Can one hear the shape of a concept?

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红杏枝头春意闹

Among the red-apricot branches spring runs riot!

闹=making noise

Why is blossoming connecting to making noise?

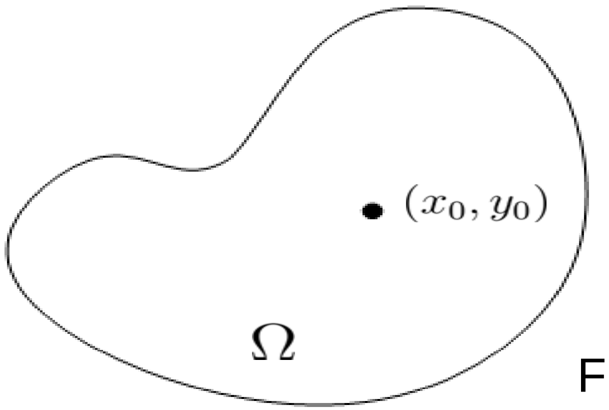
Outline

- Abstract concepts might be coded as complex dense neuron networks, or graphs, in brain
- Methods for large graph similarity finding is crucial
- Possibilities for implementing these algorithms in networked computers could lead to concept abstraction capabilities (intelligence)

methodology

- Intelligence is a complex adaptive dynamic system
- Mathematical guidance is necessary
- Wigner: “the unreasonable effectiveness of mathematics” in physics, engineering, …
- Einstein: “Mathematics is not accurate when it is real, and it is not real when accurate.”

example: diffusion equation



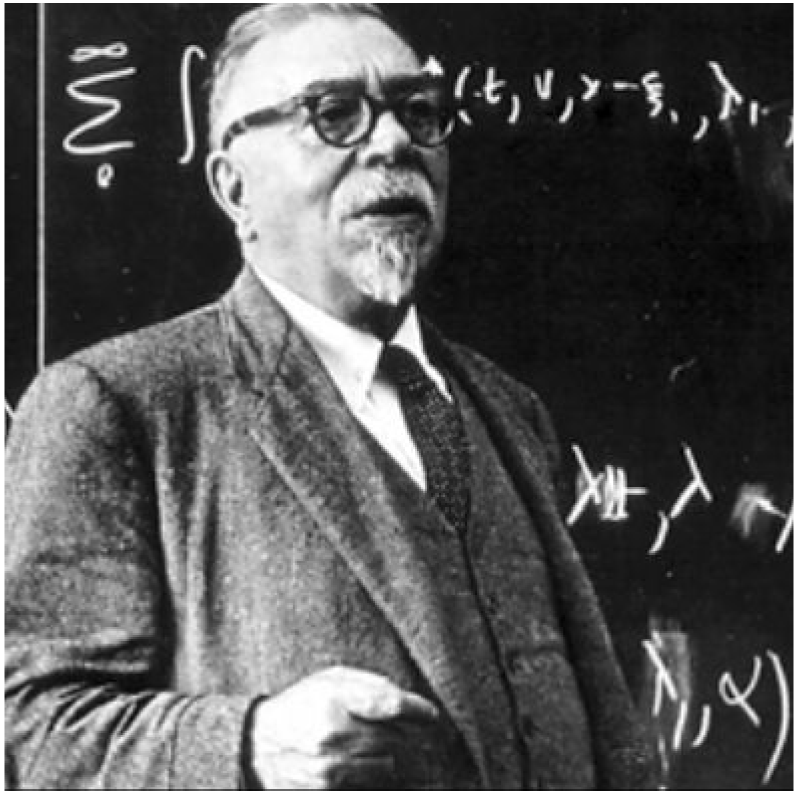
$$\frac{\partial \rho(x,y,t)}{\partial t} = \frac{1}{2}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\rho(x,y,t)$$

From the naïve Fick’s law to infinite propagation speed

Where are the information and automation technologies going next?

- Intelligence machinery for the next technology revolution?
- A forgotten hero in our discipline –

Norbert Wiener (1894-1964)

