Synthesis of Boolean Networks via Semi-tensor Product

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Outline

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I. An Introduction to Boolean Network

☞ Historical Review — Boolean Network

- McCulloch and Pitts (1943): “the **brain could be modeled as a network of logical operations** such as and or not and so forth.”
- Jacob and Monod (Nobel Prize winners) (1961-1963): “Any cell contains a number of **‘regulatory’ genes** that act as switches and **can turn one another on and off**. ...then you can have **genetic circuits**.” (M.M. Waldrop, *Complexity*, 1992)
- Kauffman (1969): “The **Boolean rules describing the activities of different genes** ...” (S. Kauffman, *At Home in the Universe*,1994)

☞ Boolean Control Network

- Ideker, et al (2001): "**Gene-regulatory networks** are defined by trans and cis logic. $\cdots$ Both of these types of regulatory networks **have input and output**." (*Annu. Rev. Genomics Hum. Genet.*, 2001)
- Akutsu, et al (2007): "**One of the major goals of systems biology is to develop a control theory** for complex biological systems." (*J. Theoretical Biology*, 2007)

☞ Some Other Applications

- Dynamic Games;
- Logic-based Control;
- Cryptography and Secure Community;
- Circuit Failure Detection, etc.

Boolean Network

☞ Network Graph

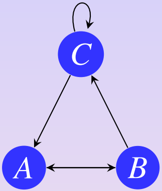


Figure 1: A Boolean network

☞ Network Dynamics

$$\begin{cases} A(t+1) = B(t) \wedge C(t) \\ B(t+1) = \neg A(t) \\ C(t+1) = B(t) \vee C(t) \end{cases} \tag{1}$$

Boolean Control Network

☞ Network Graph

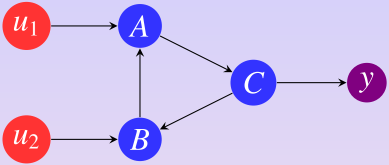


Figure 2: A Boolean control network

☞ Network Dynamics

Its logical equation is

$$\begin{cases} A(t+1) = B(t) \wedge u_1(t) \\ B(t+1) = C(t) \vee u_2(t) \\ C(t+1) = A(t) \end{cases} \qquad (2)$$
$$y(t) = \neg C(t)$$

☞ Dynamics of Boolean Network

$$\begin{cases} x_1(t+1) = f_1(x_1(t), \cdots, x_n(t)) \\ \vdots \\ x_n(t+1) = f_n(x_1(t), \cdots, x_n(t)), \quad x_i \in \mathcal{D}, \end{cases} \tag{3}$$

where

$$\mathcal{D} := \{0, 1\}.$$

☞ Dynamics of Boolean Control Network

$$\begin{cases} x_1(t+1) = f_1(x_1(t), \cdots, x_n(t), u_1(t), \cdots, u_m(t)) \\ \vdots \\ x_n(t+1) = f_n(x_1(t), \cdots, x_n(t), u_1(t), \cdots, u_m(t)), \end{cases} \tag{4}$$
$$y_j(t) = h_j(x(t)), \quad j = 1, \cdots, p,$$

where $x_i, u_k, y_j \in \mathcal{D}$.

Some Generalizations

☞ k-valued and Mix-valued Logical Network

- Boolean: $x_i \in \mathcal{D} = \{0, 1\}$;
- k-valued: $x_i \in \mathcal{D}_k = \{0, \frac{1}{k-1}, \cdots, 1\}$;
- mix-valued: $x_i \in \mathcal{D}_{k_i}$.

  (Example: For a game, player i has k_i strategies.)

