{0ij}^*}^{\mathrm{T}} = 1$ and $\lambda_{\max}(S_{0i}) = 0$, where i associates with the ρ_i^* which is equal to ρ^*, and $S_{0i} = \mathrm{He}\left(P(A + B(D_iF + D_i^- H_{0i}^*))\right)$. The following theorem, which generalizes Theorem 2, provides a new criterion for determining the maximal ellipsoidal invariant set for a linear system with multiple inputs subject to actuator saturation.

Theorem 3 *Let $(\rho^*, H_{01}^*, H_{02}^*, \cdots, H_{02^m}^*)$ and (ρ_i^*, H_i^*), $i \in I[1, 2^m]$, be the optimal solutions to the optimization problems (7) and (8), respectively. Denote $J_i = \{j : d_{ij} \neq 0, j \in I[1, m]\}$, where d_{ij} is the jth diagonal element of D_i. Suppose that*

1) there is only one i such that $\rho_i^* = \rho^*$;

2) for the i in 1), all h_{ij}^*'s that satisfy $\rho^* h_{ij}^* P^{-1} {h^*}_{ij}^{\mathrm{T}} = 1$ for all $j \in J_i$ are equal to each other.

Denote $h_i^* := h_{ij}^*$, $\forall\, j \in J_i$, and let $x_0 = \rho^* P^{-1} h_i^{*\mathrm{T}}$.

3) If $|f_j x_0| \le 1$ for all $j \notin J_i$,

then $\rho^* = \rho_c$.

Differently from Theorem 2, which is used to determine the maximal contractively invariant ellipsoid $\mathcal{E}(P, \rho_c)$ in the case that only one input saturates when $\dot V(x_0) = 0$, $x_0 \in \partial\mathcal{E}(P, \rho_c)$, Theorem 3 works in the generalized case where more than one input could saturate synchronously at x_0. Clearly, if the conditions in Theorem 2 are satisfied, the conditions in Theorem 3 will also be satisfied.

3.2 A new criterion

Theorem 3 refers to the same partitions of the state space as Theorem 2, each of which is represented as D_i, $i \in I[1, 2^m]$. In this subsection, we first introduce another partitioning of the state space. The closed-loop system (2) can be rewritten as

$$\dot x = Ax + \sum_{j\in\bar N(x)} b_j f_j x + \sum_{j\in\bar N^c(x)} (-1)^{p_j(x)} b_j, \tag{9}$$

where $\bar N(x) := \{j \in [1, m] : |f_j x| < 1\} \subseteq I[1, m]$, $\bar N^c(x)$ is the complement of $\bar N(x)$, and

$$p_j(x) = \begin{cases} 0, & \mathrm{sat}(f_j x) = 1, \ j \in \bar N^c(x), \\ 1, & \mathrm{sat}(f_j x) = -1, \ j \in \bar N^c(x). \end{cases}$$

Depending on whether an input saturates or not and whether it saturates at 1 or -1, there are 3^m different saturation statuses of the m inputs. The state space can thus be accordingly divided into 3^m regions, denoted as Ω_i, $i \in I[0, 3^m - 1]$. For example, for a two-dimensional input $\mathrm{sat}(Fx)$, that is $F \in \mathbf{R}^{2\times n}$, we have 9 state regions as follows, $\Omega_0 = \{x : |f_1x| < 1,\ |f_2x| < 1\}$, $\Omega_1 = \{x : |f_1x| < 1,\ f_2x \ge 1\}$, $\Omega_2 = \{x : f_1x \ge 1,\ |f_2x| < 1\}$, $\Omega_3 = \{x : f_1x \ge 1,\ f_2x \ge 1\}$, $\Omega_4 = \{x : f_1x \le -1,\ f_2x \ge 1\}$, $-\Omega_1, -\Omega_2, -\Omega_3, -\Omega_4$. Two state regions are said to be adjacent if only one input has different statuses on the two regions, and one of statuses must be non-saturation. For example, Ω_0 and Ω_1 are adjacent to each other since f_2x is unsaturated in Ω_0, while f_2x saturates in Ω_1.

By the piecewise symmetry of these state regions with respect to the one where none of the m inputs saturates, we only need to consider $\frac{1}{2}(3^m - 1)$ regions. We relabel these regions as Ω_i, $i \in I[1, \frac{1}{2}(3^m - 1)]$. Define $\mathcal{N} = \{N_l : N_l \subseteq I[1, m]\}$ and $q_j = 0 \text{ or } 1$, $j \in N_l^c$, where N_l denotes the set of inputs that do not saturate, N_l^c is the complement of N_l, $q_j = 0$ denotes the saturation value of the jth input is 1, and $q_j = 1$ means the saturation value of the jth input is -1. We use N_l and q_j to denote Ω_i.

Let $N(\Omega_i)$ be N_l associated with Ω_i, and Ω_i^o be the interior of Ω_i. Denote $\mathcal{A}_i = A + \sum_{j\in N(\Omega_i)} b_j f_j$ and $\mathcal{B}_i = \sum_{j\in N^c(\Omega_i)} (-1)^{q_j} b_j$. In each of Ω_i's, the dynamic of system (2) is equivalent to that of the following linear system with constant inputs:

$$\dot x = \mathcal{A}_i x + \mathcal{B}_i, \ \ i \in I[1, 0.5(3^m - 1)]. \tag{10}$$

We consider the following group of $\frac{1}{2}(3^m - 1)$ optimization problems, each of which corresponds to a system in (10):

$$\begin{aligned} &\sup_{h_i} \ \rho_i, \quad i = 1, 2, \cdots, 0.5(3^m - 1), \\ &\text{s.t. a) } \mathrm{He}(P(\mathcal{A}_i + \mathcal{B}_i h_i)) \le 0, \\ &\quad\ \ \text{b) } \rho_i h_i P^{-1} h_i^{\mathrm{T}} \le 1, \end{aligned} \tag{11}$$

where $h_i \in \mathbf{R}^{1\times n}$.