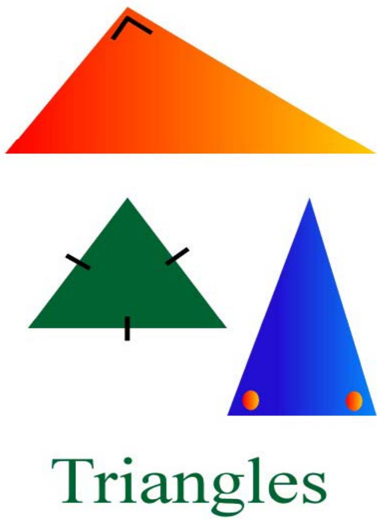


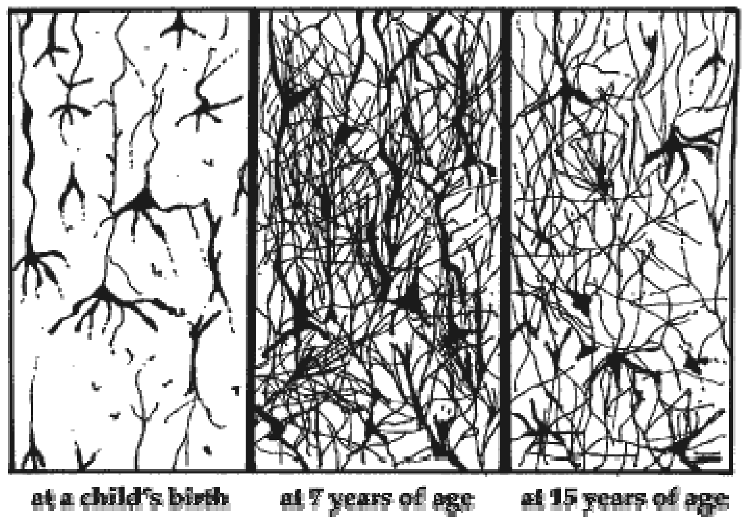
Develop brain function motived algorithms

Mathematical Challenge One: ***The Mathematics of the Brain*** --- Develop a mathematical theory to build a functional model of the brain that is mathematically consistent and predictive rather than merely biologically inspired.

Our agenda: We would like to learn from the brain, but not to mimic the real brain. Our brain is formed from a trial and error process and inevitably accumulates many illogical constructions that are hard to reason with.

How do we form the abstract concept of triangle as a young child?





Now an example of more complex “concept” ….

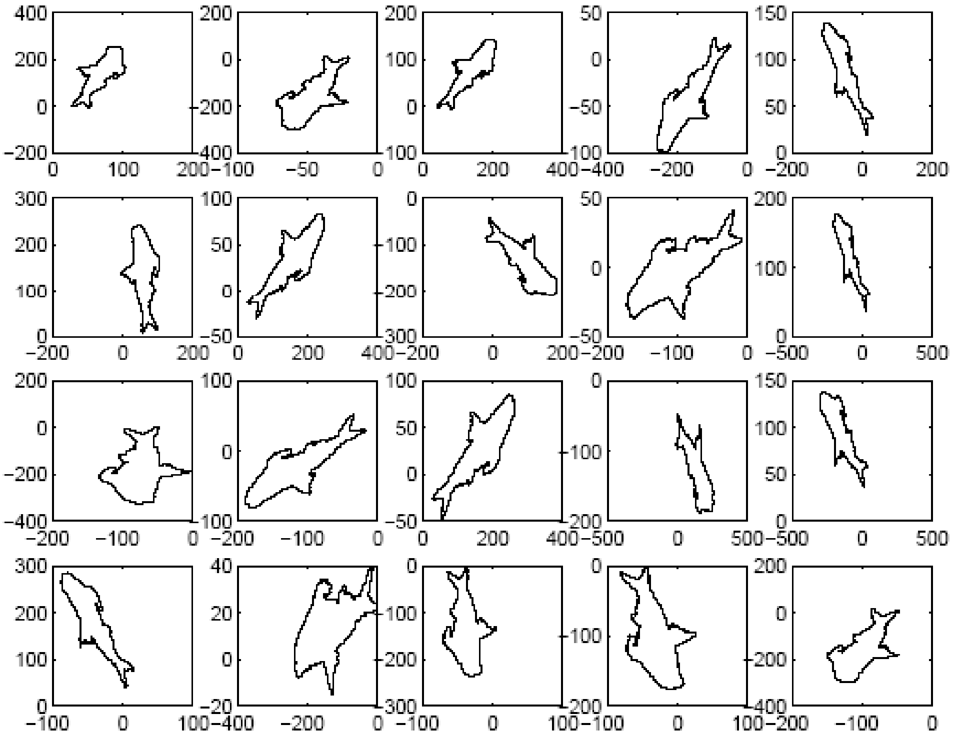
Immediate recognition of shapes/concepts

We can categorize the tree leaves very quickly.

Why are we so good in distinguishing tree leaves ?