☞ Probabilistic Boolean Network

$$f_i = \begin{cases} f_i^1, & P(f_i = f_i^1) = p_i^1; \\ \vdots & \\ f_i^{k_i}, & P(f_i = f_i^{k_i}) = p_i^{k_i}, \end{cases} \tag{5}$$

where

$$\sum_{j=1}^{k_i} p_i^j = 1, \quad i = 1, \cdots, n.$$

Outline

II.1 Algebraic Form of Logic

☞ Semi-tensor Product of Matrices

$$A \in \mathcal{M}_{m\times n}, \quad B \in \mathcal{M}_{p\times q}, \quad A \times B =?$$

Definition 2.1.1

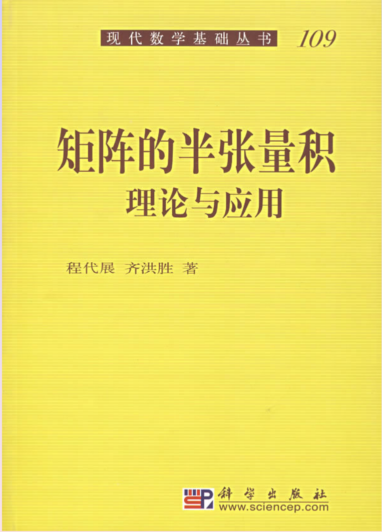
Let $t = \mathrm{lcm}\{n, p\}$. Then the **semi-tensor product** (STP) of A and B is defined as

$$A \ltimes B := \left(A \otimes I_{t/n}\right)\left(B \otimes I_{t/p}\right) \tag{6}$$

Remark 2.1.2

- It is a generalization of conventional matrix product (CMP);
- All the properties of the CMP remain true;
- Pseudo-commutativity.

☞ Reference Book



☞ Some Notations:

- δ_k^i: the i-th column of I_k;
- Δ_k: $\{\delta_k^1, \delta_k^2, \cdots, \delta_k^k\}$; $\Delta := \Delta_2$;
- $\mathcal{L}_{m\times n}$: the set of logical matrices. $A \in \mathcal{L}_{m\times n}$ means $A = [\delta_m^{i_1}\ \delta_m^{i_2}\ \cdots\ \delta_m^{i_n}]$. Briefly denote it as

$$A = \delta_m[i_1\ i_2\ \cdots\ i_n].$$

☞ Vector Form of Boolean Variables (Functions)

Setting Equivalence:

$$1 \sim \delta_2^1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad 0 \sim \delta_2^2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

then $x_i \in \Delta$ and

$$f : \mathcal{D}^n \to \mathcal{D} \quad \Rightarrow f : \Delta^n \to \Delta.$$

☞ Algebraic Form of Boolean (Control) Networks

Theorem 2.1.3 (Cheng & Qi, IEEE TAC, 55(10), 2010)

Let $F : \mathcal{D}^n \to \mathcal{D}^m$ be determined by

$$y_i = f_i(x_1, \cdots, x_n), \quad i = 1, \cdots, m. \tag{7}$$

Then in vector form we have

$$y_i = M_i \ltimes_{j=1}^{n} x_j := M_i x, \quad i = 1, \cdots, m, \tag{8}$$

where $M_i \in \mathcal{L}_{2\times 2^n}$. Moreover,

$$y := \ltimes_{k=1}^{m} y_k := M_F x, \tag{9}$$

where $M_F = M_1 * \cdots * M_m \in \mathcal{L}_{2^m \times 2^n}$. ($*$: Khatri-Rao Prod.)

☞ Algebraic Form of Boolean (Control) Networks

Theorem 2.1.4 (Cheng & Qi, IEEE TAC, 55(10), 2010)

1. There exists a unique $L \in \mathcal{L}_{2^n \times 2^n}$ such that (3) can be expressed as

$$x(t+1) = Lx(t), \tag{10}$$

where $x = \ltimes_{i=1}^{n} x_i$.

2. There exist unique $L \in \mathcal{L}_{2^n \times 2^{n+m}}$ and unique $H \in \mathcal{L}_{2^p \times 2^n}$, such that (4) can be expressed as

$$\begin{cases} x(t+1) = Lx(t)u(t) \\ y(t) = Hx(t), \end{cases} \tag{11}$$

where $u = \ltimes_{i=1}^{m} u_i$, $y = \ltimes_{i=1}^{p} y_i$.