Note that it is not guaranteed that each of the optimization problems in (11) is solvable. Suppose that the ith optimization problem in (11) is solvable and (ρ_i^*, h_i^*) is its optimal solution. If an ith optimization problem in (18) is not solvable, let $\rho_i^* = 0$. Let $\mathcal{E}(P, \rho_i^\sharp)$ be the minimal ellipsoid that intersects with Ω_i.

Lemma 3 *Suppose that $\rho_i^* \neq 0$. Then, $\mathcal{E}(P, \rho_i)$ is contractively invariant on Ω_i^o for the corresponding system in (10), where $\rho_i \in (\rho_i^\sharp, \rho_i^*)$. Moreover, $\mathcal{E}(P, \rho_i^*)$ is the maximal contractively invariant ellipsoid on Ω_i if $x_{0i} \in \Omega_i$, where $x_{0i} = \rho_i^* P^{-1} h_i^{*\mathrm{T}}$.*

Lemma 3 provides a sufficient condition to guarantee the contractive invariance of $\mathcal{E}(P, \rho_i)$ on Ω_i^o. Next, we will present the main result in this subsection, the proof of which is mainly established on Lemma 3.

Theorem 4 *Let (ρ_i^*, h_i^*) be the solutions to the solvable optimization problems in (11), and $x_{0i} = \rho_i^* P^{-1} h_i^{*\mathrm{T}}$. Let $\rho_i^* = 0$ for the unsolvable optimization problems. Suppose that there exist some $x_{0i} \in \Omega_i$. Denote $\rho^* = \min\{\rho_i^* : x_{0i} \in \Omega_i\}$. Define $\mathbf{\Omega} = \{\Omega_i : \rho_i^* < \rho^*\}$. If all of Ω_i's in $\mathbf{\Omega}$ are not adjacent to each other, then $\rho^* = \rho_c$.*

Theorem 4 generalizes Theorem 2 when $m = 2$. In this case, we only consider the following $\frac{1}{2}(3^2 - 1) = 4$ regions of state space, $\Omega_1 = \{N_1 = \{1\}, q_2 = 0\}$, $\Omega_2 =$

$\{N_2 = \{2\}, q_1 = 0\}$, $\Omega_3 = \{N_3 = \emptyset, q_1 = 0, q_2 = 0\}$, and $\Omega_4 = \{N_3 = \emptyset, q_1 = 0, q_2 = 1\}$. Let (ρ^*, H^*) and (ρ_i^*, h_i^*), $i \in I[1,4]$, be the optimal solutions to (5) and (11), respectively. Suppose that conditions in Theorem 2 are satisfied. Then there exists only $j = 1$ corresponding to $\Omega_1(D_1 = \mathrm{diag}\{1,0\})$ (or, $j = 2$ corresponding to $\Omega_2(D_2 = \mathrm{diag}\{0,1\})$) such that $x_{01} = \rho^* P^{-1} h_1^* \in \Omega_1$ (or $x_{02} = \rho^* P^{-1} h_2^* \in \Omega_2$). Thus, we have $\rho_1^* = \rho^*$ and $\rho_2^* > \rho^*$ (or, $\rho_2^* = \rho^*$ and $\rho_1^* > \rho^*$). Clearly, $\Omega_2 \notin \mathbf{\Omega}$ (or, $\Omega_1 \notin \mathbf{\Omega}$). Note that Ω_3 and Ω_4 are not adjacent to each other. In whatever case that $\mathbf{\Omega} = \emptyset$, $\mathbf{\Omega} = \{\Omega_3\}$, $\mathbf{\Omega} = \{\Omega_4\}$ or $\mathbf{\Omega} = \{\Omega_3, \Omega_4\}$, the condition in Theorem 4 is satisfied. Hence, Theorem 2 is a special case of Theorem 4 when $m = 2$.

4 The Maximal Contractively Invariant Ellipsoid: Algebraic Computational Approach

Section 3 provides two new criterions to determine if the optimal ellipsoid obtained from (7) or (11) is the maximal contractively invariant ellipsoid of a linear system with multiple inputs subject to actuator saturation. In fact, in these criterions, the following condition is necessarily satisfied for the extreme state $x_0 \in \partial\mathcal{E}(P, \rho_c)$. Without loss of generality, we assume that $x_0 \in \Omega_i$.

I) There exists a $g_0 > 0$ such that for all $g > g_0$, $\mathrm{He}(P\mathcal{A}_i^{\mathrm{T}}) - gP\mathcal{B}_i\mathcal{B}_i^{\mathrm{T}}P < 0$.

The satisfaction of Condition I) guarantees that the maximal value of $\dot{V}(x)$ on $\partial\mathcal{E}(P, \rho_c)$ occurs at x_0 and $\frac{d\dot{V}(x)}{dx}\Big|_{x=x_0} = 0$, which implies that the maximal contractively invariant ellipsoid $\mathcal{E}(P, \rho_c)$ can be obtained through the solutions of a group of LMIs. However, Condition I) is not always easy to satisfy, particularly when the dimension of the input is high. In this section, we will adopt an algebraic computational approach to obtain the maximal contractively invariant ellipsoid for system (2).