Nature forced quick similarity finding algorithm for shapes

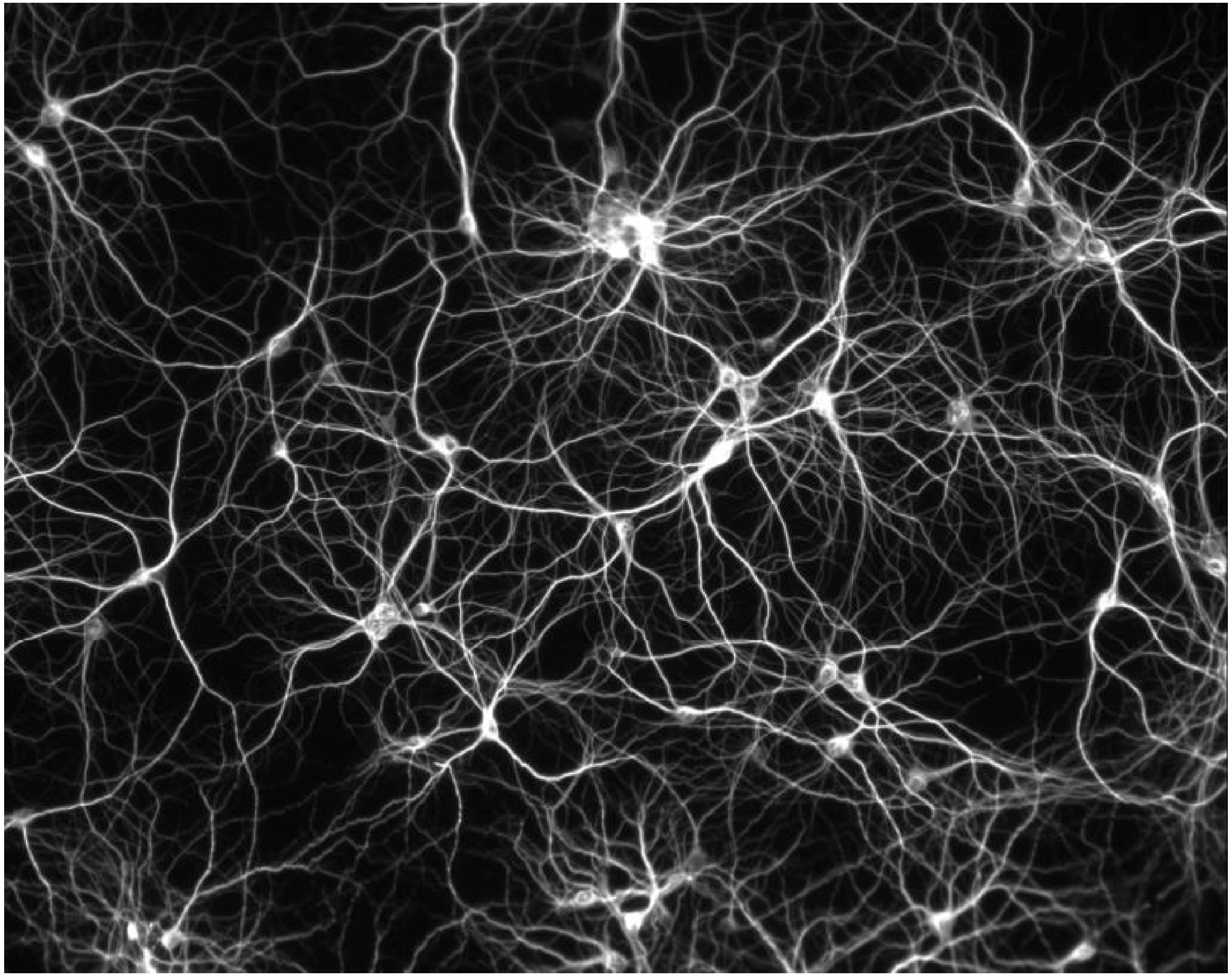
Once upon a time we were fishes –



Shape similarity finding
is a life skill for fishes

Proposed Requirements for Concept Abstraction Algorithms

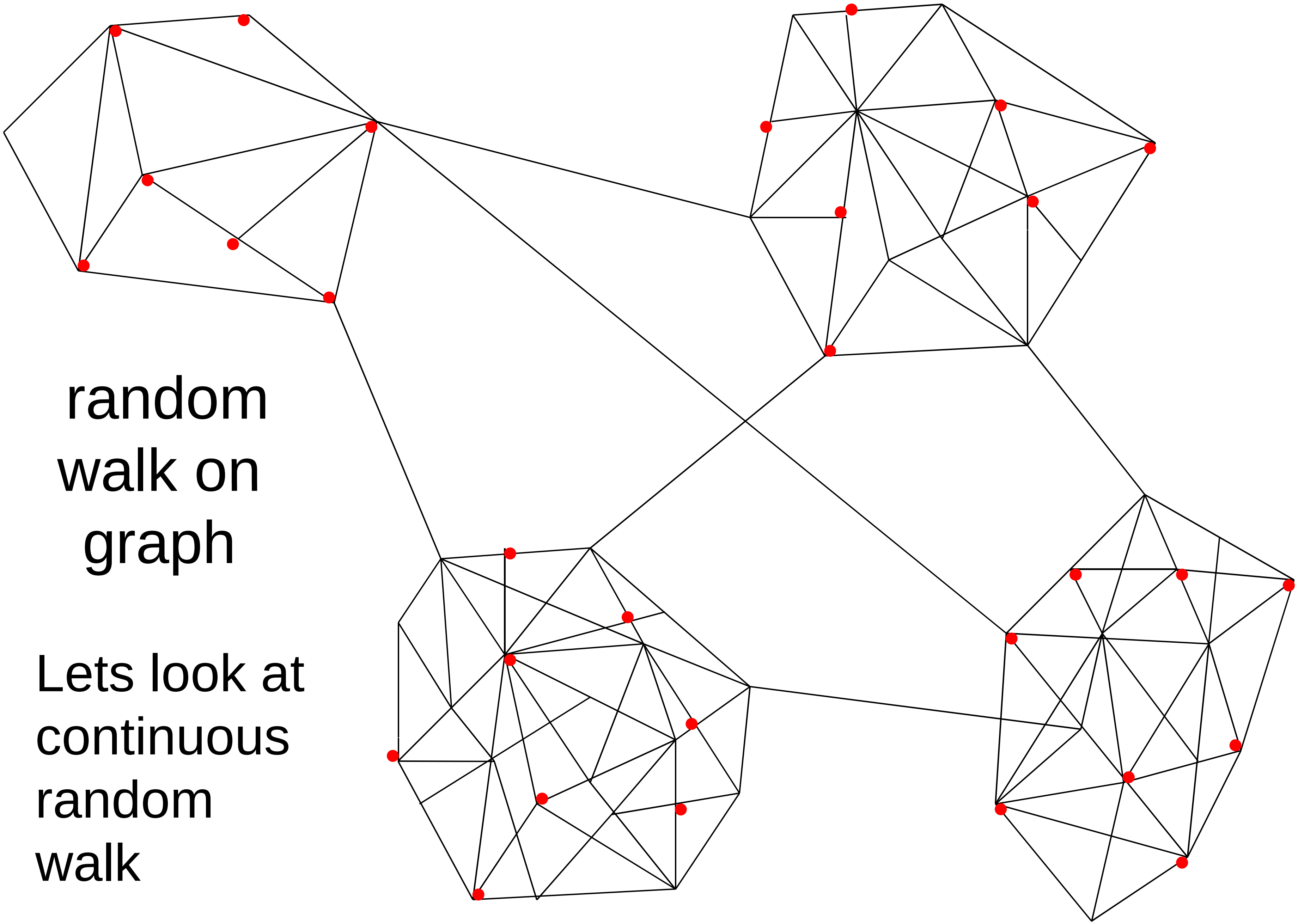
- Concepts and their instances are represented by network connectivity, or graphs;
- Concept representations should be invariant against the rotation/permutation changes of the input signal representations;