Example

Example 2.1.4

- Consider Boolean network (1) for Fig. 1. We have

$$L = \delta_8[3\ 7\ 7\ 8\ 1\ 5\ 5\ 6].$$

- Consider Boolean control network (2) for Fig. 2. We have

$$\begin{aligned} L &= \delta_8[1\ 1\ 5\ 5\ 2\ 2\ 6\ 6\ 1\ 3\ 5\ 7\ 2\ 4\ 6\ 8 \\ &\qquad 5\ 5\ 5\ 5\ 6\ 6\ 6\ 6\ 5\ 7\ 5\ 7\ 6\ 8\ 6\ 8]; \\ H &= \delta_2[2\ 1\ 2\ 1\ 2\ 1\ 2\ 1]. \end{aligned}$$

Outline

II.2 State Space Approach

☞ State Space and Subspace

Definition 2.2.1

Consider Boolean network (3)

(1) State space:

$$\mathcal{X} = F_\ell(x_1, \cdots, x_n). \tag{12}$$

(2) Subspace: Let $y_1, \cdots, y_k \in \mathcal{X}$.

$$\mathcal{Y} = F_\ell(y_1, \cdots, y_k) \subset \mathcal{X}. \tag{13}$$

(3) Regular Subspace: Let $\{x_{i_1}, \cdots, x_{i_k}\} \subset \{x_1, \cdots, x_n\}$.

$$\mathcal{Z} = F_\ell(x_{i_1}, \cdots, x_{i_k}). \tag{14}$$

☞ Physical Meaning: Dual Space

In $\mathbb{R}^n$, let $\{x_1, x_2, \cdots, x_n\}$ be the coordinate frame.

To describe the state subspace generalized by $\{x_{i_1}, \cdots, x_{i_k}\}$, we may use the linear functions over this subspace as

$$V^* = \{c_1 x_{i_1} + \cdots + c_k x_{i_k} | c_1, \cdots, c_k \in \mathbb{R}\}.$$

☞ Coordinate Transformation

Definition 2.2.2

Let $\mathcal{X} = F_\ell(x_1, \cdots, x_n)$ be the state space of (3). Assume there exist $z_1, \cdots, z_n \in \mathcal{X}$, such that

$$\mathcal{X} = F_\ell(z_1, \cdots, z_n),$$

then the logical mapping $T : (x_1, \cdots, x_n) \mapsto (z_1, \cdots, z_n)$ is a **coordinate transformation**.

Proposition 2.2.3

A mapping $T : \mathcal{D}^n \to \mathcal{D}^n$ is a coordinate transformation, if and only if, T is bijective.

☞ Algebraic Form of Coordinate Transformation

Theorem 2.2.4 (Cheng & Qi, IEEE TNN, 21(4), 2010)

Assume $z_1, \cdots, z_n \in \mathcal{X}$, with

$$\begin{cases} z_1 = f_1(x_1, \cdots, x_n) \\ \vdots \\ z_n = f_n(x_1, \cdots, x_n). \end{cases} \tag{15}$$

Moreover, the algebraic form of (15) is

$$z = Tx, \tag{16}$$

where $z = \ltimes_{i=1}^n z_i$, and $x = \ltimes_{i=1}^n x_i$, $T \in \mathcal{L}_{2^n \times 2^n}$. Then $\pi : (x_1, \cdots, x_n) \mapsto (z_1, \cdots, z_n)$ is a coordinate transformation, if and only if, T is nonsingular.

☞ Regular Subspace

Definition 2.2.5

Let $\mathcal{Z}_0 = F_\ell(z_1, \cdots, z_k) \subset \mathcal{X}$. If there exist $\{z_{k+1}, \cdots, z_n\}$ such that

$$\mathcal{X} = F_\ell(z_1, \cdots, z_n),$$

then $\mathcal{Z}_0$ is a **regular subspace**.

