Finite-time stability and input-to-state stability of stochastic nonlinear systems

KANG Yu[1], ZHAO Ping[2,*]

1. Department of Automation, University of Science and Technology of China, Hefei 230026, Anhui, P. R. China
E-mail: kangduyu@ustc.edu.cn

2. School of Electrical Engineering, University of Jinan, Jinan 250022, P. R. China
E-mail: cse_zhaop@ujn.edu.cn

Abstract: In this paper, finite-time globally asymptotical stability in probability (FTGASiP) and finite-time stochastic input-to-state stability (FTSISS) for stochastic nonlinear (SNL) systems are investigated. For the study of FTGASiP, there is a generalized $\mathcal{KL}$ ($\mathcal{GKL}$) function in the definition which we considered. Correspondingly, based on this definition, some sufficient conditions are provided for SNL systems. Further more, the definition of FTSISS is introduced and corresponding criterion is presented for SNL systems. To prove the results of the above, some lemmas about $\mathcal{GKL}$ functions and their properties are provided. Finally, some simulation examples are given to demonstrate the effectiveness of our results.

Key Words: Stochastic nonlinear system, Finite-time input-to-state stability, Finite-time globally asymptotically stable in probability, FTSISS-Lyapunov function

1 Introduction

Since the performance of a real control system is affected more or less by uncertainties such as unmodelled dynamics, parameter perturbations, exogenous disturbances, measurement errors etc., the research on robustness of control systems do always have a vital status in the development of control theory and technology. Aiming at robustness analysis of nonlinear control systems, a new method from the point of view of input-to-state stability (ISS), input-to-output stability (IOS) and integral input-to-state stability (iISS) are developed and a series of fundamental results centralizing on the theory of ISS-, IOS-Lyapunov functions are obtained by many scholars, Sontag, Wang and Lin, etc [13, 15, 16, 17, 18, 9, 10, 11, 1, 8]. ISS focuses on the design of smooth controllers to tackle stabilization of various classes of nonlinear systems or their robust and adaptive control in the presence of various uncertainties arising from control engineering applications.

On the other hand, non-smooth (including discontinuous and continuous but not Lipschitz continuous) control approaches have drawn increasing attention in nonlinear control system design. One of the main benefits of the non-smooth finite-time control strategy is that it can force a control system to reach a desirable target in finite time. This approach was first studied in the literature of optimal control. In recent years, finite-time ISS and its's applications to finite-time controller design have been considered in many literatures [7, 5, 6, 19]. But, for stochastic systems, these problems have not been studied.

From the definition of finite-time input-to-state stability (FTISS), we can find that, if the input $u = 0$, an FTISS system is necessarily finite-time GAS. So, the study of finite-time GAS is very helpful to the study of FTISS. In [2, 3, 20], the definition of finite-time globally asymptotical stability in probability (FTGASiP) is provided and some criteria have been given. But, in [2, 3, 20], the definition of FTGASiP is defined in the form of stability in probability plus attractivity in probability. But, for study the FTSISS of stochastic systems, a definition of FTGASiP in the form of $\mathcal{GKL}$ function ($|x| \le \beta(|x_0|, t)$, where x is the system state and x_0 is the initial value and β is a $\mathcal{GKL}$ function) is needed. The definition of this form is much more elegant and easier to work with.

In this paper, the FTGASiP and FTSISS will be considered for SNL systems, and the definition of FTGASiP and FTSISS are both in the form of $\mathcal{GKL}$ function. Firstly, we provide some lemmas to make the proof of our main results easier. Then, the criteria on FTGASiP and FTSISS are provided. To illustrate the effectiveness of our main results, some simulation examples will be given at the last.

The remainder of this paper is organized as following: Section 2 provides some notations and introduces the definitions of FTGASiP, FTSISS and FTSISS-Lyapunov function. Section 3 investigates the FTGASiP and FTSISS property of SNL. In section 4, some simulation examples are provided to illustrate the results. Section 5 includes some concluding remarks.

2 Notations and preliminary results

Throughout this paper, $\mathbb{R}_+$ denotes the set of all nonnegative real numbers; $\mathbb{R}^n$ and $\mathbb{R}^{n\times m}$ denote, respectively, n-dimensional real space and $n \times m$ dimensional real matrix space. For vector $x \in \mathbb{R}^n$, $|x|$ denotes the Euclidean norm $|x| = (\sum_{i=1}^n x_i^2)^{1/2}$. All the vectors are column vectors unless otherwise specified. The transpose of vectors and matrices are denoted by superscript T. $\mathcal{C}([-\mu, 0]; \mathbb{R}^n)$ denotes continuous $\mathbb{R}^n$-valued function space defined on $[-\mu, 0]$; $\mathcal{C}^i$ denotes all the ith continuous differential functions; $\mathcal{C}^{i,k}$ denotes all the functions with ith continuously differentiable first component and kth continuously differentiable second component. $E(x)$ denotes the expectation of stochastic variable x. The composition of two functions $\varphi : A \to B$ and

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*Corresponding author. Tel.: +86 15806620986; fax: +86 0531 82765980.

$\psi : B \to C$ is denoted by $\psi \circ \varphi : A \to C$.