4.1 Algebraic computation in Ω_i's

Reconsider the systems (10) with constant inputs. We start this subsection by presenting the following lemma, which characterizes the contractive invariance of an ellipsoid $\mathcal{E}(P, \rho_i)$ on a state region for the corresponding system in (10).

Lemma 4 *Let $(\lambda_{ik}, \rho_{ik}, \eta_{ik})$, $k \in I[1, \mathcal{K}_i]$, be all the solutions to the following nonlinear equations and matrix inequalities:*

$$\mathcal{B}_i^{\mathrm{T}} P(\mathrm{He}(P\mathcal{A}_i) + \lambda P)^{-1} P(\mathrm{He}(P\mathcal{A}_i) + \lambda P)^{-1} P\mathcal{B}_i = \rho, \tag{12}$$

$$\lambda\rho + \mathcal{B}_i^{\mathrm{T}} P(\mathrm{He}(P\mathcal{A}_i) + \lambda P)^{-1} P\mathcal{B}_i = 0, \tag{13}$$

$$\mathrm{He}(P\mathcal{A}_i) + \lambda P + \mathrm{He}(\eta\mathcal{B}_i^{\mathrm{T}} P(\mathrm{He}(P\mathcal{A}_i) + \lambda P)^{-1} P) \leq 0, \tag{14}$$

and

$$\lambda < 0, \tag{15}$$

where $\rho \in \mathbf{R}$ and $\eta \in \mathbf{R}^{n\times 1}$. Let $x_{0ik} = -(\mathrm{He}(P\mathcal{A}_i) + \lambda_{ik}P)^{-1}P\mathcal{B}_i$. Denote

$$\rho_i^* = \begin{cases} \min\{\rho_{ik} : k \in I[1, \mathcal{K}_i]\}, & \forall\, x_{0ik} \notin \Omega_i, \\ \xi, & \textit{otherwise}. \end{cases} \tag{16}$$

where $\xi = \min\{\rho_{ik} : x_{0ik} \in \Omega_i, k \in I[1, \mathcal{K}_i]\}$. Let λ_i^* and x_{0i}^* be the λ_{ik} and x_{0ik} associated with ρ_i^*, respectively. Then, for any $\rho_i \in (\rho_i^\sharp, \rho_i^*)$, $\mathcal{E}(P, \rho_i)$ is contractively invariant on Ω_i^o for the corresponding system in (10). Moreover, $\mathcal{E}(P, \rho_i^*)$ is the maximal contractively invariant ellipsoid on Ω_i if $x_{0i}^* \in \Omega_i$.

We consider the computational issue for nonlinear equations (12) and (13). Substituting (12) into (13), we have

$$\begin{aligned} &\lambda\mathcal{B}_i^{\mathrm{T}} P(\mathrm{He}(P\mathcal{A}_i) + \lambda P)^{-1} P(\mathrm{He}(P\mathcal{A}_i) + \lambda P)^{-1} P\mathcal{B}_i \\ &+\mathcal{B}_i^{\mathrm{T}} P(\mathrm{He}(P\mathcal{A}_i) + \lambda P)^{-1} P\mathcal{B}_i = 0. \end{aligned}$$

It is a $(2n+1)$th order polynomial equation of λ, which is very difficult to solve directly. By matrix inverse theorem, one can obtain $(\mathrm{He}(P\mathcal{A}_i) + \lambda P)^{-1} = \mathcal{S}_i - \mathcal{S}_i\Theta_i\mathcal{S}_i$, where $\mathcal{S}_i = (\mathrm{He}(P\mathcal{A}_i))^{-1}$, and $\Theta_i = \left(\frac{1}{\lambda}P^{-1} + \mathcal{S}_i\right)^{-1}$. Then (13) is equivalent to

$$\lambda\rho - \mathcal{B}_i^{\mathrm{T}}\mathcal{S}_i\Theta_i\mathcal{S}_i\mathcal{B}_i = -\mathcal{B}_i^{\mathrm{T}}\mathcal{S}_i\mathcal{B}_i. \tag{17}$$

From (12), we have

$$\mathcal{B}_i^{\mathrm{T}}\mathcal{S}_i\Theta_i P^{-1}\Theta_i\mathcal{S}_i\mathcal{B}_i = \rho\lambda^2. \tag{18}$$

Substituting (18) into (17), we have

$$\begin{aligned} &\lambda\mathcal{B}_i^{\mathrm{T}}\mathcal{S}_i\Theta_i P^{-1}\Theta_i\mathcal{S}_i\mathcal{B}_i - \mathcal{B}_i^{\mathrm{T}}\mathcal{S}_i\Theta_i\mathcal{S}_i\mathcal{B}_i = -\lambda^2\mathcal{B}_i^{\mathrm{T}}\mathcal{S}_i\mathcal{B}_i \\ \Longleftrightarrow\quad &\mathcal{B}_i^{\mathrm{T}}\mathcal{S}_i\Theta_i\mathcal{S}_i\Theta_i\mathcal{S}_i\mathcal{B}_i = \mathcal{B}_i^{\mathrm{T}}\mathcal{S}_i\mathcal{B}_i \\ \Longleftrightarrow\quad &\det\begin{bmatrix} \mathcal{B}_i^{\mathrm{T}}\mathcal{S}_i\mathcal{B}_i & \mathcal{B}_i^{\mathrm{T}}\mathcal{S}_i\Theta_i \\ \Theta_i\mathcal{S}_i\mathcal{B}_i & \mathcal{S}_i^{-1} \end{bmatrix} = 0 \\ \Longleftrightarrow\quad &\det\begin{bmatrix} \Theta_i^{-1} & \mathcal{S}_i \\ \frac{1}{\mathcal{B}_i^{\mathrm{T}}\mathcal{S}_i\mathcal{B}_i}\mathcal{S}_i\mathcal{B}_i\mathcal{B}_i^{\mathrm{T}}\mathcal{S}_i & \Theta_i^{-1} \end{bmatrix} = 0. \end{aligned}$$