Theorem 2.2.6 (Cheng & Qi, IEEE TNN, 21(4), 2010)

Let the algebraic form of $\mathcal{Z}_0$ be

$$z^0 = Gx,$$

where $G = [g_{i,j}] \in \mathcal{L}_{2^k \times 2^n}$. Then $\mathcal{Z}_0$ is a regular subspace, if and only if,

$$\sum_{j=1}^{2^n} g_{i,j} = 2^{n-k}, \quad i = 1, \cdots, 2^k. \tag{17}$$

Outline

III. Topological Structure of BN/BCN

☞ Fix Points and Cycles

Theorem 3.1 (Cheng & Qi, IEEE TAC, 55(10), 2010)

Let the algebraic form of a Boolean network be

$$x(t+1) = Lx(t). \tag{18}$$

Then the number of cycles of length s (denoted by N_s) are

$$\begin{cases} N_1 = \mathrm{tr}(L), \\ N_s = \dfrac{\mathrm{tr}(L^s) - \sum\limits_{t \in \mathcal{P}(s)} t N_t}{s}, \quad 2 \le s \le 2^n. \end{cases} \tag{19}$$

- There are similar formulas for BCN (**Zhao & Cheng, IEEE TAC, to appear.**)

☞ Rolling Gear Structure

Theorem 3.2 (Cheng, IEEE TNN, 20(3), 2009)

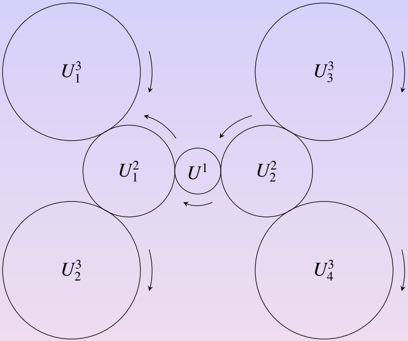
If the network has cascading form as

$$\begin{cases} z^1(t+1) = F^1(z^1(t)) \\ z^2(t+1) = F^2(z^1(t), z^2(t)), \end{cases} \tag{20}$$

then

$$C_Z = C_{Z^1} \circ C_{Z^2}. \tag{21}$$

☞ Rolling Gear Structure ($V_1 \subset V_2 \subset V_3 = \mathcal{X}$)



- RGS explains why "**tiny cycles decide vast order**"(Kaufman, *At Home in the Universe*)

Outline

IV.1 Controllability, Observability, Realization

☞ Controllability Matrix

Assume a Boolean control network has its algebraic form as

$$\begin{cases} x(t+1) = Lu(t)x(t) \\ y(t) = Hx(t). \end{cases} \tag{22}$$

Let

$$M = \sum_{i=1}^{2^m} \mathrm{Blk}_i(L).$$

Controllability Matrix

$$\mathcal{M}_{\mathcal{C}} := \sum_{s=1}^{2^{m+n}} M^{(s)} := (c_{ij}) \in \mathcal{B}_{2^n \times 2^n}. \tag{23}$$

☞ Controllability

Theorem 4.1.1 (Zhao & Cheng, SCL, 59(12), 2010)

Consider Boolean control network (22).

(i) $x^0 = \delta_{2^n}^j \Rightarrow x^d = \delta_{2^n}^i$ is reachable, iff, $c_{ij} > 0$;

(ii) (22) is controllable at $x^0 = \delta_{2^n}^j$, iff, $\mathrm{Col}_j(\mathcal{M}_\mathcal{C}) > 0$;

(iii) (22) is controllable, iff, $\mathcal{M}_\mathcal{C} > 0$.

Related Works:

- Controllability and Observability (**Cheng & Qi, Automatica, 45(7), 2009**)
- Realization (**Cheng, Li & Qi, Automatica, 46(1), 2010**)

Outline

IV.2 Disturbance Decoupling Problem (DDP)

☞ Network Model

$$\begin{cases} x_1(t+1) &= f_1(x_1(t),\cdots,x_n(t),u_1(t),\cdots,u_m(t), \\ & \qquad \xi_1(t),\cdots,\xi_q(t)) \\ \vdots & \\ x_n(t+1) &= f_n(x_1(t),\cdots,x_n(t),u_1(t),\cdots,u_m(t), \\ & \qquad \xi_1(t),\cdots,\xi_q(t)), \end{cases}$$
$$y_j(t) = h_j(x(t)), \quad j=1,\cdots,p, \tag{24}$$

where $\xi_i(t)$, $i=1,\cdots,q$ are disturbances.