A function $\varphi(u)$ is said to belong to the class $\mathcal{K}$ if $\varphi \in \mathcal{C}(\mathbb{R}_+, \mathbb{R}_+)$, $\varphi(0) = 0$ and $\varphi(u)$ is strictly increasing in u. $\mathcal{K}_\infty$ is the subset of $\mathcal{K}$ functions that are unbounded. A function $\beta : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ is of class $\mathcal{KL}$, if $\beta(\cdot, t)$ is of class $\mathcal{K}$ in the first argument for each fixed $t \geq 0$ and $\beta(s, t)$ decreases to 0 as $t \to +\infty$ for each fixed $s \geq 0$.

A function $h : \mathbb{R}_+ \to \mathbb{R}_+$ is said to belong to the class generalized $\mathcal{K}$ ($\mathcal{GK}$) if it is continuous with $h(0) = 0$, and satisfies

$$\begin{cases} h(r_1) > h(r_2), & \text{if } h(r_1) \neq 0; \\ h(r_1) = h(r_2) = 0, & \text{if } h(r_1) = 0, \end{cases} \quad \forall r_1 > r_2. \tag{1}$$

Note that a class $\mathcal{GK}$ function is a (conventional) class $\mathcal{K}$ function, which is defined as a continuous and strictly increasing function with $h(0) = 0$ because a strictly increasing function satisfies (1). A function $\beta : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ is of class generalized $\mathcal{KL}$ function ($\mathcal{GKL}$ function) if, for each fixed $t \geq 0$, the function $\beta(s, t)$ is a generalized $\mathcal{K}$-function, and for each fixed $s \geq 0$ it decreases to zero as $t \to T$ for some $T \leq \infty$.

Consider the following n-dimensional SNL system

$$dx = f(t, x, u)dt + g(t, x, u)dw, \ t \geq t_0, \tag{2}$$

where $x \in \mathbb{R}^n$ and $u \in \mathcal{L}_\infty^m$ are system state and input, respectively; $\mathcal{L}_\infty^m$ denotes the set of all the measurable and locally essentially bounded input $u \in \mathbb{R}^m$ on $[t_0, \infty)$ with norm

$$\|u\| = \sup_{t \geq t_0} \inf_{\mathcal{A} \subset \Omega, P(\mathcal{A})=0} \sup\{|u(t, \omega))| : \omega \in \Omega \backslash \mathcal{A}\}. \tag{3}$$

$w(t)$ is an r-dimensional Brownian motion defined on the complete probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq t_0}, P)$, with Ω being a sample space, $\mathcal{F}$ being a σ-field, $\{\mathcal{F}_t\}_{t \geq t_0}$ being a filtration and P being a probability measure. $f : [t_0, \infty) \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$, $g : [t_0, \infty) \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^r$ are continuous and satisfies $f(\cdot, 0, 0) = g(\cdot, 0, 0) \equiv 0$. Moreover, system (2) is assumed to has a pathwise unique strong solution[12], denoted by $x(t, t_0, x_0), t_0 \leq t < +\infty$, for any given $x_0 \in \mathbb{R}^n$.

For convenience, we denote the the system (2) with input $u = 0$ as follows

$$dx = f(t, x)dt + g(t, x)dw, \ t \geq t_0, \tag{4}$$

and introduce some corresponding definitions on FTGASiP and FTSISS.

Definition 2.1 *(Stochastic Settling Time Function). For system (2), define* $T_0(t_0, x_0, w) = \inf\{T \geq 0 : x(t, t_0, x_0) = 0, \forall t \geq t_0 + T\}$, *which is called the stochastic settling time function. Especially,* $T_0(t_0, x_0, w) =: +\infty$ *if* $x(t, t_0, x_0) \neq 0, \forall t \geq t_0$.

Definition 2.2 *The equilibrium $x = 0$ of the system (4) is finite-time globally asymptotically stable in probability (FT-GASiP) if for any $\varepsilon > 0$ there exists a class $\mathcal{GKL}$ function $\beta(\cdot, \cdot)$ such that*

$$P\{|x(t)| < \beta(|x_0|, t - t_0)\} \geq 1 - \varepsilon, \quad \forall t \geq t_0, \forall x_0 \in \mathbb{R}^n \backslash \{0\}.$$

Remark 2.1 The deterministic definition of GAS [13] is of course equivalent to the usual one (stability plus attractivity) and is much more elegant and easier to work with. But, for stochastic case, due to the dependence of events, they are not equivalent any more. In [4], the definition of GASiP was given in the form of stability plus attractivity. Here, using the $\mathcal{GKL}$ function, we give the definition of FTGASiP. The corresponding criterion will also be given in the following.

Definition 2.3 *System* (2) *is said to be finite-time stochastic input-to-state stable (FTSISS), if for any $\varepsilon > 0$, there exist functions $\beta \in \mathcal{GKL}$ and $\gamma \in \mathcal{K}$ such that*

$$P\{|x(t)| < \beta(|x_0|, t - t_0) + \gamma(\|u\|)\} \geq 1 - \varepsilon, \quad \forall \ t \geq t_0, \ \forall \ x_0 \in \mathbb{R}^n \setminus \{0\}.$$

Remark 2.2 *Similar to the explanation of the relation of ISS and GAS in [14], FTSISS says basically that for bounded initial state and control, a trajectory that bounded in probability results, and further (since β decays to zero at finite stochastic settling time) that the state is bounded in probability by a function of the control alone after the finite stochastic settling time (and this bound is small if the control is small). This is much stronger than that asking GASiP plus "bounded-point bounded-state" stability.*