- Concept representations could check similarity “immediately” (meaning “almost instantaneously”);
- The spatial/temporal signal conversion mechanisms should be plausible for emerging from randomness.



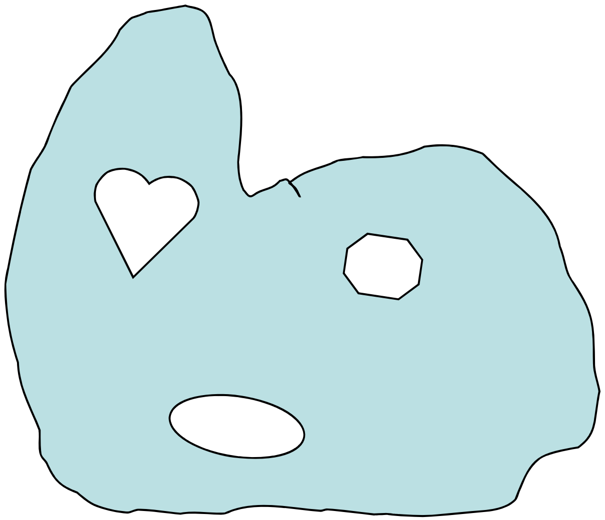
Could random walk over graph help recognizing network connectivity quickly?

random walk on graph

Lets look at continuous random walk



Can one hear the shape of a drum?