Definition 4.2.1

The DDP is: Finding, if possible, state feedback controls

$$u_i(t) = g_i(x_1(t),\cdots,x_n(t)), \quad i=1,\cdots,m, \tag{25}$$

such that for the closed-loop system the outputs are not affected by the disturbances.

☞ $\mathcal{Y}$-friendly Subspace

Definition 4.2.2

Let $\mathcal{Y} = F_\ell\{y_1, \cdots, y_p\}$. $\mathcal{S} \subset \mathcal{X}$ is called the $\mathcal{Y}$-friendly subspace, if $\mathcal{S}$ is a regular subspace and

$$y_i \in \mathcal{S}, \quad i = 1, \cdots, p.$$

☞ $\mathcal{Y}$-friendly Form

Proposition 4.2.3 (Cheng, IEEE TAC, 56(1), 2011)

Let $\mathcal{S} = F_\ell\{z^2\}$ be an $\mathcal{Y}$-friendly subspace. Then we have the follow $\mathcal{Y}$-friendly Form:

$$\begin{cases} z^1(t+1) = F_1(z_1(t), \cdots, z_n(t), u_1(t), \cdots, u_m(t), \\ \qquad\qquad \xi_1(t), \cdots, \xi_q(t)) \\ z^2(t+1) = F_2(z_1(t), \cdots, z_n(t), u_1(t), \cdots, u_m(t), \\ \qquad\qquad \xi_1(t), \cdots, \xi_q(t)), \end{cases}$$
$$y_j(t) = h_j(z^2(t)), \quad j = 1, \cdots, p. \tag{26}$$

☞ DDP Solvability

Theorem 4.2.4 (Cheng, IEEE TAC, 56(1), 2011)

Consider Boolean control network (24) with disturbances. The disturbance decoupling problem is solvable, if and only if,

(i) there exists a $\mathcal{Y}$-friendly subspace with $\mathcal{Y}$-friendly form (26);

(ii) there exist state-feedback controls

$$u_i(t) = \phi_i(z(t)), \quad i = 1, \cdots, m,$$

such that

$$\begin{aligned} F_2(z_1(t), \cdots, z_n(t), \phi_1(z(t)), \cdots, \phi_m(z(t)), \\ \xi_1(t), \cdots, \xi_q(t)) = \tilde{F}_2(z^2(t)). \end{aligned}$$

Outline

IV.3 Stability and Stabilization

Definition 4.3.1

1. The Boolean network (3) is stable, if there exists an $N > 0$ such that $x_i(t) = \text{const.},\ t \geq N,\ i = 1, \cdots, n$.
2. The Boolean control network (4) is stabilizable, if there exists controls $\{u_i\}$ such that the closed-loop network is stable. Particularly, if

$$u_i = \text{const.}, \quad i = 1, \cdots, m, \tag{27}$$

it is said to be stabilized by constant control; if

$$u_i(t) = g_i(x_1(t), \cdots, x_n(t)), \quad i = 1, \cdots, m, \tag{28}$$

it is said to be stabilized by state feedback control.

☞ Constant Mapping

Definition 4.3.2

1. Let $F : \mathcal{D}^n \to \mathcal{D}^k$ be

$$z_i = f_i(x_1, \cdots, x_n), \quad i = 1, \cdots, k.$$

   F is called a **constant mapping**, if $f_i = \text{const.}$, $i = 1, \cdots, k$.

2. A logical matrix, M, is called a **constant mapping matrix** (CMM), if

$$\mathrm{Col}_i(M) = M_0, \quad \forall i.$$

☞ Constant Mapping ⇔ Constant Matrix

Proposition 4.3.3

Let $F : \mathcal{D}^n \to \mathcal{D}^k$ be a constant mapping and its algebraic form is

$$z = M_F x, \quad M_F \in \mathcal{L}_{2^k \times 2^n}.$$

Then M_F is a CMM.