Definition 2.4 *For system (2), a function $V(t, x) \in \mathcal{C}^{1,2}([t_0, \infty) \times \mathbb{R}^n; \mathbb{R}_+)$ is called an* FTSISS-Lyapunov *function, if there exist functions $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$, $\alpha, \chi \in \mathcal{K}$ such that, $\alpha(V) \sim V(t, x)^a$ for some positive constant $a < 1$, for all $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$ and $t \geq t_0$,*

$$\alpha_1(|x|) \leq V(t, x) \leq \alpha_2(|x|), \tag{5}$$

$$\begin{aligned} &|x| \geq \chi(\|u\|) \Rightarrow \\ &\mathcal{L}V(t, x) = \tfrac{\partial V(t,x)}{\partial t} + \tfrac{\partial V(t,x)}{\partial x} f(t, x, v) \\ &+ \tfrac{1}{2} trace\{g^T(t, x) \tfrac{\partial^2 V(t,x)}{\partial x^2} g(t, x)\} \leq -\alpha(V), \end{aligned} \tag{6}$$

where $\mathcal{L}$ is infinitesimal generator. For short, they will be denoted as $(V; \alpha_1, \alpha_2, \alpha, \chi)$.

3 Main results

In this section, the criteria on FTGASiP and FTSISS will be given. Firstly, some useful lemmas will be provided as follows.

Lemma 3.1 *Assume that $\alpha(\cdot) : \mathbb{R} \to \mathbb{R}$ and $\beta(\cdot, \cdot) : [t_0, +\infty) \times \mathbb{R}^n \to \mathbb{R}$ are two smooth functions, and $x(t)$ is the solution of system (2), then the following equation holds.*

$$\mathcal{L}[\alpha(\beta(t, x))] = \frac{d\alpha}{d\beta} \mathcal{L}[\beta(t, x)] + \frac{1}{2} \frac{d^2\alpha}{d\beta^2} Tr\left[\left(\frac{\partial \beta}{\partial x} g\right)^T \left(\frac{\partial \beta}{\partial x} g\right)\right]$$

Proof. Using the definition of the infinitesimal generator in (6),

$$\begin{aligned}
&\mathcal{L}[\alpha(\beta(t,x))]\\
=\;& \frac{d\alpha}{d\beta}\frac{\partial\beta}{\partial t}+\frac{d\alpha}{d\beta}\frac{\partial\beta}{\partial x}f(t,x)\\
&+\frac{1}{2}Tr\left[g^T\left(\frac{d^2\alpha}{d\beta^2}(\frac{\partial\beta}{\partial x})^T\frac{\partial\beta}{\partial x}+\frac{d\alpha}{d\beta}\frac{\partial^2\beta}{\partial x^2}\right)g\right]\\
=\;& \frac{d\alpha}{d\beta}\left[\frac{\partial\beta}{\partial t}+\frac{\partial\beta}{\partial x}f(t,x)+\frac{1}{2}Tr\left[g^T\left(\frac{\partial^2\beta}{\partial x^2}\right)g\right]\right]\\
&+\frac{1}{2}\frac{d^2\alpha}{d\beta^2}Tr\left[\left(\frac{\partial\beta}{\partial x}g\right)^T\left(\frac{\partial\beta}{\partial x}g\right)\right]\\
=\;& \frac{d\alpha}{d\beta}\mathcal{L}[\beta(t,x)]+\frac{1}{2}\frac{d^2\alpha}{d\beta^2}Tr\left[\left(\frac{\partial\beta}{\partial x}g\right)^T\left(\frac{\partial\beta}{\partial x}g\right)\right].
\end{aligned}$$

Remark 3.1 *Lemma 3.1 is a natural generalization of Lemma 2 in [2, 3].*

Lemma 3.2 *Let $\eta(r)=\int_0^r\frac{1}{h(v)}dv$, $r\in[0,+\infty)$, with $h\in\mathcal{K}$ and such that $\eta(r)<+\infty$, and define*

$$\beta(r,s)=\begin{cases}0, & r=0,\ s\geq 0;\\ 0, & r\neq 0,\ s\geq\eta(r);\\ \eta(r)-s, & r\neq 0,\ s<\eta(r).\end{cases}\tag{7}$$

Then, the function $\beta(r,s)$ is of class $\mathcal{GKL}$.

Proof. Let

$$0<a=\lim_{r\to\infty}\eta(r)<+\infty.$$

From the definition of η and $h\in\mathcal{K}$, we can get that $\eta(0)=0$, $\eta\in\mathcal{K}$ and η^{-1} can be defined, where η^{-1} denotes the inverse of η. The range of η, i.e. the definition domain of η^{-1} is $[0,a)$. Because η is continuous, the function $\beta(r,s)$ is continuous. Further, for fixed $s\in[0,+\infty)$, if $s\geq a$, then, it's obviously that $\beta(\cdot,s)\equiv 0\in\mathcal{GK}$; if $s<a$, we can find that

$$\beta(r,s)=\begin{cases}0, & \text{when } r\leq\eta^{-1}(s);\\ \eta(r)-s, & \text{when } r>\eta^{-1}(s).\end{cases}$$

So, $\beta(0,s)=0$, and for any $r_1>r_2$,

$$\begin{cases}\beta(r_1,s)>\beta(r_2,s), & \text{if } \beta(r_1,s)\neq 0;\\ \beta(r_1,s)=\beta(r_2,s)=0, & \text{if } \beta(r_1,s)=0.\end{cases}$$

So, for every fixed s, $\beta(\cdot,s)$ satisfies (1) and is a $\mathcal{GK}$ function.