It follows that λ^{-1} is an eigenvalue of the following matrix

$$\begin{bmatrix} -P^{\frac{1}{2}}\mathcal{S}_i P^{\frac{1}{2}} & P^{\frac{1}{2}}\mathcal{S}_i P^{\frac{1}{2}} \\ (\mathcal{B}_i^{\mathrm{T}}\mathcal{S}_i\mathcal{B}_i)^{-1} P^{\frac{1}{2}}\mathcal{S}_i\mathcal{B}_i\mathcal{B}_i^{\mathrm{T}}\mathcal{S}_i P^{\frac{1}{2}} & -P^{\frac{1}{2}}\mathcal{S}_i P^{\frac{1}{2}} \end{bmatrix}. \tag{19}$$

If (19) has no negative real eigenvalues or (14) is not satisfied, which means that (12)-(15) have no solution, let $\rho_i^* = 0$. Otherwise, compute ρ_i^* as (16). Let

$$\rho^* = \begin{cases} \beta, & \exists\, i \text{ such that } x_{0i}^* \in \Omega_i, \\ +\infty, & \text{otherwise}, \end{cases} \tag{20}$$

where $\beta = \min\{\rho_i^* : x_{0i}^* \in \Omega_i, \rho_i^* \neq 0\}$. Assume that $\rho^* \neq +\infty$. Define $\mathbf{\Omega} = \{\Omega_i : \rho_i^* < \rho^*\}$.

Theorem 5 *If every element in $\mathbf{\Omega}$ is not adjacent to each other, then $\rho_c = \rho^*$.*

In the procedure to solve the nonlinear equations (12) and (13), it is naturally considered that $\mathrm{He}(P\mathcal{A}_i)$ is invertible, and $\mathcal{B}_i^{\mathrm{T}}\mathcal{S}_i\mathcal{B}_i$ is not equal to 0. However, these might not always be true, in which case, we can choose a λ_o such that $\mathrm{He}(P\mathcal{A}_i) + \lambda_o I$ is nonsingular, and $\mathcal{B}_i^{\mathrm{T}}(\mathrm{He}(P\mathcal{A}_i) + \lambda_o I)^{-1}\mathcal{B}_i \neq 0$. Let $\mathcal{S}_{oi} = \mathrm{He}(P\mathcal{A}_i) + \lambda_o I$ and $\lambda_e = \lambda - \lambda_o$. Then λ_e^{-1} is an eigenvalue of the alternative matrix

$$\begin{bmatrix} -P^{\frac{1}{2}}\mathcal{S}_{oi}P^{\frac{1}{2}} & P^{\frac{1}{2}}\mathcal{S}_{oi}P^{\frac{1}{2}} \\ \Lambda & -P^{\frac{1}{2}}\mathcal{S}_{oi}P^{\frac{1}{2}} \end{bmatrix}, \tag{21}$$

where $\Lambda = (\mathcal{B}_i^{\mathrm{T}}\mathcal{S}_{oi}\mathcal{B}_i)^{-1} P^{\frac{1}{2}}\mathcal{S}_{oi}\mathcal{B}_i\mathcal{B}_i^{\mathrm{T}}\mathcal{S}_{oi}P^{\frac{1}{2}}$. Thus, $\lambda = \lambda_o + \lambda_e$.

4.2 Algebraic computation in the intersections between Ω_i's

All criterions presented above apply to the systems, whose extreme state $x_0 \in \Omega_i$ satisfies that $\frac{d\dot{V}(x)}{dx} \to 0$ as $x \to x_0$, $x \in \Omega_i$. However, this condition is not always satisfied either. In this case, x_0 resides in the intersection between two Ω_i's, where one input critically saturates, i.e., there exists a $j \in I[1,m]$ such that $\mathrm{sat}(f_j x) = f_j x$.

Denote $\mathbf{\Omega} = \{\Omega_i : \rho_i^* < \rho^*\}$, where ρ_i^* and ρ^* have been defined in the previous subsection. Suppose that there are $\mathcal{R}$ elements in $\mathbf{\Omega}$. We relabel them as Ω_r, $r \in I[1,\mathcal{R}]$. Assume that there exist $\mathcal{R}_1$ intersections between these Ω_r's. Denote the intersection between Ω_r and Ω_l, $r,l \in I[1,\mathcal{R}]$, as $\mathcal{X}_{r,l}$. Let $h(r,l)$ represent that the $h(r,l)$th input saturates critically on $\mathcal{X}_{r,l}$. Then $\mathcal{X}_{r,l} = \{x : \ f_{h(r,l)}x = \mathrm{sat}(f_{h(r,l)}x), x \in \Omega_r \cap \Omega_l, r,l \in I[1,\mathcal{R}]\}$. Define a region of state space $\mathcal{X}_{h(r,l)} = \{x : f_{h(r,l)}x = \mathrm{sat}(f_{h(r,l)}), r,l \in I[1,\mathcal{R}]\}$. Clearly, $\mathcal{X}_{r,l} \subset \mathcal{X}_{h(r,l)}$. We use the state transformation of the form $z = Tx$, where $T = UP^{\frac{1}{2}}$ for the unitary matrix U such that $\bar{f}_{h(r,l)} = f_{h(r,l)}T^{-1} = [\bar{f}_{h(r,l)1}, 0_{1\times(n-1)}]$, where $\bar{f}_{h(r,l)1} \neq 0$. Let