☞ Power-reducing Matrix

$$\Phi_k := \prod_{i=1}^{k} I_{2^{k-1}} \otimes \left[(I_2 \otimes W_{[2,2^{k-i}]} \delta_4[1,4] \right],$$

where $W_{[m,n]}$ is a swap matrix.

☞ Stability

Theorem 4.3.4 (Cheng, Qi, Li & Liu, IJRNC, 21(2), 2011)

- The network (3) is stable if there exists a $k \leq n$, such that ($\mathcal{I}(F)$: incidence matrix)

$$[\mathcal{I}(F)]^{(k)} = 0. \tag{29}$$

- The network (3) is stable if there exists a $k \leq 2^n$, such that the system structure matrix satisfies that L^k is a CMM. That is,

$$\mathrm{Col}_i(L^k) = M_0, \quad \forall i. \tag{30}$$

☞ Stabilization

Theorem 4.3.5 (Cheng, Qi, Li & Liu, IJRNC, 21(2), 2011)

1. Define
$$L\left[I_{2^m} \otimes L)\Phi_m\right]^{k-1} := \left[L_1^k \; L_2^k \; \ldots \; L_{2^m}^k\right].$$
The network (4) is stablizable by constant constrols, if and only if, there exists at least a $L_j^k, \; 1 \le k \le 2^n, \; 1 \le j \le 2^m$, which is a CMM.
2. The network (4) is stablizable by state feedback controls $u = Gx$, if and only if, there exists a $1 \le k \le 2^n$, such that $(LG\Phi_n)^k$ is a CMM.

Outline

IV.4 Optimal Control of BCN

☞ Problem Formulation

Definition 4.4.1

Consider the Boolean control network (4). The payoff function is assumed to be

$$p(t) = P(x(t), u(t)), \quad t = 1, 2, \cdots .$$

The optimization problem is: Finding, if possible, an optimal control sequence, which maximize the average payoff

$$J(u) = \overline{\lim_{T\to\infty}} \frac{1}{T} \sum_{t=1}^{T} P(x(t), u(t)). \tag{31}$$

☞ Optimal Control

Theorem 4.4.2 (Zhao & Cheng, IEEE TAC, to appear)

Consider Boolean control network (4). There exists an optimal control $u^*(t)$ of the form

$$\begin{cases} u_1^*(t+1) = g_1(x_1(t), \cdots, x_n(t), u_1^*(t), \cdots, u_n^*(t)) \\ \vdots \\ u_m^*(t+1) = g_m(x_1(t), \cdots, x_n(t), u_1^*(t), \cdots, u_n^*(t)), \end{cases} \tag{32}$$

which maximize (31). Moreover, the corresponding optimal trajectory $w^*(t) = u^*(t)x^*(t)$ becomes periodic after finite steps.

- The key issue is to calculate the cycles for BCN (refer to: **Zhao & Cheng, IEEE TAC, to appear**)

Outline

IV.5 Identification of BN/BCN

☞ Identification of BN

- Observed data:

$$\{X(0), X(1), \cdots, X(N)\}.$$

- Network:

$$x_i(t+1) = f_i(x_1(t), \cdots, x_n(t)), \quad i = 1, \cdots, n. \tag{33}$$

- Component-wise algebraic form:

$$x_i(t+1) = M_i x(t), \quad i = 1, \cdots, n, \tag{34}$$

  where $M_i \in \mathcal{L}_{2\times 2^n}$.
- Purpose: Identify M_i, $i = 1, \cdots, n$.

☞ Identifying Column

Theorem 4.5.1 (Cheng, Qi & Li, IEEE TNN, 22(4), 2011)

Let $x(t) = \delta_{2^n}^j$ and $x_i(t+1) = \delta_2^k$. Then

$$\mathrm{Col}_j(M_i) = \delta_2^k. \tag{35}$$

Assume we have partly identified structure matrices M_i, $i = 1, \cdots, n$.
Define

$$M_{i,j} := M_i W_{[2,2^{j-1}]}, \quad j = 1, 2, \cdots, n. \tag{36}$$

Then split it into two equal-size blocks as

$$M_{i,j} = \begin{bmatrix} M_{i,j}^1 & M_{i,j}^2 \end{bmatrix}. \tag{37}$$

☞ Least In-degree Model

Theorem 4.5.2 (Cheng, Qi & Li, IEEE TNN, 22(4), 2011)

f_i has a realization which is independent of x_j, if and only if

$$M^1_{i,j} = M^2_{i,j} \tag{38}$$

has solution for unidentified columns.