On the other hand, for every fixed r, $\beta(r,\cdot)$ decreases to zero as $s\to\eta(r)$ ($\eta(r)<+\infty$). So, $\beta\in\mathcal{GKL}$. This completes the proof.

Lemma 3.3 *Let α_1 and α_2 be class $\mathcal{K}$ functions on $[0,a),(a\leq+\infty)$, and β be a class $\mathcal{GKL}$ function. Then $\sigma(r,s)=\alpha_1(\beta(\alpha_2(r),s))$ belongs to class $\mathcal{GKL}$.*

Remark 3.2 *Lemma 3.3 is a generalization to the last conclusion of Lemma 3.2 in (Khalil, 1996). From the monotonicity of the functions in Lemma 3.3, the proof of Lemma 3.3 is easy. Here, we omit it.*

Now, based on the above lemmas on the infinitesimal generator, $\mathcal{GKL}$ function and its properties, the criteria on FTGASiP and FTSISS will be provided.

Theorem 3.1 *Consider the system (4) and suppose the pathwise uniqueness be satisfied, and there exists a $\mathcal{C}^{1\times 2}$ function $V:[t_0,+\infty)\times\mathbb{R}^n\to\mathbb{R}_+$, class $\mathcal{K}_\infty$ functions α_1, and a continuous differentiable $h:\mathbb{R}_+\to\mathbb{R}_+$ such that for all $x\in\mathbb{R}^n$, $t\geq t_0$,*

$$\text{(i)}\quad V(t,x)\geq\alpha_1(|x|)\tag{8}$$

$$\text{(ii)}\quad \mathcal{L}V(t,x)\leq -h(V(t,x)),\tag{9}$$

$$\text{(iii)}\quad \int_0^\epsilon\frac{1}{h(v)}dv<+\infty,\forall\,\epsilon\in[0,+\infty),$$

$$\text{(iv)}\quad h'(v)>0,\ \forall\,v>0,$$

then the origin of system (4) is FTGASiP, and the settling time function $T_0(t_0,x_0,\omega)$ satisfies $E[T_0(t_0,x_0,\omega)]\leq\int_0^{V_0}\frac{1}{h(v)}dv$, where $V_0=V(t_0,x_0)$, which implies $T_0(t_0,x_0,\omega)<+\infty$ a.s.

Proof. Condition (iii) implies that there exists $\int_0^V\frac{1}{r(v)}dv$. For convenience, we define a function $\eta(V)=\int_0^V\frac{1}{h(v)}dv$, $V\in[0,\infty)$. Applying Itô formula along with system (4), for all $t\geq t_0$

$$\begin{aligned}\eta(V(t,x(t)))\;&=\;\eta(V(t_0,x_0))+\int_{t_0}^t\mathcal{L}\eta(V(s,x(s)))ds\\ &\quad+\int_{t_0}^t\frac{d\eta}{dV}\frac{\partial V}{\partial x}g(x)dw.\end{aligned}\tag{10}$$

If t is replaced by $t_r=\min\{t,\tau_r\}$ in the above, where $\tau_r=\inf\{s\geq 0:|x(s)|\geq r\}$, then the stochastic integral(second integral) in (10) defines a martingale (with r fixed and t varying), not just a local martingale. Thus, on taking expectations in (10) with t_r in place of t, we obtain

$$E[\eta(V(t_r,x(t_r)))]=\eta(V(x_0))+E[\int_{t_0}^{t_r}\mathcal{L}\eta(V(s,x(s)))ds]$$

On letting $r\to\infty$ and using Fatou's lemma on the left and monotone convergence on the right, we obtain

$$E[\eta(V(t,x(t)))]=\eta(V(t_0,x_0))+E[\int_{t_0}^{t}\mathcal{L}\eta(V(s,x(s)))ds]$$

From condition (ii) and (iv), we have that, when $h(V)\neq 0$, i.e. $V\neq 0$,

$$\mathcal{L}\eta(V(t,x(t)))\leq -1\tag{11}$$

When $V=0$, due to the positive definiteness of V, and condition (ii), we have $V\equiv 0$. Let

$$\tilde\beta(r,s)=\begin{cases}0, & r=0,\ s\geq 0;\\ 0, & r\neq 0,\ s\geq\eta(r);\\ \eta(r)-s, & r\neq 0,\ s<\eta(r).\end{cases}$$

Then

$$E[\eta(V(t,x(t)))]\leq\tilde\beta(V_0,t-t_0),\quad\forall\,t\geq t_0,$$

where $V_0 = V(t_0, x_0)$ and $\tilde{\beta} \in \mathcal{GKL}$ can be known from Lemma 3.2. For any $\varepsilon \in (0, 1)$, take $\bar{\beta} = \frac{\tilde{\beta}}{\varepsilon}$. The function $\bar{\beta}$ is the composition of $\alpha(s) = \frac{1}{\varepsilon} s \in \mathcal{K}$ and $\widetilde{\beta}$. From Lemma 3.3, $\bar{\beta} \in \mathcal{GKL}$. Using the Chebyshev's inequality and the above inequality, we have for any $t \in [t_0, \infty)$,

$$P\{\eta(V(t, x)) \geq \bar{\beta}(V_0, t - t_0)\} \leq \frac{E[\eta(V(t, x))]}{\bar{\beta}(V_0, t - t_0)} < \varepsilon.$$

