Global Stability and Controllability of Switched Boolean Networks

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Abstract: This paper investigates the global stability and controllability of switched Boolean networks (BNs) by using the semi-tensor product method. First, the model of switched Boolean networks is introduced and expressed into the algebraic form by the semi-tensor product. Second, the concept of the switching reachability of points for switched BNs is defined and a necessary and sufficient condition is presented to verify the switching reachability of points by constructing the switching-state incidence matrix. Third, a necessary and sufficient condition is proposed to check the global stability of switched BNs under arbitrary switching signals by using the switching reachability of points. Finally, the concept of the controllability for switched Boolean control networks is given, and a kind of switching-input-state incidence matrix is constructed by defining the switching-input-state transfer graph, based on which several necessary and sufficient conditions are obtained for the controllability of switched Boolean control networks. The study of two illustrative examples shows that the new results obtained in this paper are very effective in investigating the global stability and controllability of switched BNs.

Key Words: Switched Boolean Network, Switching Reachability of Points, Global Stability, Controllability, Switching-State Incidence Matrix, Switching-Input-State Incidence Matrix, Semi-Tensor Product.

1 Introduction

With the development of systems biology, many cell models have been proposed to investigate gene regulatory networks, which include Boolean networks [10], Bayesian networks [15] and differential equations [7, 17]. In Boolean networks, gene expressions are quantized as 0 and 1 to represent inactive and active, respectively. Since BNs are structurally simple, the study of BNs has attracted a great attention of scholars and many excellent results were proposed [1, 10]. In spite of this, due to the shortage of systematic tool to deal with logical dynamic systems, there are only few results about the control of BNs.

It is well worth pointing out that while BNs are typically based on purely discrete dynamics, the dynamics of biological networks in practice are often governed by different switching models [7]. For example, at the cellular level, the cell growth and division in a eukaryotic cell is usually described as a sequence of four processes, each of which is triggered by a set of conditions or events. At the inter-cellular level, cell differentiation can also be viewed as a switched system [7, 11]. In addition to naturally occurred switchings, switched dynamics can be the result of external intervention that attempts to re-engineer a given network by turning on and off. When modeling these biological networks as Boolean networks, the dynamics become switched Boolean networks. Thus, it is necessary for us to investigate switched BNs. In the past two decades, due to the great importance of switched systems in both theoretical development and practical applications, the study of ordinary switched systems has drawn a great deal of attention, and a large number of excellent results have been obtained on the stability and controllability of ordinary switched systems [2, 9, 14, 16, 19]. However, to our best knowledge, there are no results available to investigate switched Boolean networks.

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Recently, a novel matrix product, namely, the semi-tensor product of matrices, has been proposed [4] and then successfully applied to express and analyze BNs. By this method, it is very convenient to convert a logical expression into an algebraic form, and many fundamental and landmark results about BNs [3–6, 12, 20] and other control problems [13, 18] have been presented.

In this paper, using the semi-tensor product method, we investigate the global stability and controllability of switched BNs. The main contributions of this paper are as follows. (i) The semi-tensor product method is first applied to the investigation of switched BNs, and a new framework to study the global stability and controllability is established. (ii) The concept of the switching reachability of points for switched BNs is first given in this work, and a necessary and sufficient condition is presented to verify the switching reachability of points by constructing a kind of switching-state incidence matrix. (iii) A neat necessary and sufficient condition is proposed to check the global stability of switched BNs under arbitrary switching signals by using the switching reachability of points, which is just based on the switching-state incidence matrix and easily checked. (iv) A kind of switching-input-state incidence matrix is proposed for switched Boolean control networks (BCNs), based on which some necessary and sufficient conditions for the controllability of switched BCNs are presented. The main feature of these conditions is that they only use the structural properties of switched BCNs, not being based on the existence of switching sequences, to determine the controllability of the systems. We believe that many control problems of switched BNs can be studied based on our obtained methods/results in further works.

The rest of this work is structured as follows. Section 2 contains some preliminaries on the semi-tensor product of matrices. Section 3 investigates the switching reachability of points and global stability of switched BNs and presents some necessary and sufficient conditions. In Section 4, we propose some necessary and sufficient conditions for the controllability by constructing the switching-input-state incidence matrix for switched BCNs. Two illustrative examples are given in Section 5 to support our results, which is followed by a brief conclusion in Section 6.

2 Preliminaries

In this section, we give some necessary preliminaries on the semi-tensor product of matrices, which will be used in the sequel.

Definition 2.1 *([4]) The semi-tensor product of two matrices $A \in \mathbb{R}^{m\times n}$ and $B \in \mathbb{R}^{p\times q}$ is defined as*

$$A \ltimes B = (A \otimes I_{\frac{\alpha}{n}})(B \otimes I_{\frac{\alpha}{p}}), \tag{1}$$

where $\alpha = lcm(n, p)$ is the least common multiple of n and p, and $\otimes$ is the Kronecker product.

Remark 2.2 *When $n = p$, the semi-tensor product of A and B becomes the conventional matrix product. Thus, the semi-tensor product of matrices is a generalization of the conventional matrix product. We can simply call it "product" and omit the symbol "$\ltimes$" if no confusion raises.*

The following notations will be used later:

1). $\mathcal{D} := \{1, 0\}$, and $\mathcal{D}^n := \underbrace{\mathcal{D}\times\cdots\times\mathcal{D}}_{n}$.

2). $\Delta_n := \{\delta_n^k \mid 1 \le k \le n\}$, where δ_n^k denotes the k-th column of the identity matrix I_n. For compactness, $\Delta := \Delta_2$.

3). An $n \times t$ matrix M is called a logical matrix, if $M = [\delta_n^{i_1}\ \delta_n^{i_2}\ \cdots\ \delta_n^{i_t}]$. We express M briefly as $M = \delta_n[i_1\ i_2\ \cdots\ i_t]$. Denote the set of $n \times t$ logical matrices by $\mathcal{L}_{n\times t}$.

4). An $n \times t$ matrix $A = (a_{ij})$ is called a Boolean matrix, if $a_{ij} \in \mathcal{D},\ \forall\ i = 1, \cdots, n,\ j = 1, \cdots, t$. Denote the set of $n \times t$ Boolean matrices by $\mathcal{B}_{n\times t}$.

5). $Col_i(A)$ denotes the i-th column of the matrix A. $Row_i(A)$ denotes the i-th row of the matrix A.

6). $Blk_i(A)$ denotes the i-th $n \times n$ block of an $n \times mn$ matrix A.

By identifying $True \sim 1 \sim \delta_2^1$ and $False \sim 0 \sim \delta_2^2$, we have $\Delta \sim \mathcal{D}$, where $p \sim q$ denotes the logical equivalence of p and q. In most places of this work, we use δ_2^1 and δ_2^2 to express logical values and call them the vector form of logical values.

The following lemma is fundamental for the matrix expression of logical functions.