$$\bar{A}_{r,l} = T\mathcal{A}_r T^{-1}, \bar{b}_{r,l} = T\mathcal{B}_r = \begin{bmatrix} \bar{b}_{(r,l)1} \\ \bar{b}_{(r,l)2} \end{bmatrix},$$

$$\bar{A_{r,l}}^{\mathrm{T}} + \bar{A}_{r,l} = \begin{bmatrix} \bar{Q}_{(r,l)11} & \bar{Q}_{(r,l)12} \\ \bar{Q}_{(r,l)12}^{\mathrm{T}} & \bar{Q}_{(r,l)22} \end{bmatrix},$$

where $\bar{b}_{(r,l)1} \in \mathbf{R}$, $\bar{Q}_{(r,l)11} \in \mathbf{R}$. Let $z_{r,l} = [z_{(r,l)1} \ z_{(r,l)2}]^{\mathrm{T}}$, $z_{(r,l)1} \in \mathbf{R}$. Then $z_{(r,l)1} = \frac{\mathrm{sat}(f_{h(r,l)}x)}{\bar{f}_{h(r,l)1}}$, and

$$\dot{z}_{r,l} = \bar{A}_{r,l} z_{r,l} + \bar{b}_{r,l}. \tag{22}$$

The derivative of the quadratic Lyapunov function for (22) on $\mathcal{X}_{h(r,l)}$ is given as follows,

$$\begin{aligned} \dot{V}_{r,l}(x) =& x^{\mathrm{T}}\mathrm{He}(P\mathcal{A}_r)x + 2x^{\mathrm{T}}P\mathcal{B}_r \\ =& z_{r,l}^{\mathrm{T}}(\bar{A}_{r,l} + \bar{A}_{r,l}^{\mathrm{T}})z_{r,l} + 2z_{r,l}^{\mathrm{T}}\bar{b}_{r,l} \\ =& z_{(r,l)2}^{\mathrm{T}}\bar{Q}_{(r,l)22}z_{(r,l)2} \\ &+ 2z_{(r,l)2}^{\mathrm{T}}(\bar{Q}_{(r,l)12}^{\mathrm{T}}z_{(r,l)1} + \bar{b}_{(r,l)2}) + \alpha_{r,l} \\ =:& \, g_{r,l}(z_{(r,l)2}), \end{aligned}$$

where $\alpha_{r,l} = \bar{Q}_{(r,l)11}(z_{(r,l)1})^2 + 2\bar{b}_{(r,l)1}z_{(r,l)1}$.

Define $\mathcal{X}_{r,l}^o = \{x : x \in \mathcal{X}_{r,l}, \mathcal{N}(x) \subset \mathcal{X}_{r,l}\}$, where $\mathcal{N}(x)$ is any neighborhood of x on $\mathcal{X}_{h(r,l)}$. For simplicity, let $b_{r,l} = (\bar{Q}_{(r,l)12}^{\mathrm{T}}z_{(r,l)1} + \bar{b}_{(r,l)2})$ and $S_{r,l} = \bar{Q}_{(r,l)22}$.

Lemma 5 *Let $(\lambda_{(r,l)k}, \rho_{(r,l)k}, \eta_{(r,l)k})$, $k \in I[1,\mathcal{K}_r]$, be all the solutions to the following equations and inequalities:*

$$b_{r,l}^{\mathrm{T}}(S_{r,l} + \lambda I)^{-2}b_{r,l} = \rho - (z_{(r,l)1})^2, \tag{23}$$

$$\lambda(\rho - (z_{(r,l)1})^2) + b_{r,l}^{\mathrm{T}}(S_{r,l} + \lambda I)^{-1}b_{r,l} = \alpha_{r,l}, \tag{24}$$

$$S_{r,l} + \Gamma + \Gamma^{\mathrm{T}} < 0, \tag{25}$$

and

$$\lambda < 0, \tag{26}$$

where $\lambda, \rho \in \mathbf{R}$, $\eta \in \mathbf{R}^{(n-1)\times 1}$ and $\Gamma = \eta b_{r,l}^{\mathrm{T}}(S_{r,l} + \lambda I)^{-1}$.