Define $\beta(r, s) = \alpha_1^{-1} \circ \eta^{-1} \circ \bar{\beta}(\alpha_2(r), s)$. By (8),

$$P\{|x(t)| < \beta(|x_0|, t - t_0)\} \geq 1 - \varepsilon, \ \forall\, t \in [t_0, \infty), \quad (12)$$

where $\beta(r, s) \in \mathcal{GKL}$ can be known from Lemma 3.3. So, system (4) is FTGASiP.

Next, we prove that $E[T_0(t_0, x_0, \omega)] < +\infty$. From (11), we have

$$\begin{aligned} &E[\eta(V(t_0 + T_0(t_0, x_0, \omega), x(t_0 + T_0(t_0, x_0, \omega))))] \\ &\quad -E[\eta(V_0)] \leq -(E[t_0 + T_0(t_0, x_0, \omega)] - t_0). \end{aligned}$$

From the definition of $T_0(t_0, x_0, \omega)$ and condition (iii), we get

$$E[T_0(t_0, x_0, \omega)] \leq E[\eta(V_0)] = \eta(V_0) < +\infty,$$

which obviously implies $T_0(t_0, x_0, \omega) < +\infty$ a.s. This completes the proof.

Remark 3.3 *Theorem 3.1 is similar to the Theorem 1 in [2, 3]. Both of them are on the criteria of finite-time stability of SNL systems. But there are two obvious different points: 1) the system (4) which we considered is non autonomous, while the system (1) in [2, 3] is autonomous; 2) there is a $\mathcal{GKL}$ function in the proof of our Theorem 3.1 and corresponding Definition 2.3, which means that the FTGASiP problem is considered quantitatively; while, in [2, 3], the same problem is considered qualitatively.*

Remark 3.4 *In Theorem 3.1, $V(t, x) \geq \alpha_1(|x|)$ means that the the Lyapunov function $V(t, x)$ is radially unbounded, which makes the origin of system (4) is finite-time "globally" asymptotically stable in probability (FTGASiP). Moreover, if there exists a $\mathcal{K}_\infty$ function α_2 such that $V(t, x) \leq \alpha_2(|x|)$, which means that V is decrescent. Then, the origin of system (4) is finite-time globally "uniformly" asymptotically stable in probability (FTGUASiP). The stochastic settling time $T_0(t_0, x_0, \omega)$ uniformly satisfies $E[T_0(t_0, x_0, \omega)] \leq \eta(\alpha_2(|x_0|)) < +\infty$ on t_0.*

Corollary 3.1 *Consider the system (4), if the conditions* (i) *and* (ii) *in Theorem 3.1 hold and the function h in (9) equals to $k(V(t, x))^\rho$, where $k > 0$ and $0 < \rho < 1$ are real numbers, i.e.*

$$\text{(ii}'\text{)}\ \mathcal{L}V(t, x) \leq -k(V(t, x))^\rho. \quad (13)$$

Then, the origin of system (4) is FTGASiP, and $E[T_0(t_0, x_0, \omega)] \leq \frac{(V_0)^{1-\rho}}{k(1-\rho)}$, which implies $T_0(t_0, x_0, \omega) < +\infty$ a.s.

Proof. From the inequality (13), we let the function h in Theorem 3.1 is defined as $h((v) = kv^\rho$, where $k > 0$ and $0 < \rho < 1$ are real numbers. It's obvious that the function h is continuous differentiable. For any $\epsilon \in [0, +\infty)$,

$$\int_0^\epsilon \frac{1}{kv^\rho} dv = \frac{\epsilon^{1-\rho}}{k(1 - \rho)} < +\infty.$$

For any $v > 0$, $h'(v) = k\rho v^{1-\rho} > 0$. So, the conditions (iii) and (iv) in Theorem 3.1 is satisfied. From Theorem 3.1, the origin system (4) is FTGASiP. Moreover, $E[T_0(t_0, x_0, \omega)] \leq \eta(V_0) \leq \frac{(V_0)^{1-\rho}}{k(1-\rho)}$, which implies $T_0(t_0, x_0, \omega) < +\infty$ a.s.

The following theorem is on the criterion of FTSISS of SNL systems.

Theorem 3.2 *The system (2) is FTSISS if there exists a FTSISS-Lyapunov function $(V; \alpha_1, \alpha_2, \alpha, \chi)$.*

Proof. Let $\tau \in [t_0, \infty)$ denote a time at which the trajectory enters the set $\mathcal{B} = \{x \in \mathbb{R}^n : |x| < \chi(\|u\|)\}$ for the first time. Let us complete the proof by considering the following two cases: $x_0 \in \mathcal{B}^c$ and $x_0 \in \mathcal{B}\backslash\{0\}$, respectively.