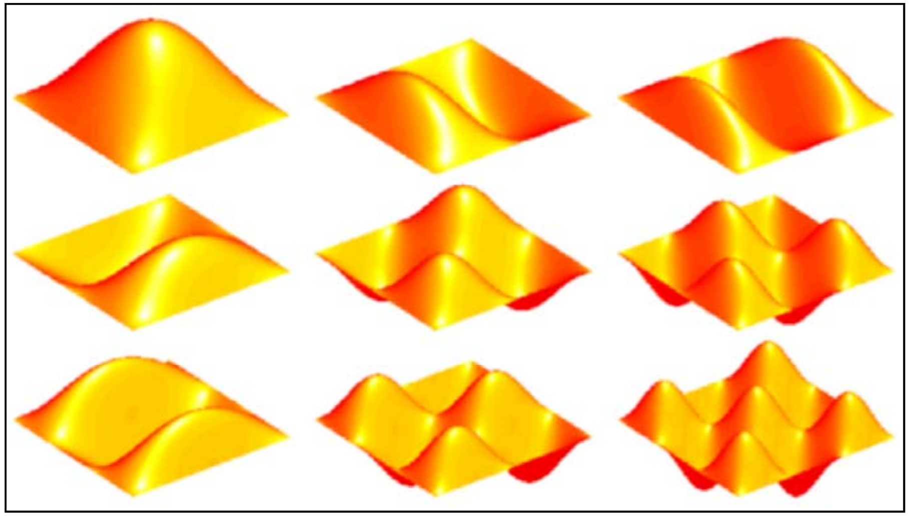
Given two planar domains D1 and D2. If the Laplacian eigenvalues $\lambda_k(D_1) = \lambda_k(D_2)$, does it follow that D1 is congruent to D2?

Mark Kac (1914-1984)

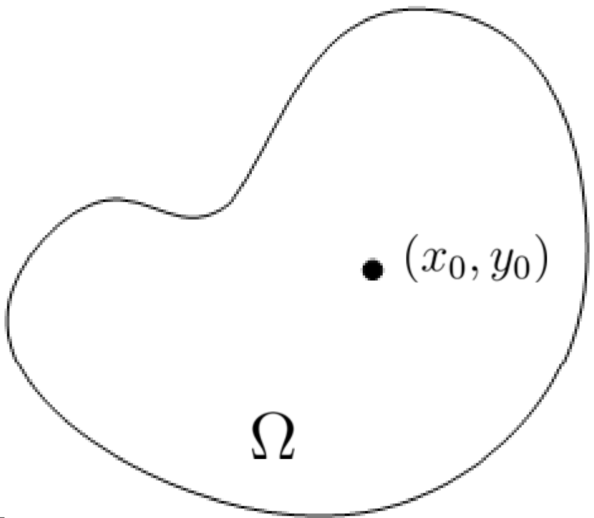
Relaxed version: “how much can we know about the shape (geometric information) of the domain from the Laplacian eigenvalues”.

Laplacian Eigenvalues and Eigenfunctions

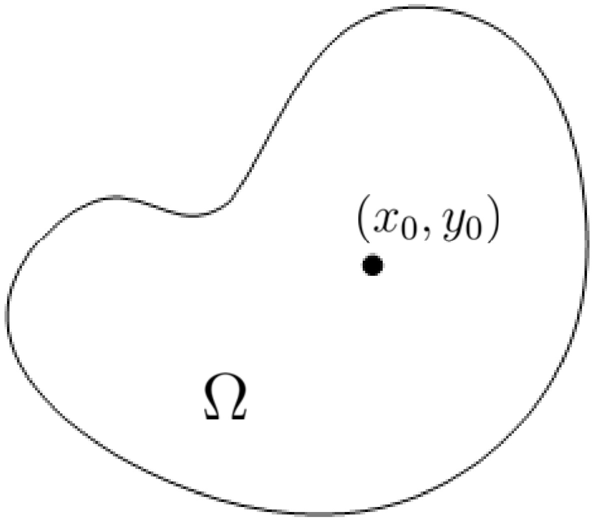
$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) u_k(x, y, t) = \lambda_k u_k(x, y, t), \quad k = 1, 2, \ldots$$



First 9 eigen-function of a square drum head



Diffusion on the drum head



$$\rho(x,y,t) = 0 \text{ if } (x,y) \in \partial\Omega$$

$$\rho(x,y,0) = \delta(x - x_0, y - y_0)$$

$$\frac{\partial \rho(x,y,t)}{\partial t} = \frac{1}{2}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\rho(x,y,t)$$

For a bounded $\Omega \subset R^2$

$$t \downarrow 0 : \sum_{n=1}^{\infty} e^{-\lambda_n t} \sim \frac{|\Omega|}{2\pi t} - \frac{|\partial\Omega|}{4\sqrt{2\pi t}}$$

more results like ….