- Using (38), we can find a least in-degree realization.
- Identification of BCN: Refer to **Cheng & Zhao, Automatica, 47(4), 2011**.

Example 9.3

Observed Data:

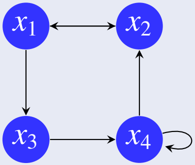
$$\begin{array}{cccccc} \bullet\circ\bullet\bullet & \bullet\bullet\bullet\bullet & \circ\bullet\bullet\bullet & \circ\circ\circ\bullet & \bullet\circ\circ\circ & \bullet\circ\bullet\bullet \\ t=0 & t=1 & t=2 & t=3 & t=4 & t=5 \end{array}$$

Partly identified structure matrix:

$$\begin{aligned} M_1 &= \delta_2[* \; 2 \; * \; * \; * \; * \; * \; 1 \; * \; 2 \; * \; 2 \; * \; * \; * \; 1] \\ M_2 &= \delta_2[* \; 1 \; * \; * \; * \; * \; * \; 1 \; * \; 1 \; * \; 2 \; * \; * \; * \; 2] \\ M_3 &= \delta_2[* \; 1 \; * \; * \; * \; * \; * \; 1 \; * \; 2 \; * \; 2 \; * \; * \; * \; 2] \\ M_4 &= \delta_2[* \; 1 \; * \; * \; * \; * \; * \; 2 \; * \; 2 \; * \; 2 \; * \; * \; * \; 2]. \end{aligned}$$

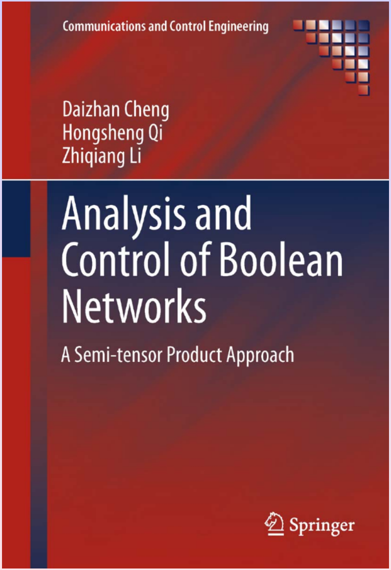
Example 9.3 (Cont'd)

Lease In-degree Realization:



$$
\begin{cases}
x_1(t+1) = \neg x_2(t), \\
x_2(t+1) = x_4(t) \vee x_1(t), \\
x_3(t+1) = x_1(t), x_4(t+1) = x_3(t) \bar{\vee} x_4(t).
\end{cases}
\tag{39}
$$

☞ Reference Book



V. Concluding Remarks

☞ Conclusion

- Boolean network is a proper model for cellular networks;
- **Semi-tensor Product**:

$$\text{Logical Dynamics} \Rightarrow \text{Discrete-time Dynamics};$$

- **State space/subspaces**:

$$\text{Control Theory(applicable)} \Rightarrow \text{Systems Biology}.$$

- Control Problems Considered: Controllability, Observability, Realization, DDP, Stabilization, Optimal control, Identification, etc.
- There are many open problems, and several follow up papers (IEEE TAC, Automatica, IEEE TNN ..., CCC Invited Session)

☞ Topics for Further Study

- Properties and control of probabilistic Boolean networks;
- Application to general biological systems (size problem);
  - Multi-agent Boolean network (Metabolic network: module + network motif)
  - Protein network (self similar, scale-free network)
- Dynamic games with finite strategies and finite memories;
- Fuzzy control (Fuzzy relational equations);
- Logic-based control;
- Cryptography, Coding by Boolean function, Secure community;
- Circuit Design, Failure Detection, etc.
- $\cdots$

Thank you!

Question?