Case 1. $x_0 \in \mathcal{B}^c$. In this case, for any $t \in [t_0, \tau]$, $|x(t)| > \chi(\|u\|)$. From (6), we have

$$\mathcal{L}V(t, x) \leq -\alpha(V).$$

According to Theorem 3.1 and Corollary 3.1, for any $\varepsilon' > 0$, there exists a class $\mathcal{GKL}$ function β, such that

$$\begin{aligned} P\{|x(t)| < \beta(|x_0|, t - t_0)\} \geq 1 - \varepsilon', \quad (14) \\ \forall\, t \in [t_0, \tau], \forall\, x_0 \in \mathbb{R}^n\backslash\{0\}, \end{aligned}$$

and the settling time function $T_0(t_0, x_0, \omega)$ satisfies $E[T_0(t_0, x_0, \omega)] \leq \int_0^{\alpha_2(|x_0|)} \frac{1}{\alpha(v)} dv$, which implies $T_0(t_0, x_0, \omega) < +\infty$ a.s. Let us now turn our attention to the interval $t \in (\tau, \infty)$. Since

$$\frac{d}{dt}\{E[V(t, x)]\} = E[\mathcal{L}V(t, x)],$$

which is negative for $x(t)$ outside the set $\{x \in \mathbb{R}^n : |x| \leq \chi(\|u\|)\} \subseteq \{x \in \mathbb{R}^n : V(x, t) \leq \alpha_2(\chi(\|u\|))\}$, we have

$$E[V(t, x)] \leq \alpha_2(\chi(\|u\|)), \quad \forall\, t \geq \tau.$$

By Chebyshev's inequality, it follows that

$$\begin{aligned} &P\{\sup_{t \in [\tau, \infty)} V(t, x) \geq \delta(\alpha_2(\chi(\|u\|)))\} \\ \leq\ &\frac{\alpha_2(\chi(\|u\|))}{\delta(\alpha_2(\chi(\|u\|)))} \leq \varepsilon'', \end{aligned}$$

where ε'' can be made arbitrarily small by an appropriate choice of $\delta \in \mathcal{K}_\infty$. Hence, for all $\varepsilon'' > 0$, there exists $\gamma = \alpha_1^{-1} \circ \delta \circ \alpha_2 \circ \chi$ such that

$$P\{|x(t)| < \gamma(\|u\|)\} \geq 1 - \varepsilon'', \quad \forall\, t \geq \tau. \quad (15)$$

Thus, we get

$$\begin{aligned} &P\{|x(t)| < \beta(|x_0|, t - t_0) + \gamma(\|u\|)\} \\ \geq\ &\max\{1 - \varepsilon', 1 - \varepsilon''\} \\ =\ &1 - \min\{\varepsilon', \varepsilon''\} \\ =\ &1 - \varepsilon, \quad \forall\, t \geq t_0, \forall\, x_0 \in \mathcal{B}^c. \quad (16) \end{aligned}$$

Case 2. $x_0 \in \mathcal{B}\backslash\{0\}$. In this case $\tau = t_0$ a.s.

When $t > t_0$, $P\{t \in (\tau, \infty)\} = P\{t \in (t_0, \infty)\} = 1$. Following the proof of **Case 1.**, we know that (15) still holds, and then,

$$\begin{aligned} & P\{|x(t)| < \beta(|x_0|, t - t_0) + \gamma(\|u\|)\} \\ \geq \quad & P\{|x(t)| < \gamma(\|u\|)\} \geq 1 - \varepsilon''. \end{aligned} \qquad (17)$$

When $t = t_0$, by the definition of the set $\mathcal{B}$ and the definition of γ, we obtain

$$\begin{aligned} & P\{|x(t_0)| < \beta(|x_0|, 0) + \gamma(\|u\|)\} \\ \geq \quad & P\{|x(t_0)| < \gamma(\|u\|)\} = 1, \end{aligned}$$

which implies

$$P\{|x(t_0)| < \beta(|x_0|, 0) + \gamma(\|u\|)\} = 1. \qquad (18)$$

Thus, by (17) and (18) we have

$$\begin{aligned} & P\{|x(t)| < \beta(|x_0|, t - t_0) + \gamma(\|u\|)\} \geq 1 - \varepsilon, \quad (19) \\ & \qquad \forall\, t \geq t_0,\ x_0 \in \mathcal{B}\backslash\{0\}. \end{aligned}$$

In conclusion, by (16) and (19) we have

$$\begin{aligned} & P\{|x(t)| < \beta(|x_0|, t - t_0) + \gamma(\|u\|)\} \geq 1 - \varepsilon, \\ & \qquad \forall\, t \geq t_0,\ x_0 \in \mathbb{R}^n\backslash\{0\}. \end{aligned}$$

So, system (2) is FTSISS.

4 Simulation examples

In this section, two examples are presented to demonstrate the effectiveness of our main results. For convenience, only some first-order SNL systems will be considered.