Lemma 2.3 *([4]) Let $f(x_1,\ x_2,\ \cdots,\ x_s)$ be a logical function. Then, there exists a unique matrix $M_f \in \mathcal{L}_{2\times 2^s}$, called the structural matrix of f, such that*

$$f(x_1, x_2, \cdots, x_s) = M_f \ltimes_{i=1}^{s} x_i, \quad x_i \in \Delta, \tag{2}$$

where $\ltimes_{i=1}^{s} x_i = x_1 \ltimes \cdots \ltimes x_s$.

3 Global Stability of Switched BNs under Arbitrary Switching Signals

In this section, we study the global stability of switched BNs under arbitrary switching signals. Firstly, we give the concept of the switching reachability of points for switched BNs. Then, we define the so-called switching-state incidence matrix to check the switching reachability of points. Finally, based on the switching reachability of points, we present a necessary and sufficient condition for the global stability of switched BNs under arbitrary switching signals.

3.1 Switching Reachability of Points for Switched BNs

A switched Boolean network with n nodes and w sub-networks is described as

$$\begin{cases} x_1(t+1) = f_1^{\sigma(t)}(x_1(t), \cdots, x_n(t)), \\ x_2(t+1) = f_2^{\sigma(t)}(x_1(t), \cdots, x_n(t)), \\ \quad\vdots \\ x_n(t+1) = f_n^{\sigma(t)}(x_1(t), \cdots, x_n(t)), \end{cases} \tag{3}$$

where $\sigma : \mathbb{N} \to W = \{1,\ 2,\ \cdots,\ w\}$ is the switching signal, $x_i \in \mathcal{D},\ i = 1, 2, \cdots, n$ are logical variables, $f_i^k : \mathcal{D}^n \to \mathcal{D},\ i = 1, \cdots, n,\ k = 1, \cdots, w$ are logical functions, and $\mathbb{N}$ stands for the set of non-negative integers.

Given a finite-time switching signal $\sigma : \{0, 1, \cdots, l\} \to W$ with l a given positive integer, set $\sigma(k) = i_k,\ k = 0, 1, \cdots, l$. Then, we obtain the following switching sequence:

$$\pi := \{(0,\ i_0),\ (1,\ i_1),\ \cdots,\ (l,\ i_l)\}. \tag{4}$$

Consider the switched BN (3). We give the definition of the switching reachability of points for switched BNs as follows.

Definition 3.1 *Consider the switched BN (3). Let $X_0 = (x_1(0), ..., x_n(0)) \in \mathcal{D}^n$. $X = (x_1, ..., x_n) \in \mathcal{D}^n$ is said to be switching reachable from X_0, if we can find an integer $l > 0$ and a switching sequence $\pi = \{(0,\ i_0), \cdots,\ (l-1,\ i_{l-1})\}$, such that under the switching sequence π, the trajectory of the system (3) starting from X_0 reaches X at time l.*

To investigate the switching reachability of points, we propose the concept of the switching-state product space for the switched BN (3).

Definition 3.2 *The switching-state product space of the switched BN (3) is defined as*

$$\xi = \{(\lambda, X) | \lambda \in W,\ X = (x_1, \cdots, x_n) \in \mathcal{D}^n\}.$$

1. Let $Q_i = (\lambda^i, X^i) \in \xi$ and $Q_j = (\lambda^j, X^j) \in \xi$. Denote by $X^i = (x_1^i, \cdots, x_n^i)$. (Q_i, Q_j) is said to be a directed edge, if X^i and X^j satisfy (3). Precisely,

$$x_k^j = f_k^{\lambda^i}(x_1^i, \cdots, x_n^i),\ k = 1, \cdots, n.$$

The set of edges is denoted by $\varepsilon \subset \xi \times \xi$.

2. The pair $(\xi,\ \varepsilon)$ forms a directed graph, which is called the switching-state transfer graph.

3. $(Q_1,\ Q_2,\ \cdots,\ Q_l)$ is called a path, if $(Q_i,\ Q_{i+1}) \in \varepsilon,\ i = 1, 2, \cdots, l-1$.

Now, we give the algebraic form of the switching-state product space.

We first express the switched BN (3) into an algebraic form by the semi-tensor product. Using the vector form of logical variables, and setting $x = \ltimes_{i=1}^{n} x_i$, by Lemma 2.3, the system (3) can be expressed as

$$
\begin{cases}
x_1(t+1) = S_1^{\sigma(t)} x(t), \\
x_2(t+1) = S_2^{\sigma(t)} x(t), \\
\quad \vdots \\
x_n(t+1) = S_n^{\sigma(t)} x(t),
\end{cases} \tag{5}
$$

where $S_i^{\sigma(t)} \in \mathcal{L}_{2\times 2^n}$. Multiplying the equations in (5) together yields the following algebraic form:

$$
x(t+1) = \overline{L}_{\sigma(t)} x(t), \tag{6}
$$

where $\overline{L}_{\sigma(t)} \in \mathcal{L}_{2^n\times 2^n}$ and $Col_i(\overline{L}_{\sigma(t)}) = \ltimes_{j=1}^{n} Col_i(S_j^{\sigma(t)})$, $\forall\, i = 1, 2, \cdots, 2^n$.

By identifying the switching signal $\sigma = i \sim \delta_w^i \in \Delta_w,\ i \in W$, the switching-state product space becomes $\Delta_w \times \Delta_{2^n}$. In this case, a state in the switching-state product space can be expressed as $Q_i = \delta_w^{i_1} \times \delta_{2^n}^{i_2}$. We set the states in the switching-state product space in the order of

$$
Q_1 = \delta_w^1 \times \delta_{2^n}^1, \cdots, Q_{2^n} = \delta_w^1 \times \delta_{2^n}^{2^n}, \cdots, Q_{w2^n} = \delta_w^w \times \delta_{2^n}^{2^n}, \tag{7}
$$

where the states are arranged according to the ordered multi-index $Id(i_1, i_2; w, 2^n)$ defined in [4]. This ordered multi-index arranges the states as follows: Let i_1 and i_2 run from 1 to w and 2^n, respectively, where i_2 runs first and i_1 second. Hence, $Q_i = \delta_w^{i_1} \times \delta_{2^n}^{i_2}$ is ahead of $Q_j = \delta_w^{j_1} \times \delta_{2^n}^{j_2}$, if and only if there exists an integer $k \in \{1, 2\}$ such that $i_p = j_p,\ p = 1, \cdots, k-1$ and $i_k < j_k$.

In the following, we propose the so-called switching-state incidence matrix for the switched BN (3) to express the switching-state transfer graph in an algebraic form.

Definition 3.3 *A $w2^n \times w2^n$ matrix $\mathcal{J}$ is called the switching-state incidence matrix for the switched BN (3), if*

$$
\mathcal{J}_{ij} = \begin{cases} 1, & \text{if } (Q_j, Q_i) \in \varepsilon, \\ 0, & \text{otherwise}, \end{cases} \tag{8}
$$

where ε is the same as that in Definition 3.2, $\mathcal{J}_{ij}$ denotes the (i,j)-th element of $\mathcal{J}$, and Q_i and Q_j are given in (7).