Let $z_{0(r,l)k2} = -(S_{r,l} + \lambda_{(r,l)k}I)^{-1}b_{r,l}$, and $x_{0(r,l)k} = T^{-1}[z_{(r,l)1}, z_{0(r,l)k2}^{\mathrm{T}}]^{\mathrm{T}}$. Let

$$\rho_{r,l}^{\natural} = \begin{cases} \min\{\rho_{(r,l)k} : k \in I[1,\mathcal{K}_r]\}, & \forall\, x_{0(r,l)k} \notin \mathcal{X}_{r,l}, \\ \pi, & \textit{otherwise}, \end{cases}$$

where $\pi = \min\{\rho_{(r,l)k} : x_{0(r,l)k} \in \mathcal{X}_{r,l}, k \in I[1,\mathcal{K}_r]\}$. Let $\lambda_{r,l}$ and $x_{0(r,l)}$ be the $\lambda_{(r,l)k}$ and $x_{0(r,l)k}$ associated with $\rho_{r,l}^{\natural}$. Then for any $\rho_{r,l} \in (\rho_{r,l}^{\sharp}, \rho_{r,l}^{\natural})$, $\mathcal{E}(P,\rho_{r,l})$ is contractively invariant on $\mathcal{X}_{r,l}^o$ for the corresponding system in (10). Moreover, $\mathcal{E}(P,\rho_{r,l}^{\natural})$ is the maximal contractively invariant ellipsoid on $\mathcal{X}_{r,l}$ if $x_{0(r,l)} \in \mathcal{X}_{r,l}$.

We proceed to solve the nonlinear equations (23) and (24). Substituting (23) into (24), we have

$$\lambda b_{r,l}^{\mathrm{T}}\Upsilon_{r,l}\Upsilon_{r,l}b_{r,l} + b_{r,l}^{\mathrm{T}}\Upsilon_{r,l}b_{r,l} = \alpha_{r,l}, \tag{27}$$

where $\Upsilon_{r,l} = (S_{r,l} + \lambda I)^{-1}$. We consider two following cases:

Case 1) $\alpha_{r,l} \neq 0$. It follows from (27) that

$$\begin{aligned} & b_{r,l}^{\mathrm{T}}\Upsilon_{r,l}(S_{r,l} + 2\lambda I)\Upsilon_{r,l}b_{r,l} = \alpha_{r,l} \\ \Longleftrightarrow\ & \det\begin{bmatrix} \alpha_{r,l} & -b_{r,l}^{\mathrm{T}}\Upsilon_{r,l} \\ -\Upsilon_{r,l}b_{r,l} & (S_{r,l} + 2\lambda I)^{-1} \end{bmatrix} = 0 \\ \Longleftrightarrow\ & \det\begin{bmatrix} S_{r,l} + \lambda I & S_{r,l} \\ -\alpha_{r,l}^{-1}b_{r,l}b_{r,l}^{\mathrm{T}} & S_{r,l} - 2\alpha_{r,l}^{-1}b_{r,l}b_{r,l}^{\mathrm{T}} + \lambda I \end{bmatrix} = 0, \end{aligned}$$

which implies that λ is an eigenvalue of the following matrix

$$\begin{bmatrix} -S_{r,l} & -S_{r,l} \\ \alpha_{r,l}^{-1}b_{r,l}b_{r,l}^{\mathrm{T}} & -S_{r,l} + 2\alpha_{r,l}^{-1}b_{r,l}b_{r,l}^{\mathrm{T}} \end{bmatrix}. \tag{28}$$

Case 2) $\alpha_{r,l} = 0$. It follows from (27) that

$$\begin{aligned} & b_{r,l}^{\mathrm{T}}\Upsilon_{r,l}(S_{r,l} + 2\lambda I)\Upsilon_{r,l}b = 0 \\ \Longleftrightarrow\ & \det\begin{bmatrix} \frac{1}{\lambda}I + S_{r,l}^{-1} & S_r^{-1} \\ \Psi & \frac{1}{\lambda}I + S_{r,l}^{-1} \end{bmatrix} = 0, \end{aligned}$$

where $\Psi = (b_{r,l}^{\mathrm{T}}S_{r,l}^{-1}b_{r,l})^{-1}S_{r,l}^{-1}b_{r,l}b_{r,l}^{\mathrm{T}}S_{r,l}^{-1}$. Then, $\frac{1}{\lambda}$ is an eigenvalue of the following matrix

$$\begin{bmatrix} -S_{r,l}^{-1} & S_{r,l}^{-1} \\ (b_{r,l}^{\mathrm{T}}S_{r,l}^{-1}b_{r,l})^{-1}S_{r,l}^{-1}b_{r,l}b_{r,l}^{\mathrm{T}}S^{-1} & -S_{r,l}^{-1} \end{bmatrix}. \tag{29}$$

If (28) (or (29)) has no negative real eigenvalues or (25) is not satisfied, that is, (23)-(26) have no solutions, let $\rho_{r,l}^{\natural} = 0$. Otherwise, compute $\rho_{r,l}^{\natural}$ as (28) (or (29)). Let

$$\rho^{\natural} = \begin{cases} \kappa, & \exists\, r \text{ such that } x_{0(r,l)} \in \mathcal{X}_{r,l}, \\ +\infty, & \text{otherwise}. \end{cases} \tag{30}$$

where $\kappa = \min\{\rho_{r,l}^{\natural} : x_{0(r,l)} \in \mathcal{X}_{r,l}, \rho_{r,l}^{\natural} \neq 0, r,l \in I[1,\mathcal{R}]\}$.

Theorem 6 *Let ρ^* be defined as (20), then $\rho_c = \min\{\rho^*, \rho^{\natural}\}$.*

We consider a special class of system (2), the planar systems, that is, $n = 2$. Since the intersection of $\partial\mathcal{E}(P,\rho)$ and the surface $f_j x = 1$, $j = 1,2,\cdots,m$, only includes isolated states, Lemma 4 and Lemma 5 cannot be used to determine the maximal contractively invariant ellipsoid. In this case, we only need to solve the quadratic equations $g_{r,l}(z_{(r,l)2}) = 0$ with unknown $z_{(r,l)2}$. Thus, we have $x_{0(r,l)} = T^{-1}[z_{(r,l)1}, z_{(r,l)2}]^{\mathrm{T}}$. Then $\rho^{\natural}$ can be determined from (30).