Example 4.1 *Consider the following first-order SNL non autonomous system*

$$dx = (-\frac{1}{2}x^2 - |x|^{\frac{1}{2}})sign(x)dt + \sin(t)|x|^{\frac{3}{2}}dw \qquad (20)$$

where $x \in \mathbb{R}$. Let $V(x) = \frac{1}{2}x^2$ and then

$$\begin{aligned} \mathcal{L}V(x) &= x(-\frac{1}{2}x^2 - |x|^{\frac{1}{2}})sign(x) + \frac{1}{2}\sin^2(t)|x|^3 \\ &\leq -2^{\frac{3}{4}}(V(x))^{\frac{3}{4}}. \end{aligned}$$

Based on Theorem 3.1 (or corollary 3.1), we can conclude that the origin $x = 0$ is FTGASiP. The simulation curves of $w(t)$ and $x(t)$ with $x_0 = \pm 1$ are shown in Figs. 1 and 2. Furthermore, we easily conclude that the stochastic setting time function $T_0(t_0, x_0, w)$ uniformly satisfies $ET_0(t_0, x_0, w) \leq \int_0^{\frac{1}{2}} \frac{1}{2^{\frac{3}{4}} v^{\frac{3}{4}}} dv = 2$. From Fig. 2, it can be seen that $x(t)$ with $x_0 = \pm 1$ indeed converges to zero at about 2s, which accords with Theorem 3.1 (or Corollary 3.1).

Example 4.2 *Consider the following first-order SNL system*

$$dx = (-x - x^{\frac{1}{3}} + u)dt + xdw \qquad (21)$$

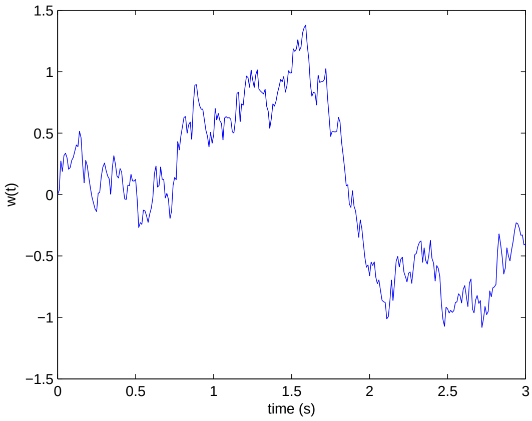

Fig. 1: Response curve of $w(t)$ in system (20)

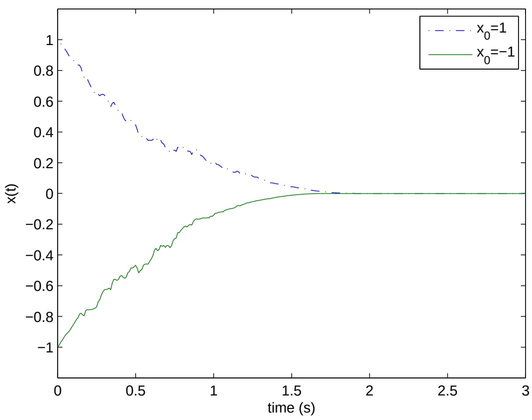

Fig. 2: Response curve of $x(t)$ in system (20)

where $x \in \mathbb{R}$ and $u \in \mathbb{R}$ are the system state and input, respectively. Let $V(x) = \frac{1}{2}x^2$ and then

$$\begin{aligned} |x| \geq |u|^{\frac{3}{2}} \quad \Rightarrow \quad \mathcal{L}V(x) &= x(-x - x^{\frac{1}{3}} + u) + \frac{1}{2}x^2 \\ &\leq -\frac{1}{2} \cdot 2^{\frac{2}{3}}(V(x))^{\frac{2}{3}}. \end{aligned}$$

Based on Theorem 3.2, system (21) is FTSISS. The simulation curves of $x(t)$ with $x_0 = \pm 0.5$ and input $u \equiv 0.5$ are shown in Fig. 3. From Fig. 3, it can be seen that, due to the effect of input u, $x(t)$ with $x_0 = \pm 0.5$ will not converge to zero. But, we can find that afte $ET_0(x_0, w) \leq \int_0^{\frac{1}{2}} \frac{1}{\frac{1}{2} 2^{\frac{2}{3}} v^{\frac{2}{3}}} dv \approx 1.9s$, the two solutions $x(t)$ with $x_0 = \pm 0.5$ will be equal all along and remain bounded, which means that the state is bounded by a function of the input alone. This is consistent with Definition 2.3 (or Remark 2). On the other hand, if the input $u \equiv 0$, we can get that

$$\mathcal{L}V(x) = x(-x - x^{\frac{1}{3}}) + \frac{1}{2}x^2 \leq -\frac{1}{2}x^2 - x^{\frac{4}{3}}.$$

Roughly speaking,

$$\mathcal{L}V(x) \leq -x^{\frac{4}{3}} = -2^{\frac{2}{3}}(V(x))^{\frac{2}{3}}.$$

Based on Theorem 3.1 (or corollary 3.1), we can conclude that the origin $x = 0$ is FTGASiP. The simulation curves of