We have the following property about $\mathcal{J}$.

Proposition 3.4 *Consider the switched BN (3) with its algebraic form (6). The switching-state incidence matrix of the system (3) is*

$$
\mathcal{J} = \left.\begin{bmatrix} \widehat{\overline{L}} \\ \widehat{\overline{L}} \\ \vdots \\ \widehat{\overline{L}} \end{bmatrix}\right\} w \in \mathcal{B}_{w2^n\times w2^n}, \tag{9}
$$

where $\widehat{\overline{L}} = [\overline{L}_1 \ \cdots \ \overline{L}_w]$.

Proof: For any two states $Q_i,\ Q_j \in \mathcal{J}$, suppose that $Q_i = \delta_w^{i_1} \times \delta_{2^n}^{i_2}$ and $Q_j = \delta_w^{j_1} \times \delta_{2^n}^{j_2}$.

By Definition 3.2, (Q_j, Q_i) is an edge if and only if $x(t+1) = \delta_{2^n}^{i_2}$ is reachable from $x(t) = \delta_{2^n}^{j_2}$ under $\sigma(t) = \delta_w^{j_1}$, that is, $\delta_{2^n}^{i_2} = \overline{L}_{j_1} \ltimes \delta_{2^n}^{j_2} = Col_{j_2}(\overline{L}_{j_1})$. Thus,

$$
\mathcal{J}_{ij} = \begin{cases} 1, & \text{if } \delta_{2^n}^{i_2} = Col_{j_2}(\overline{L}_{j_1}), \\ 0, & \text{otherwise}. \end{cases} \tag{10}
$$

Since i_1 is independent of Q_j, without loss of generality, we let $i_1 = 1$ first. Then, from (7), it is easy to see that $i = i_2$. Set $Col_{j_2}(\overline{L}_{j_1}) = \delta_{2^n}^{\zeta}$, then, (10) implies that

$$
\mathcal{J}_{ij} = \begin{cases} 1, & \text{if } i = i_2 = \zeta, \\ 0, & \text{otherwise}. \end{cases}
$$

Hence,

$$
\begin{bmatrix} \mathcal{J}_{1j} \\ \mathcal{J}_{2j} \\ \vdots \\ \mathcal{J}_{2^n j} \end{bmatrix} = \delta_{2^n}^{\zeta} = Col_{j_2}(\overline{L}_{j_1}).
$$

For $Q_j,\ 1 \le j \le w2^n$, one can obtain from the order of (7) that $j = (j_1 - 1)2^n + j_2$. Then, for fixed integer $1 \le j_1 \le w$, we have

$$
\begin{aligned}
&\begin{bmatrix}
\mathcal{J}_{1,(j_1-1)2^n+1} & \cdots & \mathcal{J}_{1,(j_1-1)2^n+2^n} \\
\mathcal{J}_{2,(j_1-1)2^n+1} & \cdots & \mathcal{J}_{2,(j_1-1)2^n+2^n} \\
\vdots & \vdots & \vdots \\
\mathcal{J}_{2^n,(j_1-1)2^n+1} & \cdots & \mathcal{J}_{2^n,(j_1-1)2^n+2^n}
\end{bmatrix} \\
&= [Col_1(\overline{L}_{j_1}) \ \cdots \ Col_{2^n}(\overline{L}_{j_1})] = \overline{L}_{j_1}.
\end{aligned}
$$

Letting $j_1 = 1, 2, \cdots, w$, respectively, we obtain

$$
\begin{aligned}
&\begin{bmatrix} Row_1(\mathcal{J}) \\ Row_2(\mathcal{J}) \\ \vdots \\ Row_{2^n}(\mathcal{J}) \end{bmatrix} = \left[\begin{matrix}
\mathcal{J}_{1,1} & \cdots & \mathcal{J}_{1,2^n} \\
\mathcal{J}_{2,1} & \cdots & \mathcal{J}_{2,2^n} \\
\vdots & \vdots & \vdots \\
\mathcal{J}_{2^n,1} & \cdots & \mathcal{J}_{2^n,2^n}
\end{matrix}\right. \\
&\qquad \left.\begin{matrix}
\cdots & \mathcal{J}_{1,(w-1)2^n+1} & \cdots & \mathcal{J}_{1,w2^n} \\
\cdots & \mathcal{J}_{2,(w-1)2^n+1} & \cdots & \mathcal{J}_{2,w2^n} \\
\vdots & \vdots & \vdots & \vdots \\
\cdots & \mathcal{J}_{2^n,(w-1)2^n+1} & \cdots & \mathcal{J}_{2^n,w2^n}
\end{matrix}\right] \\
&= [\overline{L}_1 \ \cdots \ \overline{L}_w] = \widehat{\overline{L}}.
\end{aligned}
$$

Similarly, let $i_1 = 2, ..., w$, then we can obtain the rest (block) rows of $\mathcal{J}$, each of which is equal to $\widehat{\overline{L}}$. With this, we conclude that (9) holds true. □

A matrix $A \in \mathbb{R}^{m\times n}$ is called a row-periodic one with period τ, if $Row_{i+\tau}(A) = Row_i(A),\ 1 \le i \le m - \tau$. Equivalently, $A \in \mathbb{R}^{m\times n}$ is a row-periodic matrix with period τ, if and only if $A = \mathbf{1}_k A_0$, where $A_0 \in \mathbb{R}^{\tau\times n}$ is called the basic block of A, $k = \frac{m}{\tau}$, and $\mathbf{1}_k = [\underbrace{1\ 1\ \cdots\ 1}_{k}]^T$. Moreover, if $A \in \mathbb{R}^{m\times m}$ is a row-periodic matrix with period τ, so is $A^s,\ s \in \mathbb{Z}_+$ ($\mathbb{Z}_+$ is the set of positive integers). Denote the basic block of A^s by A_0^s, then $A_0^{s+1} = A_0 A^s = A_0 \mathbf{1}_k A_0^s = \sum_{i=1}^{k} Blk_i(A_0) A_0^s$. For details about the row-periodic matrix, please refer to [20]. Based on the definition of the row-periodic matrix and by a straightforward computation, we have the following property.

Proposition 3.5 *Consider the switched BN (3) with its switching-state incidence matrix $\mathcal{J}$ given in (9). Then, $\mathcal{J}$ is a row-periodic matrix with period 2^n, and $\mathcal{J} = \mathbf{1}_w \widehat{\overline{L}}$. In addition, $\mathcal{J}^l,\ l \in \mathbb{Z}_+$ is also a row-periodic matrix with period 2^n, and the basic block of $\mathcal{J}^l$ is*

$$
\mathcal{J}_0^l = \widehat{\overline{M}}^{\,l-1} \widehat{\overline{L}}, \tag{11}
$$

where $\widehat{\overline{M}} = \sum_{i=1}^{w} \overline{L}_i$.