4.3 An algorithm for the determination of ρ_c

In this subsection, we summarize all the criterions presented in this paper in a comprehensive algorithm to determine the maximal contractively invariant ellipsoid $\mathcal{E}(P,\rho_c)$ for a linear system with multiple inputs subject to actuator saturation.

Algorithm 1: The determination of ρ_c.

Step 1. Solve the optimization problem (7). Set the optimal solution $(\rho^*, H_1^*, H_2^*, \cdots, H_{2^m}^*)$. If the optimal solution satisfies conditions in Theorem 3, then $\rho_c = \rho^*$. Else go to Step 2.

Step 2. Solve the optimization problems (11). Set the optimal solutions (ρ_i^*, h_i^*), $i = 1,2,\cdots, \frac{1}{2}(3^m - 1)$. Compute $x_{0i} = \rho_i^* P^{-1}h_i^{*\mathrm{T}}$ for the solvable optimization problems. If all $x_{0i} \notin \Omega_i$, set $\rho^* = +\infty$. Else, denote

$\rho^* = \min\{\rho_i^* : x_{0i} \in \Omega_i\}$. If the condition in Theorem 4 is satisfied, then $\rho_c = \rho^*$. Else go to Step 3.

Step 3. If $n = 2$, solve the quadratic equations $g_{r,l}(z_{(r,l)2}) = 0$ with unknown $z_{(r,l)2}$. Compute $x_{0(r,l)} = T^{-1}[z_{(r,l)1}, z_{(r,l)2}]^{\mathrm{T}}$ and determine $\rho^\natural$ from (30). Go to Step 8. Else go to Step 4.

Step 4. Let $\rho_i = \rho_i^*$ and $\rho = \rho^*$. Define $\mathbf{\Omega} = \{\Omega_i : \rho_i < \rho\}$. For every $\Omega_i \in \mathbf{\Omega}$, compute ρ_i^*, and determine ρ^* from (20). If $\rho^* = +\infty$, go to Step 6. Else go to Step 5.

Step 5. Set $\rho^* = \min\{\rho, \rho^*\}$. If the condition in Theorem 5 is satisfied, then $\rho_c = \rho^*$. Else go to Step 6.

Step 6. Set $\mathbf{\Omega} = \{\Omega_i : \rho_i^* < \rho^*\}$. Denote the intersections between Ω_i's in $\mathbf{\Omega}$ as $\mathcal{X}_{r,l}$, where $r, l \in \{i : \Omega_i \in \mathbf{\Omega}\}$.

Step 7. For every pair (r, l), compute $\rho_{r,l}^\natural$. Determine $\rho^\natural$ from (30).

Step 8. $\rho_c = \min\{\rho^*, \rho^\natural\}$.

5 A Numerical Example

Consider system (2) with the following matrices:

$$A = \begin{bmatrix} -5 & -2 & 10 \\ -4 & -1 & -1 \\ 0 & 3 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} -4 & -1 \\ 1 & -2 \\ -5 & 3 \end{bmatrix},$$
$$F = \begin{bmatrix} 9.3733 & -0.3965 & 9.0277 \\ 6.1005 & 7.4549 & -7.1289 \end{bmatrix}, \quad P = I.$$

Solving the optimization problem (5), we obtain $\rho^* = 2.4605$, $H^* = \begin{bmatrix} 0.2567 & 0.2204 & 0.5402 \\ -0.3290 & 0.5425 & -0.0404 \end{bmatrix}$, and $\rho^* h_1^* P^{-1} h_1^{*\mathrm{T}} = 1$. Moreover, $\lambda_{\max}(\mathrm{He}(P(A + BD_iF + BD_i^- H^*))) = 0$, where $D_i^- = \mathrm{diag}\{1, 0\}$. Conditions 1) and 2) in Theorem 2 are both satisfied. However, $x_0 = \rho^* P^{-1} h_1^{*\mathrm{T}} = [0.6315, 0.5425, 1.3292]^{\mathrm{T}}$, and $Fx_0 = [17.7041, -1.5804]^{\mathrm{T}}$. Condition 3) in Theorem 2 is not satisfied. Thus we have $\rho_c \neq \rho^* = 2.4605$. Next, solving the optimization problem (7), we obtain $\rho^* = 2.8799$ and $\lambda_{\max}(\mathrm{He}(P(A + BD_iF + BD_i^- H_i^*))) = 0$, where $D_i^- = \mathrm{diag}\{1, 0\}$, and $H_i^* = \begin{bmatrix} -0.0538 & 0.0014 & -0.0971 \\ 0 & 0 & 0 \end{bmatrix}$. Moreover, $x_0 = \rho^* P^{-1} h_1^{*\mathrm{T}} = [0.6287, 0.9601, 1.2499]^{\mathrm{T}}$, and $Fx_0 = [16.7955, 2.0825]^{\mathrm{T}}$. Condition 3) in Theorem 3 is not satisfied either. Hence, $\rho_c \neq \rho^* = 2.8799$.