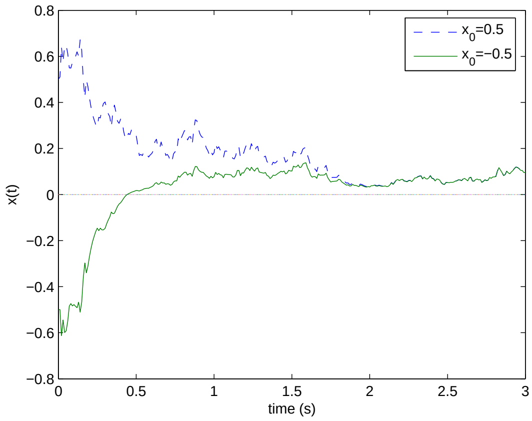

Fig. 3: Response curve of $x(t)$ in system (21) with $u \equiv 0.5$

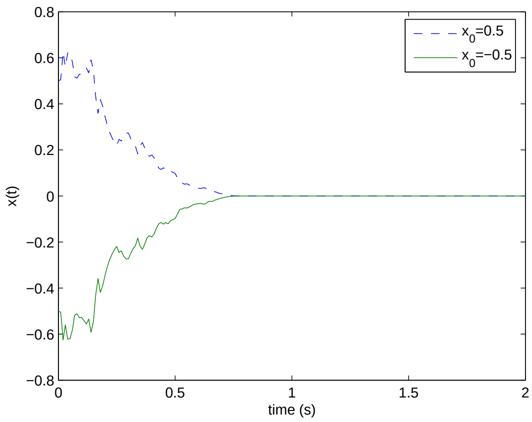

Fig. 4: Response curve of $x(t)$ in system (21) with $u \equiv 0$

$x(t)$ with $x_0 = \pm 0.5$ are shown in Fig. 4. From Fig. 4, it can be seen that $x(t)$ with $x_0 = \pm 0.5$ indeed converges to zero at finite time.

5 Conclusion

In this paper, the notions of FTGASiP and FTSISS are introduced for SNL systems. By the Lypunov-like function method, some corresponding criteria on FTGASiP and FTSISS are provided. In the proof of the criteria, to describe the finite-time asymptotic property of the solutions to the systems, a $\mathcal{GKL}$ function β is obtained. To illustrate the effectiveness of our criteria, some examples are also provided.

References

[1] D.E. Angeli, D. Sontag, Y. Wang, A Lyapunov characterization of integral input-to-state stability, IEEE Transactions on Automatic Control 45(6) (2000) 1082-1097.

[2] W.S. Chen, L.C. Jiao, Finite-time stability theorem of stochastic nonlinear systems, Automatica 46(12) (2010) 2105-2108.

[3] W.S. Chen, L.C. Jiao, Authors' reply to "Comments on 'Finite-time stability theorem of stochastic nonlinear systems [Automatica 46 (2010) 2105-2108]'", Automatica 47(7) (2011) 1544-1545.

[4] H. Deng, M. Kristić, R.J. Williams, Stabilization of stochastic nonlinear systems driven by noise of unknown covariance, IEEE Transactions on Automatic Control 46(8)(2001) 1237-1253.

[5] Y.G. Hong, Z.P. Jiang, G. Feng, Finite-time input-to-state stability and applications to finite-time control, Proceedings of the 17th World Congress The International Federation of Automatic Control (2008) 2466-2471 Seoul, Korea.

[6] Y.G. Hong, Z.P. Jiang, G. Feng, Finite-time input-to-state stability and applications to finite-time control design, SIAM J. Control Optim. 48(7)(2010) 4395-4418.

[7] Y.G. Hong, Z.P. Jiang, G. Feng, Finite-time input-to-state stability and related Lyapunov analysis, Proceedings of the 26th Chinese Control Conference (2007) 26-31 Zhangjiajie, Hunan, China.

[8] D. Liberzon, A.S. Morse, E.D. Sontag, Output-input stability and minimum-phase nonlinear systems, IEEE Transactions on Automatic Control 47(3)(2002) 422-436.

[9] Y.D. Lin, E.D. Sontag, Y. Wang, Input to state stabilizability for parameterized families of systems, Int. J. Robust Nonlinear Control 5(3)(1995) 187-205.

[10] Y.D. Lin, E.D. Sontag, Y. Wang, A smooth converse Lyapunov theorem for robust stability, SIAM J. Control Optim. 34(1996) 124-160.

[11] L. Praly, Y. Wang, Stabilization in spite of matched unmodeled dynamics and an equivalent definition of input-to-state stability, Mathematics of Control, Signals, and Systems 9(1)(1996) 1-33.

[12] R. Situ, Theory of stochastic diffferential equations with jumps and applications: Mathematical and analysis techniques with applications to engineering. USA, New York: Springer, 2005.

[13] E.D. Sontag, Smooth stabilization implies coprime factorization, IEEE Transactions on Automatic Control, 34(4)(1989) 435-443.

[14] E.D. Sontag, Further facts about input to state stabilization, IEEE Transactions on Automatic Control, 35(4)(1990), 473-476.

[15] E.D. Sontag, Y. Wang, On characterization of the input-to-state stability property, Syst Contr Lett. 24(5)(1995), 351-359.

[16] E.D. Sontag, Y. Wang, New characterizations of input to state stability, IEEE Transactions on Automatic Control 41(9)(1996), 1283-1294.

[17] E.D. Sontag, Y. Wang, A notion of input to output stability, In: Proc of European Control Conference (1997) Brussels.

[18] E.D. Sontag, Y. Wang, Lyapunov characterizations of input to output stability, SIAM J. Control Optim. 39(1)(2001) 226-249.

[19] X.L. Wang, Y.G. Hong, Z.P. Jiang, Finite-time input-to-state stability and optimization of switched systems, Proceedings of the 27th Chinese Control Conference (2008) 479-483 Kunming, Yunnan, China.

[20] J.L. Yin, S.Y. Khoo, Comments on "Finite-time stability theorem of stochastic nonlinear systems" [Automatica 46 (2010) 2105-2108], Automatica 47(7)(2011) 1542-1543.