Next, let us explain the physical meaning of the switching-state incidence matrix given in (9). By the definition of $\mathcal{J}$, it is easy to see that $\mathcal{J}_{ij} = 1 > 0$ means that there exists a proper switching sequence such that Q_i is reachable from Q_j in one step. Similarly, for the physical meaning of $(\mathcal{J}^l)_{ij}$, $l \in \mathbb{Z}_+$, we have the following result.

Proposition 3.6 *Consider the switched BN (3) with its switching-state incidence matrix given in (9), and assume that the (i,j)-th element of the l-th power of $\mathcal{J}$, denoted by $(\mathcal{J}^l)_{ij}$, is equal to c, $l \in \mathbb{Z}_+$. Then, there are c paths from the state Q_j to Q_i at the l-th step with proper switching sequences, where Q_i and Q_j are given in (7).*

Proof: We prove it by the mathematical induction.

When $l = 1$, the conclusion follows from Definition 3.3.

Assume that when $l = k$, the conclusion holds true, that is, $(\mathcal{J}^k)_{ij}$ is the number of paths from the state Q_j to Q_i at the k-th step with proper switching sequences. Now, we consider the case of $l = k+1$. In this case, a path from Q_j to Q_i at the $(k+1)$-th step can be decomposed into a path from Q_j to Q_v at the k-th step and a path from Q_v to Q_i at one step. Noting that Q_v has $w2^n$ choices, we know that the number of paths from Q_j to Q_i at the $(k+1)$-th step is given as

$$\sum_{v=1}^{w2^n} \mathcal{J}_{iv} (\mathcal{J}^k)_{vj} = (\mathcal{J}^{k+1})_{ij} = c,$$

which means that the conclusion holds true for $l = k+1$.

By the mathematical induction, the conclusion holds for any l. □

From Proposition 3.6, one can easily see that $\{\mathcal{J}^l | \forall\ l \in \mathbb{Z}_+\}$ contains the entire switching reachability information of the system (3). By Cayley-Hamilton theorem [8], it is easy to see that if $(\mathcal{J}^l)_{ij} = 0,\ \forall\ l \le w2^n$, then $(\mathcal{J}^l)_{ij} = 0,\ \forall\ l \in \mathbb{Z}_+$. From Proposition 3.5, $\mathcal{J}^l$ is row-periodic. Thus, we only need to consider the basic block of $\mathcal{J}^l$, $\forall\ l \le w2^n$. By the construction of the switching-state incidence matrix, it is easy to see that $Blk_i(\mathcal{J}_0^l)$ corresponds to $\sigma = \delta_w^i$, and the j-th column of $Blk_\mu(\mathcal{J}_0^l)$ corresponds to the initial value $x(0) = \delta_{2^n}^j$.

For ease of expression, we now give a new definition. Let $A = (a_{ij}) \in \mathbb{R}^{n\times m}$ be a matrix, then we call $A > 0$ if and only if $a_{ij} > 0$ for any i and j.

Based on Proposition 3.6 and the above analysis, and using the above definition, we have the following result to verify the switching reachability of points for the system (3).

Theorem 3.7 *Consider the switched BN (3) with its switching-state incidence matrix given in (9). Then, $x = \delta_{2^n}^\alpha$ is switching reachable from $x(0) = \delta_{2^n}^j$, if and only if*

$$\sum_{l=1}^{w2^n} \sum_{i=1}^{w} \Big(Blk_i(\mathcal{J}_0^l) \Big)_{\alpha j} = \sum_{l=1}^{w2^n} \Big(\widehat{\overline{M}}^l \Big)_{\alpha j} > 0, \qquad (12)$$

where $\Big(\widehat{\overline{M}}^l \Big)_{\alpha j}$ denotes the (α, j)-th element of $\widehat{\overline{M}}^l$, and $\mathcal{J}_0^l$ and $\widehat{\overline{M}}$ are given in Proposition 3.5.

Proof: (Necessity) Suppose that $x = \delta_{2^n}^\alpha$ is reachable from $x(0) = \delta_{2^n}^j$ at the l-th step, then there exists a switching signal with $\sigma(l) = \delta_w^{i_1}$ and $\sigma(0) = \delta_w^{i_2}$, such that $Q_\beta = \delta_w^{i_1} \times \delta_{2^n}^\alpha$ is reachable from $Q_\gamma = \delta_w^{i_2} \times \delta_{2^n}^j$. By Proposition 3.6, it is easy to see that $(\mathcal{J}^l)_{\beta\gamma} > 0$. Noticing that $\mathcal{J}^l$ is a row-periodic matrix with period 2^n, we know that $Blk_{i_2}(\mathcal{J}_0^l)_{\alpha j} > 0$, which implies that (12) holds.

(Sufficiency) Suppose that (12) holds, then there exist two positive integers l and k such that $Blk_k(\mathcal{J}_0^l)_{\alpha j} > 0$. Therefore, there exist integers i_1, i_2, β and γ with $i_2 = k$, $Q_\beta = \delta_w^{i_1} \times \delta_{2^n}^\alpha$ and $Q_\gamma = \delta_w^{i_2} \times \delta_{2^n}^j$, such that $(\mathcal{J}^l)_{\beta\gamma} > 0$. This together with Proposition 3.6 imply that $x = \delta_{2^n}^\alpha$ is reachable from $x(0) = \delta_{2^n}^j$. □

3.2 Global Stability of Switched BNs

In this part, we study the global stability of the switched BN (3) based on the switching-state incidence matrix and Theorem 3.7, and establish a necessary and sufficient condition for the global stability under arbitrary switching signals.

Consider the system (3). The equilibrium of each subsystem is described by the following assumption.

Assumption 3.8 *All the sub-networks of the system (3) are globally stable at the common fixed point $x_e = \delta_{2^n}^i$.*

For the switching-state incidence matrix given in (9), we have the following property.

Proposition 3.9 *Consider the system (3) with its switching-state incidence matrix given in (9). Then,*

$$\sum_{i=1}^{2^n} \Big(\widehat{\overline{M}}^l \Big)_{ij} = w^l,\ \forall\ j = 1, 2, \cdots, 2^n,\ l \in \mathbb{Z}_+, \qquad (13)$$

where $\widehat{\overline{M}}$ is given in Proposition 3.5, and $\Big(\widehat{\overline{M}}^l \Big)_{ij}$ denotes the (i,j)-th element of $\widehat{\overline{M}}^l$.

Proof: We prove it by the mathematical induction.