Solving the group of optimization problems in (11), we obtain $\rho_1^* = 8.1542$, $\rho_2^* = 2.8766$, and $\rho_i^* = 0$, $i = 3, 4$. Moreover, there exists no $x_{0i} \in \Omega_i$, $\forall\, i \in I[1, 4]$. Clearly, the condition in Theorem 4 is not satisfied. On the other hand, computing the eigenvalues of matrices (19) associated with Ω_3 and Ω_4, we obtain $\rho_3^* = 0.9841$, $\rho_4^* = 4.8290$, and $x_{03} \notin \Omega_3$, $x_{04} \in \Omega_4$. Moreover, both LMIs in (14) associated with Ω_3 and Ω_4 are feasible. Thus, $\rho^* = \rho_4^* = 4.8290$, and $\mathbf{\Omega} = \{\Omega_2, \Omega_3\}$. Clearly, Ω_2 is adjacent to Ω_3. Thus, Theorem 5 does not apply to this example. Next, we consider the intersection $\mathcal{X}_{2,3}$, and compute the eigenvalues of the corresponding matrices (28). Then we obtain $\rho^\natural = \rho_{2,3}^\natural = 3.2276$. Finally, by Theorem 6, we have $\rho_c = \min\{\rho^*, \rho^\natural\} = \min\{4.8290, 3.2276\} = \rho^\natural = 3.2276$.

We depict $\dot{V}(x)$ on $\partial\mathcal{E}(P, 3.2276)$ in the left plot of Fig. 1, where the maximal value of $\dot{V}(x)$ is not larger than 0. Shown in the right plot of Fig. 1 is $\dot{V}(x)$ on the intersection between $\partial\mathcal{E}(P, 3.2276)$ and $x_1 = 0.7018$. Clearly, the maximal value of $\dot{V}(x)$ on $\partial\mathcal{E}(P, 3.2276)$ reaches 0. This verifies that $\rho_c = 3.2276$. Moreover, $\frac{d\dot{V}(x)}{dx}|_{x_0} \neq 0$ can be also observed, which implies that $x_0 \in \mathcal{X}_{2,3}$.

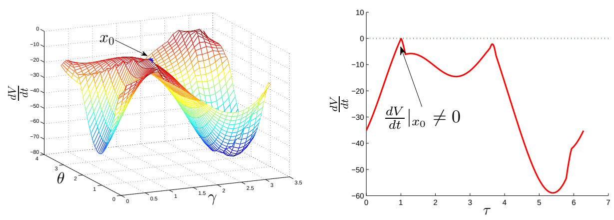

Fig. 1: The derivative $\dot{V}(x)$ on $\partial\mathcal{E}(P, \rho_c)$.

6 Conclusions

This paper revisits the problem of determining the maximal contractively invariant ellipsoid for a linear system with multiple inputs subject to actuator saturation. A comprehensive algorithm was developed to determine the maximal ellipsoidal invariant set by using LMIs and algebraic computational approaches. Simulation examples, including the one in Section 5 and those in the journal version of this paper [11], illustrated that this algorithm is capable of determining the maximal ellipsoidal invariant set for any linear system with multiple inputs subject to actuator saturation.

References

[1] T. Alamo, A. Cepeda and D. Limon, "Improved computation of ellipsoidal invariant sets for saturated control systems," in *Proceedings of the 44th IEEE Conference on Decision and Control, and European Control Conference 2005*, Seville, Spain, 2005: 6216-6221.

[2] T. Alamo, A. Cepeda, D. Limon and E.F. Camacho, "A new concept of invariance for saturated systems," *Automatica*, 42(9): 1515-1521, 2006.

[3] G. Chesi, "Computing output feedback controllers to enlarge the domain of attraction in polynomial systems," *IEEE Transactions on Automatic Control*, 49(10): 1846-1850, 2004.

[4] D. Dai, T. Hu, A.R. Teel and L. Zaccarian, "Piecewise-quadratic Lyapunov functions for systems with deadzones or saturations," *Systems & Control Letters*, 58(5): 365-371, 2009.

[5] M. Fiacchini, T. Alamo and E.F. Camacho, "Invariant sets computation for convex difference inclusions systems," *Systems & Control Letters*, 61(8): 819-826, 2012.

[6] T. Hu and Z. Lin, *Control Systems with Actuator Saturation: Analysis and Design*, Birkhauser, Boston, 2001.

[7] T. Hu and Z. Lin, "Exact characterization of invariant ellipsoids for linear systems with saturating actuators," *IEEE Transactions on Automatic Control*, 47(1): 164-169, 2002.

[8] T. Hu and Z. Lin, "On the tightness of a recent set invariance condition under actuator saturation," *Systems & Control Letters*, 49(5): 389-399, 2003.

[9] T. Hu, Z. Lin and Y. Shamash, "On maximizing the convergence rate for linear systems with input saturation," *IEEE Transactions on Automatic Control*, 48(7): 1249-1253, 2003.

[10] Y. Li and Z. Lin, "Improvements to the linear differential inclusion approach to stability analysis of linear systems with saturated linear feedback," *Automatica*, 58(3): 821-828, 2013.

[11] Y. Li, and Z. Lin, "A complete characterization of the maximal ellipsoidal invariant sets of linear systems under saturated linear feedback," submitted for journal publication.

[12] Z. Lin, *Low Gain Feedback*, Springer-Verlag, London, 1998.

[13] B. Zhou, W. Zheng and G.R. Duan, "An improved treatment of saturation nonlinearity with its application to control of systems subject to nested saturation," *Automatica*, 47(2): 306-315, 2011.