If Ω has r holes and Ω and holes are polygons.

$$t \downarrow 0 : \sum_{n=1}^{\infty} e^{-\lambda_n t} \sim \frac{|\Omega|}{2\pi t} - \frac{|\partial\Omega|}{4\sqrt{2\pi t}} + \frac{1-r}{6}$$

Mark Kac's intuitive reasoning :

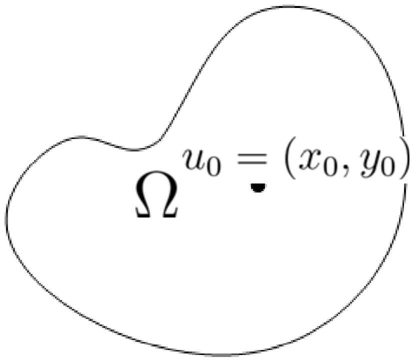
$$\frac{\partial \rho(u,t)}{\partial t} = \frac{1}{2}(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2})\rho(u,t), \qquad (x,y) = u, (x_0, y_0) = u_0$$

Heat does not see the boundary at the beginning:

$$\rho(u,t) = \sum_{n=1}^{\infty} e^{-\lambda_n t} \varphi_n(u) \varphi_n^T(u_0) \sim \frac{1}{2\pi t} e^{-\frac{\|u-u_0\|^2}{2t}} \text{ when } t \downarrow 0$$

$$\sum_{n=1}^{\infty} e^{-\lambda_n t} \varphi_n(u) \varphi_n^T(u_0) \sim \frac{1}{2\pi t} e^{-\frac{\|u-u_0\|^2}{2t}} \text{ when } t \to 0$$

$$\text{Letting } u = u_0 \quad : \quad \sum_{n=1}^{\infty} e^{-\lambda_n t} \varphi_n^2(u_0) \sim \frac{1}{2\pi t}$$



Eigenvectors are normalized :

$$\int_{\Omega} \varphi_n^2(u) du = 1 \qquad \sum_{n=1}^{\infty} e^{-\lambda_n t} \sim \frac{|\Omega|}{2\pi t}.$$

The works of van den Berg, P. Gilkey and others on heat content on closed domains in a manifold …..

Can we accomplish this for random walks on graphs?

Can we accomplish this for random walk on graphs?

- Graph $G = (V, E, W)$, complete, undirected, weighted.