When $l = 1$, since $\overline{L}_q \in \mathcal{L}_{2^n \times 2^n}$, $q = 1, \cdots, w$, we have $\sum_{i=1}^{2^n} (\overline{L}_q)_{ij} = 1$. Thus,

$$\sum_{i=1}^{2^n} (\widehat{\overline{M}})_{ij} = \sum_{i=1}^{2^n} \sum_{q=1}^{w} (\overline{L}_q)_{ij} = \sum_{q=1}^{w} \sum_{i=1}^{2^n} (\overline{L}_q)_{ij} = w,$$

which implies that (13) holds for $l = 1$.

Assume that (13) holds true for $l = k$. Now, we consider the case of $l = k+1$. In this case, one can obtain

$$\sum_{i=1}^{2^n} \Big(\widehat{\overline{M}}^{k+1} \Big)_{ij} = \sum_{i=1}^{2^n} \sum_{v=1}^{2^n} \Big(\widehat{\overline{M}}^k \Big)_{iv} \Big(\widehat{\overline{M}} \Big)_{vj} =$$

$$\sum_{v=1}^{2^n} \sum_{i=1}^{2^n} \Big(\widehat{\overline{M}}^k \Big)_{iv} \Big(\widehat{\overline{M}} \Big)_{vj} = w^k \sum_{v=1}^{2^n} \Big(\widehat{\overline{M}} \Big)_{vj} = w^{k+1}.$$

Hence, (13) holds true for $l = k+1$.

By the mathematical induction, the conclusion holds for any l. □

Next, we study the global stability of the switched BN (3) under Assumption 3.8.

For the case that the system (3) is globally stable at $x_e = \delta_{2^n}^i$ under any switching signal, we have the following necessary condition.

Proposition 3.10 *Consider the system (3), and assume that Assumption 3.8 holds. If the system is globally stable at*

$x_e = \delta_{2^n}^i$ *under any switching signal, then, starting from any initial point and under any switching signal, the trajectory of the system (3) will reach* $x_e = \delta_{2^n}^i$ *within time* 2^n.

Proof: If the conclusion is not true, then there exist an initial point $x(0)$, an integer $T > 2^n$ and a finite switching sequence $\pi = \{(0, i_0), (1, i_1), \cdots, (T-1, i_{T-1})\}$ such that starting from $x(0)$ and under π, the shortest time which guarantees that the trajectory of the system (3) reaches $x_e = \delta_{2^n}^i$ is T. Denote the trajectory by $\{x(0), x(1), \cdots, x(T-1), x(T) = x_e\}$, then, it is easy to see that $x(i) \neq x_e,\ \forall\, 0 \le i \le T-1$. Since the number of different states for the system (3) is 2^n, there must exist two integers $0 \le \alpha < \beta \le T-1$ such that $x(\alpha) = x(\beta) \neq x_e$.

Now, we construct a new switching signal as

$$\sigma_1(t) = \begin{cases} i_\alpha,\ t = k(\beta-\alpha), \\ i_{\alpha+1},\ t = k(\beta-\alpha)+1, \\ \vdots \\ i_{\beta-1},\ t = (k+1)(\beta-\alpha)-1, \end{cases}$$

where $k \in \mathbb{N}$. Then, starting from the initial point $x'(0) = x(\alpha)$ and under the switching signal $\sigma_1(t)$, the trajectory of the system (3) forms a cycle $\{x(\alpha), x(\alpha+1), \cdots, x(\beta-1); x(\alpha), x(\alpha+1), \cdots, x(\beta-1); \cdots\}$, which is a contradiction with the global stability of the system (3) under any switching signal.

Therefore, starting from any initial point and under any switching signal, the trajectory of the system (3) will reach $x_e = \delta_{2^n}^i$ within time 2^n. □

Based on Theorem 3.7 and Propositions 3.9 and 3.10, we have the following result for the global stability of switched BNs under arbitrary switching signals.

Theorem 3.11 *Consider the system (3) with its algebraic form (6), and assume that Assumption 3.8 holds. Then, the system is globally stable at* $x_e = \delta_{2^n}^i$ *under any switching signal, if and only if there exists a positive integer* $\gamma^* \le 2^n$ *such that*

$$Row_i\Big(\widehat{\widehat{M}}^{\gamma^*}\Big) = \Big[\underbrace{w^{\gamma^*} \ \cdots \ w^{\gamma^*}}_{2^n}\Big], \tag{14}$$

where $\widehat{\widehat{M}}$ *is given in Proposition 3.5.*

Proof: (Necessity) Suppose that the system (3) is globally stable at $x_e = \delta_{2^n}^i$ under any switching signal. We prove that (14) holds true.

On one hand, if the system (3) is globally stable at $x_e = \delta_{2^n}^i$ under any switching signal, then, Proposition 3.10 implies that starting from any initial point $x(0) = \delta_{2^n}^j$ and under any switching signal $\sigma(t)$, the trajectory of the system (3) reaches $x_e = \delta_{2^n}^i$ in the time $\gamma(x(0), \sigma(t)) \le 2^n$, and then stays at x_e forever. Set $\gamma(x(0)) = \max_{\sigma(t)}\{\gamma(x(0), \sigma(t))\} \le 2^n$.

On the other hand, from the physical meaning of $\widehat{\widehat{M}}^l$ (Theorem 3.7), we know that $\Big(\widehat{\widehat{M}}^l\Big)_{rs} = 0$ means that $\delta_{2^n}^r$ is not switching reachable from $\delta_{2^n}^s$ at the l-th step.

Therefore, by Proposition 3.9, it is easy to see that for any integer $k \ge \gamma(x(0))$, $\Big(\widehat{\widehat{M}}^k\Big)_{rj} = 0,\ \forall\, r \neq i$, and $\Big(\widehat{\widehat{M}}^k\Big)_{ij} = w^k$.

Let $\gamma^* = \max_{x(0) \in \Delta_{2^n}}\{\gamma(x(0))\} \le 2^n$, then from the above analysis one can obtain $\Big(\widehat{\widehat{M}}^{\gamma^*}\Big)_{rj} = 0,\ \forall\, r \neq i,\ j = 1, \cdots, 2^n$, and $\Big(\widehat{\widehat{M}}^{\gamma^*}\Big)_{ij} = w^{\gamma^*},\ j = 1, \cdots, 2^n$, which implies that (14) holds true.

(Sufficiency) Suppose that (14) holds. We need to prove that the system is globally stable at $x_e = \delta_{2^n}^i$ under any switching signal.

In fact, from (14) it is easy to see that $\Big(\widehat{\widehat{M}}^{\gamma^*}\Big)_{rj} = 0,\ \forall\, r \neq i,\ j = 1, \cdots, 2^n$, and $\Big(\widehat{\widehat{M}}^{\gamma^*}\Big)_{ij} = w^{\gamma^*},\ j = 1, \cdots, 2^n$, with which and the physical meaning of $\widehat{\widehat{M}}^l$, by Proposition 3.9, we know that starting from any initial point $x(0) = \delta_{2^n}^j$ and under any switching signal $\sigma(t) : \{0, \cdots, \gamma^*-1\} \to W$, the trajectory of the system (3) will reach $x_e = \delta_{2^n}^i$ in the time $\gamma^* \le 2^n$, and stays at x_e forever for any switching signal $\sigma(t) : \{\gamma^*, \cdots\} \to W$.

Therefore, the system is globally stable at $x_e = \delta_{2^n}^i$ under any switching signal. □

Remark 3.12 *We call the smallest positive integer* $\gamma^* \le 2^n$ *which satisfies (14) the global stability index of the switched BN (3).*

Remark 3.13 *Consider the switched BN (3). If* $w = 1$*, the system reduces to the non-switching case. In this case, Theorem 3.11 gives a new way to the stability analysis of BNs.*

4 Controllability of Switched BCNs

In this section, we investigate the controllability of switched BCNs. This section contains two subsections. In Section 4.1, we propose the so-called switching-input-state incidence matrix for switched BCNs. Section 4.2 presents some necessary and sufficient conditions for the controllability of switched BCNs.

4.1 Switching-Input-State Incidence Matrix

Consider a switched Boolean control network with n nodes, m control inputs and w sub-networks, which is given as

$$\begin{cases} x_1(t+1) = f_1^{\sigma(t)}(x_1(t), \cdots, x_n(t), u_1(t), \\ \qquad\qquad \cdots, u_m(t)), \\ x_2(t+1) = f_2^{\sigma(t)}(x_1(t), \cdots, x_n(t), u_1(t), \\ \qquad\qquad \cdots, u_m(t)), \\ \vdots \\ x_n(t+1) = f_n^{\sigma(t)}(x_1(t), \cdots, x_n(t), u_1(t), \\ \qquad\qquad \cdots, u_m(t)), \end{cases} \tag{15}$$

where $\sigma : \mathbb{N} \to W = \{1, 2, \cdots, w\}$ is the switching signal, $x_i \in \mathcal{D},\ i = 1, \cdots, n$ are logical variables, $u_i \in \mathcal{D},\ i = 1, \cdots, m$ are control inputs, and $f_i^j : \mathcal{D}^{m+n} \to \mathcal{D},\ i = 1, \cdots, n,\ j = 1, \cdots, w$ are logical functions.

We give the rigorous definition for the controllability of the switched BCN (15) as follows.

Definition 4.1 *Consider the switched BCN (15). Let* $X_0 = (x_1(0), \cdots, x_n(0)) \in \mathcal{D}^n$.

1. $X = (x_1, \cdots, x_n) \in \mathcal{D}^n$ *is said to be reachable from* X_0 *at time* $l > 0$*, if we can find a switching sig-*

nal σ : $\{0,\cdots,l-1\} \to W$ and a sequence of controls $U(0) := \{u_1(0),\ \cdots,\ u_m(0)\}$, $\cdots$, $U(l-1) := \{u_1(l-1),\ \cdots,\ u_m(l-1)\}$, such that under the switching signal σ and controls $\{U(t),\ t=0,\cdots,l-1\}$, the trajectory of the system (15) starting from X_0 reaches X at time l. The reachable set at time l is denoted by $R_l(X_0)$. The reachable set of X_0 is given as $R(X_0) = \bigcup_{l=1}^{\infty} R_l(X_0)$.

2. The switched BCN (15) is said to be controllable at X_0, if $R(X_0) = \mathcal{D}^n$. The switched BCN (15) is said to be controllable, if the system is controllable at all $X_0 \in \mathcal{D}^n$.

To study the controllability of switched BCNs, we define the switching-input-state product space for switched BCNs as follows.

Definition 4.2 *The switching-input-state product space of the switched BCN (15) is defined as*

$$\begin{aligned}\mathfrak{s} = \{(\lambda, U, X) \quad | \quad & \lambda \in W,\ U = (u_1,\cdots,u_m) \in \mathcal{D}^m, \\ & X = (x_1,\cdots,x_n) \in \mathcal{D}^n\}.\end{aligned}$$

Obviously, the cardinal number of $\mathfrak{s}$ is $|\mathfrak{s}| = w2^{m+n}$.

1. Let $P_i = (\lambda^i, U^i, X^i) \in \mathfrak{s}$ and $P_j = (\lambda^j, U^j, X^j) \in \mathfrak{s}$, where $U^i = (u_1^i,\cdots,u_m^i)$ and $X^i = (x_1^i,\cdots,x_n^i)$. (P_i, P_j) is said to be a directed edge, if X^i, U^i, X^j satisfy (15), that is,

$$x_k^j = f_k^{\lambda^i}(x_1^i,\cdots,x_n^i,u_1^i,\cdots,u_m^i),\ k=1,\cdots,n.$$

The set of edges is denoted by $\epsilon \subset \mathfrak{s} \times \mathfrak{s}$.

2. The pair $(\mathfrak{s},\ \epsilon)$ forms a directed graph, which is called the switching-input-state transfer graph.

3. $(P_1,\ P_2,\ \cdots,\ P_l)$ is called a path, if $(P_i,\ P_{i+1}) \in \epsilon$, $i = 1,\ 2,\ \cdots,\ l-1$.

Next, we express the switching-input-state transfer graph into an algebraic form.

Consider the switched BCN (15). Let us first express the system (15) into an algebraic form by using the semi-tensor product. Using the vector form of logical variables x_i and u_i, and setting $x = \ltimes_{i=1}^n x_i$ and $u = \ltimes_{i=1}^m u_i$, by the technique applied in (5), the system (15) can be expressed as the following algebraic form:

$$x(t+1) = L_{\sigma(t)} u(t) x(t), \tag{16}$$

where $L_{\sigma(t)} \in \mathcal{L}_{2^n \times 2^{m+n}}$.

Identifying the switching signal $\sigma = i \sim \delta_w^i \in \Delta_w$, $i \in W$, the switching-input-state product space becomes $\Delta_w \times \Delta_{2^m} \times \Delta_{2^n}$. In this case, a state in the switching-input-state product space can be expressed as $P_i = \delta_w^{i_1} \times \delta_{2^m}^{i_2} \times \delta_{2^n}^{i_3}$. Set all the states in the switching-input-state product space in the following order:

$$\begin{aligned}&P_1 = \delta_w^1 \times \delta_{2^m}^1 \times \delta_{2^n}^1, \cdots, P_{2^n} = \delta_w^1 \times \delta_{2^m}^1 \times \delta_{2^n}^{2^n}, \\ &\cdots, P_{w2^{m+n}} = \delta_w^w \times \delta_{2^m}^{2^m} \times \delta_{2^n}^{2^n},\end{aligned} \tag{17}$$

where the states are arranged according to the ordered multi-index $Id(i_1,i_2,i_3;w,2^m,2^n)$. This ordered multi-index arranges the states as follows: Let i_1, i_2 and i_3 run from 1 to w, 2^m and 2^n, respectively, where i_3 runs first, i_2 second and i_1 last. Hence, $P_i = \delta_w^{i_1} \times \delta_{2^m}^{i_2} \times \delta_{2^n}^{i_3}$ is ahead of $P_j = \delta_w^{j_1} \times \delta_{2^m}^{j_2} \times \delta_{2^n}^{j_3}$, if and only if there exists an integer $k \in \{1,2,3\}$ such that $i_p = j_p$, $p = 1,\cdots,k-1$ and $i_k < j_k$.