  Adjacency matrix $A = [w_{u,v}]$ with $w_{uv} = \text{weight of edge}(u, v)$

  Degree matrix $D = diag[d_u]$ with $d_u = \sum_v w_{uv}$

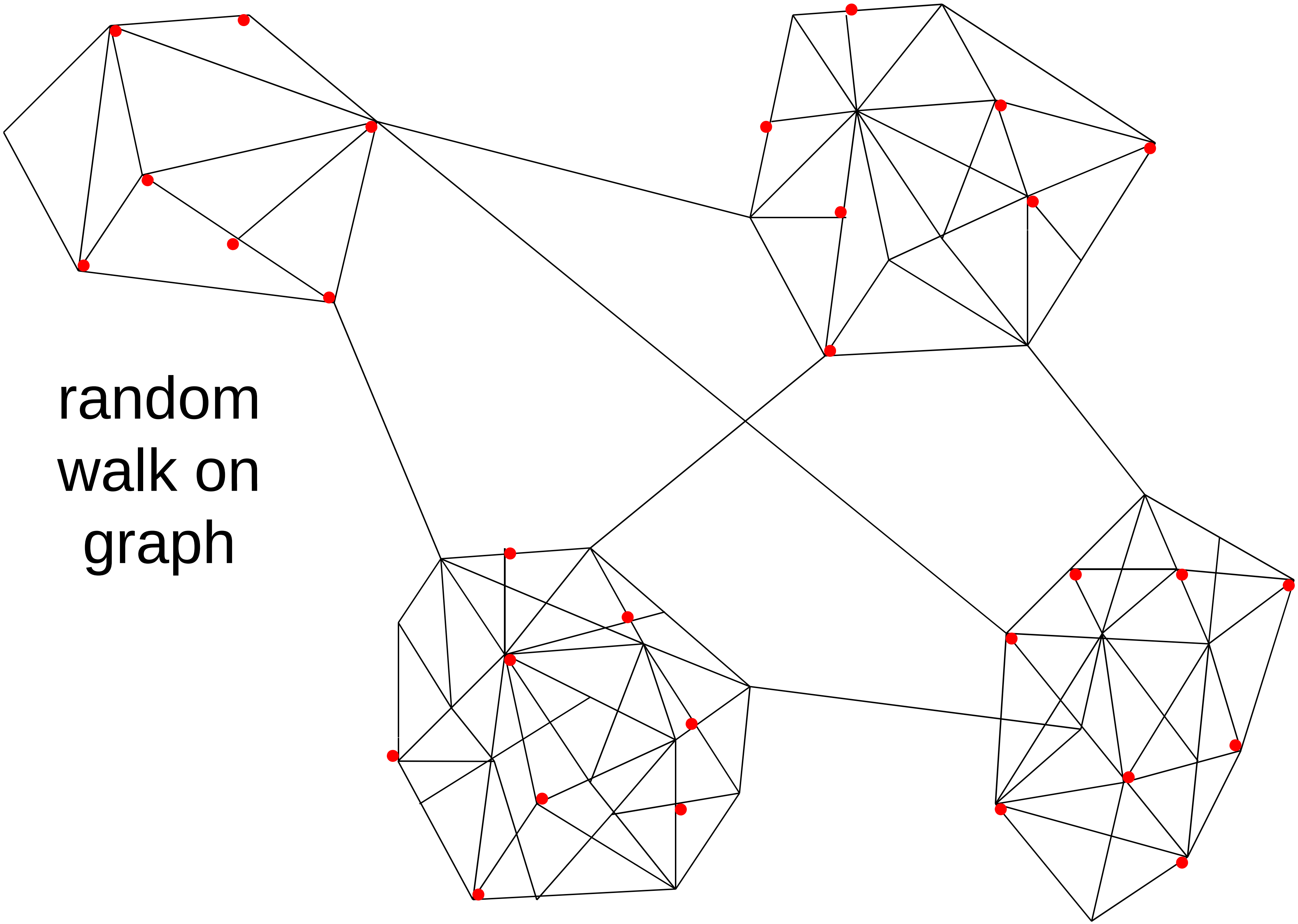
- Graph Lapacian of G = $L = D - A$

- “Walk matrix” $M = AD^{-1}$ for “naive” random walk

$$p_{k+1} = Mp_k$$

Normalized graph Laplacian $L_n = D^{-\frac{1}{2}} L D^{-\frac{1}{2}}$, symmetric.

Random Walk graph Laplacian $L_r = LD^{-1} = I - M$


random
walk on
graph

Heat content of diffusion on a graph

Patrick McDonald and Robert Meyers,
"ISOSPECTRAL POLYGONS, PLANAR GRAPHS AND HEAT CONTENT",
PROCEEDINGS OF THE AMERICAN MATHEMATICAL SOCIETY V.131, No.11, 2003

Consider a weighted graph $G(V,E)$ with boundary ∂G and its normalized graph Laplacian L_n. The heat operator is e^{-tL_n}.
Now consider the heat flow $h_t(u,v)$ (absorbed at ∂G) :

$$\frac{\partial h_t(u,v)}{\partial t} = -L_n h_t(u,v) \;\rightarrow\; h_t(u,v) = \sum_{i=1}^{|V|} e^{-\lambda_i t} \varphi_i(u) \varphi_i^T(v)$$

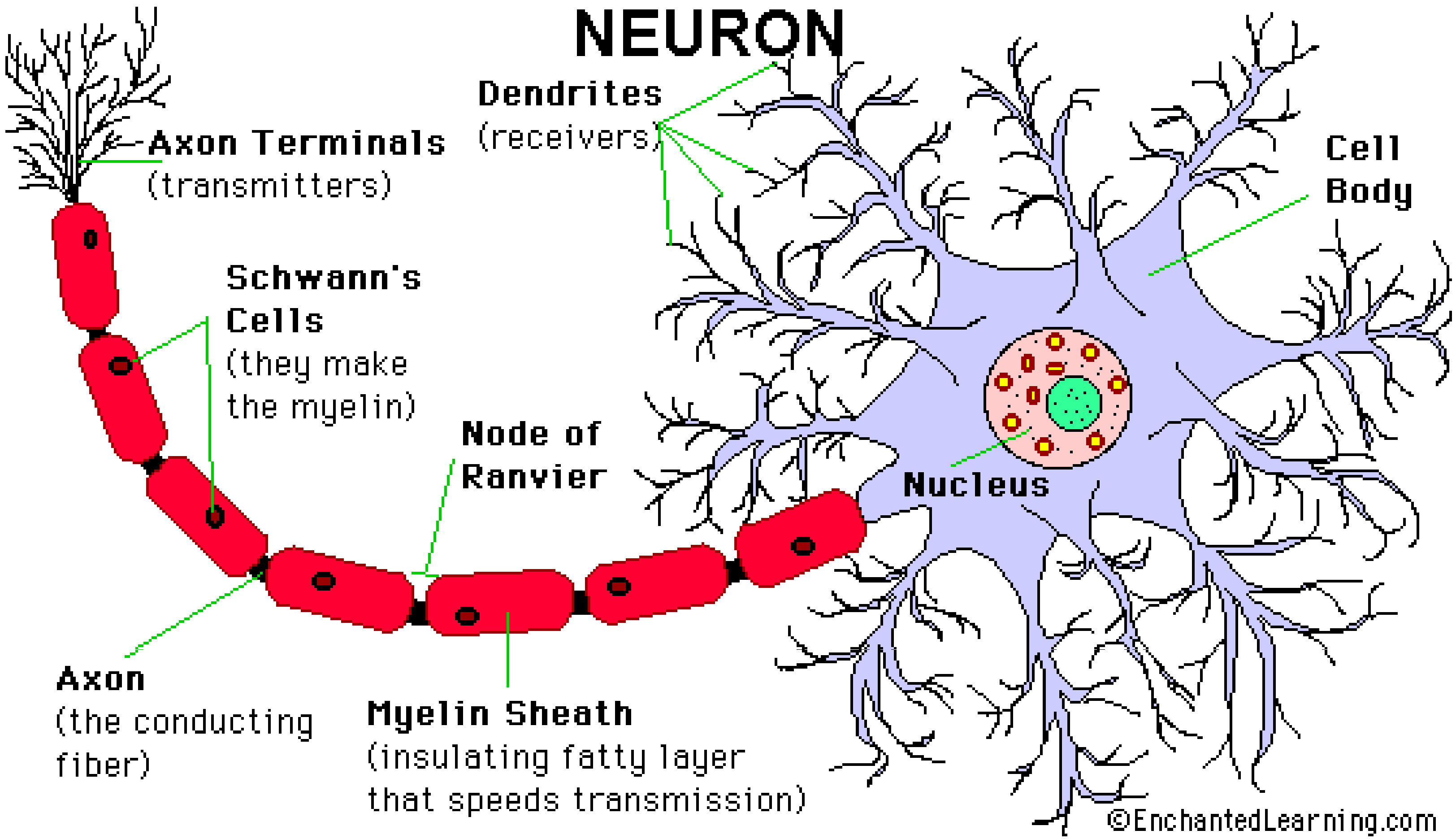
$$Q(t) = \sum_{v \in V} \sum_{u \in V} h_t(u,v)$$