Now, we define a $w2^{m+n} \times w2^{m+n}$ matrix $\mathcal{S}$ for the switched BCN (15), called the switching-input-state incidence matrix, as follows.

Definition 4.3 *A $w2^{m+n} \times w2^{m+n}$ matrix $\mathcal{S}$ is called the switching-input-state incidence matrix of the system (15), if*

$$\mathcal{S}_{ij} = \begin{cases} 1, & \text{if } (P_j, P_i) \in \epsilon, \\ 0, & \text{otherwise}, \end{cases} \tag{18}$$

where ϵ is the same as that in Definition 4.2, $\mathcal{S}_{ij}$ denotes the (i,j)-th element of $\mathcal{S}$, and P_i and P_j are given in (17).

We have the following property about $\mathcal{S}$.

Proposition 4.4 *Consider the switched BCN (15) with its algebraic form (16). The switching-input-state incidence matrix of the system (15) can be given as*

$$\mathcal{S} = \left.\begin{bmatrix} \widehat{L} \\ \widehat{L} \\ \vdots \\ \widehat{L} \end{bmatrix}\right\} w2^m \in \mathcal{B}_{w2^{m+n} \times w2^{m+n}}, \tag{19}$$

where $\widehat{L} = [L_1 \ \cdots \ L_w]$.

Proof: The proof is similar to that of Proposition 3.4, and thus omitted. □

Based on the definition of the row-periodic matrix and by a straightforward computation, we have the following property.

Proposition 4.5 *Consider the switched BCN (15) with its switching-input-state incidence matrix $\mathcal{S}$ given in (19). Then, $\mathcal{S}$ is a row-periodic matrix with period 2^n, and*

$$\mathcal{S} = \mathbf{1}_{w2^m} \widehat{L}. \tag{20}$$

In addition, $\mathcal{S}^l$, $l \in \mathbb{Z}_+$ is also a row-periodic matrix with period 2^n, and the basic block of $\mathcal{S}^l$ is

$$\mathcal{S}_0^l = \widehat{M}^{l-1} \widehat{L}, \tag{21}$$

where $\widehat{M} = \sum_{i=1}^{w2^m} Blk_i(\widehat{L})$.

4.2 Controllability

In this subsection, we investigate the controllability of the switched BCN (15) based on the switching-input-state incidence matrix $\mathcal{S}$ given in (19), and present some necessary and sufficient conditions for the controllability.

Consider the switched BCN (15) with its algebraic form (16). By the definition of $\mathcal{S}$, it is easy to see that $\mathcal{S}_{ij} = 1$ means that there exist a proper switching sequence and a control sequence such that P_i is reachable from P_j at one step. Similarly, does $(\mathcal{S}^l)_{ij} > 0$ $(l \in \mathbb{Z}_+)$ imply whether or not there exist a proper switching sequence and a sequence of controls such that P_i is reachable from P_j at the l-th step? The following result answers this question.

Theorem 4.6 *Consider the switched BCN (15) with its switching-input-state incidence matrix $\mathcal{S}$ given in (19). Assume that the (i,j)-th element of $\mathcal{S}^l$, denoted by $(\mathcal{S}^l)_{ij}$, is equal to c, $l \in \mathbb{Z}_+$. Then, there are c paths such that P_i is reached from P_j at the l-th step, where P_i and P_j are given in (17).*

Proof: The proof is similar to that of Proposition 3.6, and thus omitted. □

From Theorem 4.6, the following result is obvious.

Corollary 4.7 *Consider the switched BCN (15) with its switching-input-state incidence matrix $\mathcal{S}$ given in (19). P_i is reachable from P_j at the l-th step, if and only if $(\mathcal{S}^l)_{ij} > 0$.*

Remark 4.8 *Theorem 4.6 and Corollary 4.7 imply that $\{\mathcal{S}^l \mid \forall\, l \in \mathbb{Z}_+\}$ contains the entire controllability information of the system (15). By Cayley-Hamilton theorem [8], it is easy to see that if $(\mathcal{S}^l)_{ij} = 0,\ \forall\ l \le w2^{m+n}$, then $(\mathcal{S}^l)_{ij} = 0,\ \forall\ l \in \mathbb{Z}_+$. From Proposition 4.5, $\mathcal{S}^l$ is row-periodic. Thus, we only need to consider the basic block of $\mathcal{S}^l,\ \forall\ l \le w2^{m+n}$. By the construction of the switching-input-state incidence matrix, it is easy to see that $Blk_i(\mathcal{S}_0^l)$ corresponds to $\sigma \ltimes u = \delta_{w2^m}^i$, and the j-th column of $Blk_\mu(\mathcal{S}_0^l)$ corresponds to the initial value $x(0) = \delta_{2^n}^j$.*

Based on Theorem 4.6 and Remark 4.8, we have the following result for the controllability of the switched BCN (15).

Theorem 4.9 *Consider the switched BCN (15) with its switching-input-state incidence matrix $\mathcal{S}$ given in (19). Then,*

1). $x(l) = \delta_{2^n}^\alpha$ is reachable from $x(0) = \delta_{2^n}^j$ at the l-th step, if and only if

$$\sum_{i=1}^{w2^m} \left(Blk_i(\mathcal{S}_0^l)\right)_{\alpha j} = \left(\widehat{M}^l\right)_{\alpha j} > 0, \qquad (22)$$

where $\left(\widehat{M}^l\right)_{\alpha j}$ denotes the (α, j)-th element of $\widehat{M}^l$, and $\mathcal{S}_0^l$ and $\widehat{M}$ are given in Proposition 4.5;

2). $x = \delta_{2^n}^\alpha$ is reachable from $x(0) = \delta_{2^n}^j$, if and only if

$$\sum_{l=1}^{w2^{m+n}} \left(\widehat{M}^l\right)_{\alpha j} > 0; \qquad (23)$$

3). the system is controllable at $x(0) = \delta_{2^n}^j$, if and only if

$$\sum_{l=1}^{w2^{m+n}} Col_j\left(\widehat{M}^l\right) > 0; \qquad (24)$$

4). the system is controllable, if and only if

$$\sum_{l=1}^{w2^{m+n}} \widehat{M}^l > 0. \qquad (25)$$

Proof: Let us prove 1) first.

(Necessity) Suppose that $x(l) = \delta_{2^n}^\alpha$ is reachable from $x(0) = \delta_{2^n}^j$ at the l-th step, then there exist a switching signal with $\sigma(l) = \delta_w^{i_1}$ and $\sigma(0) = \delta_w^{i_2}$, and two controls $u(l) = \delta_{2^m}^{k_1}$, $u(0) = \delta_{2^m}^{k_2}$, such that $P_\beta = \delta_w^{i_1} \times \delta_{2^m}^{k_1} \times \delta_{2^n}^\alpha$ is reachable from $P_\gamma = \delta_w^{i_2} \times \delta_{2^m}^{k_2} \times \delta_{2^n}^j$ at the l-th step. Set $\sigma(0) \ltimes u(0) = \delta_{w2^m}^{v_2}$. By Corollary 4.7, it is easy to see that $(\mathcal{S}^l)_{\beta\gamma} > 0$. Noticing that $\mathcal{S}^l$ is a row-periodic matrix with period 2^n and by Remark 4.8, we have $Blk_{v_2}(\mathcal{S}_0^l)_{\alpha j} > 0$, which implies that (22) holds.

(Sufficiency) Suppose that (22) holds, then there exists an integer k such that $Blk_k(\mathcal{S}_0^l)_{\alpha j} > 0$. Thus, there exist integers $i_1,\ k_1,\ i_2,\ k_2,\ \beta$ and γ with $\delta_w^{i_2} \ltimes \delta_{2^m}^{k_2} = \delta_{w2^m}^k$, $P_\beta = \delta_w^{i_1} \times \delta_{2^m}^{k_1} \times \delta_{2^n}^\alpha$ and $P_\gamma = \delta_w^{i_2} \times \delta_{2^m}^{k_2} \times \delta_{2^n}^j$, such that $(\mathcal{S}^l)_{\beta\gamma} > 0$. This together with Corollary 4.7 show that $x(l) = \delta_{2^n}^\alpha$ is reachable from $x(0) = \delta_{2^n}^j$ at the l-th step.

It is easy to see that 2)-4) hold immediately from 1) and Definition 4.1, and thus the proof is completed. □

Remark 4.10 *Consider the switched BCN (15). If $w = 1$, the system reduces to the non-switching case. In this case, Theorem 4.9 becomes Theorem 3.3 in [20].*

5 Illustrative Examples

In this section, we give two illustrative examples to show how to use the obtained results to verify the global stability and controllability of switched BNs.

Example 5.1 Consider the following switched BN:

$$\begin{cases} x_1(t+1) = f_1^{\sigma(t)}(x_1(t), x_2(t)), \\ x_2(t+1) = f_2^{\sigma(t)}(x_1(t), x_2(t)), \end{cases} \qquad (26)$$

where $\sigma : \mathbb{N} \to W = \{1, 2, 3\}$ is the switching signal, and

$$f_1^1(x_1, x_2) = \neg x_1 \wedge x_2,\ f_2^1(x_1, x_2) = 0,$$
$$f_1^2(x_1, x_2) = x_1 \bar{\vee} x_2,\ f_2^2(x_1, x_2) = \neg(x_1 \to x_2),$$
$$f_1^3(x_1, x_2) = x_1 \bar{\vee} x_2,\ f_2^3(x_1, x_2) = x_1 \bar{\vee} x_2.$$

Using the vector form of logical variables and setting $x(t) = \ltimes_{i=1}^2 x_i(t)$, the system (26) can be expressed as

$$x(t+1) = \overline{L}_{\sigma(t)} x(t), \qquad (27)$$

where

$$\overline{L}_1 = \delta_4[4\ 4\ 2\ 4], \quad \overline{L}_2 = \delta_4[4\ 1\ 2\ 4], \quad \overline{L}_3 = \delta_4[4\ 1\ 1\ 4].$$

It is easy to check that all the sub-networks of the system (26) are globally stable at the common fixed point $x_e = \delta_4^4 \sim (0, 0)$, which implies that Assumption 3.8 is satisfied.

A simple calculation shows that

$$\widehat{\widehat{M}}^3 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 27 & 27 & 27 & 27 \end{bmatrix},$$

where $\widehat{\widehat{M}} = \overline{L}_1 + \overline{L}_2 + \overline{L}_3$. Thus, $Row_4\left(\widehat{\widehat{M}}^3\right) = [27\ 27\ 27\ 27]$. By Theorem 3.11, the switched BN (26) is globally stable at $(0, 0)$ under any switching signal. Moreover, it is easy to see that the global stability index of the system (26) is $\gamma^* = 3$. □

Example 5.2 *Consider the following switched BCN:*

$$\begin{cases} x_1(t+1) = f_1^{\sigma(t)}(x_1(t), x_2(t), u(t)), \\ x_2(t+1) = f_2^{\sigma(t)}(x_1(t), x_2(t), u(t)), \end{cases} \qquad (28)$$

where $\sigma : \mathbb{N} \to W = \{1, 2\}$ is the switching signal, and

$$f_1^1(x_1, x_2, u) = u \vee \neg[u \vee (x_1 \to x_2)],$$
$$f_2^1(x_1, x_2, u) = (u \wedge x_1) \vee \neg[u \vee (x_1 \wedge x_2)],$$
$$f_1^2(x_1, x_2, u) = [u \wedge (x_1 \vee x_2)] \vee \neg[u \vee (x_1 \wedge x_2)],$$
$$f_2^2(x_1, x_2, u) = \neg x_1.$$

Using the vector form of logical variables and setting $x(t) = \ltimes_{i=1}^{2} x_i(t)$, we have

$$x(t+1) = L_{\sigma(t)} u(t) x(t), \tag{29}$$

where

$$L_1 = \delta_4[1\ 1\ 2\ 2\ 4\ 1\ 3\ 3],\ L_2 = \delta_4[2\ 2\ 1\ 3\ 4\ 2\ 1\ 1].$$

Now, we investigate the controllability of the system (28). A simple calculation shows that $\sum_{l=1}^{16} \widehat{M}^l > 0$, where

$$\widehat{M} = \sum_{i=1}^{4} Blk_i(\widehat{L}) = \begin{bmatrix} 1 & 2 & 2 & 1 \\ 1 & 2 & 1 & 1 \\ 0 & 0 & 1 & 2 \\ 2 & 0 & 0 & 0 \end{bmatrix},$$

and $\widehat{L} = [L_1\ \ L_2]$. Therefore, by Theorem 4.9, we conclude that the system (28) is controllable. $\square$

6 Conclusion

In this paper, we have investigated the global stability and controllability of switched BNs by using the semi-tensor product method. We have introduced the model of switched BNs and defined the concept of the controllability for switched BCNs. The concept of the switching reachability of points for switched BNs has been given and a necessary and sufficient condition has been presented to verify the switching reachability of points by constructing the switching-state incidence matrix. A necessary and sufficient condition has been proposed to check the global stability of switched BNs under arbitrary switching signals by using the switching reachability of points. A kind of switching-input-state incidence matrix has been constructed, based on which several necessary and sufficient conditions have been obtained for the controllability. The study of two illustrative examples has shown that the new results obtained in this paper are very effective in investigating the global stability and controllability of switched BNs.

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