contains rich information about G

$$\sum \sum h_t(u,v)$$

is easy for computers,

easy for neurons too ☺



Short time asymptotics of heat content

- McDonald-Meyer have shown ($\sim$ means approximately equal when $t \downarrow 0$)

$$Q_1(t) \sim Q_2(t) \;\Rightarrow spec(G_1) = spec(G_2)$$

$$Q_1(t) = Q_2(t) \;\overset{?}{\Rightarrow}\; G_1 = G_2$$

- Relevance:

1. $Q(t)$ is more feasible in hardware structure;
2. $Q(t)$ can be estimated from lazy walk of “pulses”;
3. $Q(t)$ classification is instantaneous ($t \downarrow 0$)

Use Lazy Walk to approach diffusion

- Lazy Walk on graph:

$$M_L = (1-\alpha)I + \alpha M = (1-\alpha)I + \alpha A D^{-1}$$

- n steps in $(0, t)$ with $\alpha = \frac{t}{n}$:

$$\begin{aligned} M_L^n &= [(1-\alpha)I + \alpha A D^{-1}]^n \\ &= [I - \alpha(I - AD^{-1})]^n \\ &= [I - \alpha(D - A)D^{-1}]^n \\ &= [I - \frac{t}{N} L D^{-1}]^n \\ &\rightarrow e^{-tLD^{-1}} \\ &= e^{-tL_r} \end{aligned}$$

Heat Content Estimation

- Due to the diagonalizability of L_r

$$L_r = \Phi_r \Lambda_r \Phi_r^{-1}$$

$$\Rightarrow e^{-tL_r} = \Phi_r e^{-t\Lambda_r} \Phi_r^{-1}$$

$$= \sum_i e^{-\lambda_{ri} t} \varphi_{ri} \psi_{ri}^T$$

$$e^{-tL_n} = \sum_i e^{-\lambda_{ni} t} \varphi_{ni} \varphi_{ni}^T$$

$$L_r = LD^{-1} = D^{\frac{1}{2}} D^{-\frac{1}{2}} L D^{-\frac{1}{2}} D^{-\frac{1}{2}} = D^{\frac{1}{2}} L_n D^{-\frac{1}{2}}$$

$$\Rightarrow \ \lambda_{ri} = \lambda_{ni}, \quad \varphi_{ri} = D^{-\frac{1}{2}} \varphi_{ni}$$

$$\Rightarrow h_{nt}(u,v) = h_{rt}(u,v) (\frac{d_v}{d_u})^{\frac{1}{2}}$$

Connections to system identification

$$\frac{\partial h_t(u,v)}{\partial t} = -Lh_t(u,v) + Bx(t)$$

$$Q(t) = \sum \sum h_t(u,v)$$

$$\dot{x} = Ax + Bu$$

$$y = Cx$$

- Different plants, similar models;

- Nature prefers linear systems?

  ---- easy to code in DNA…☺

Thank you