Distributed Control Using Positive Quadratic Programming

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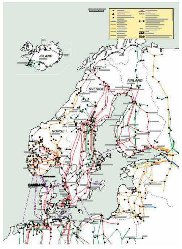
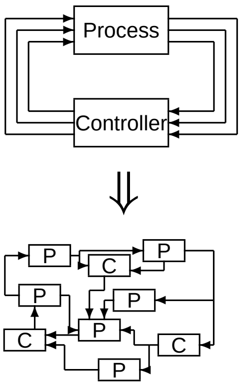
Outline

- **Why Distributed Control?**
- Distributed Control of Positive Systems
- Example: Optimizing Electrical Power Flow
- Solution using Positive Quadratic Programming
- Finding Optimum by Distributed Control

Building theoretical foundations for distributed control





We need methodology for

- Decentralized specifications
- Decentralized design
- Verification of global behavior

Example 1: A vehicle formation

Example 1: A vehicle formation

Each vehicle obeys the independent dynamics

$$\begin{bmatrix}\dot{x}_1(t)\\ \dot{x}_2(t)\\ \dot{x}_3(t)\\ \dot{x}_4(t)\end{bmatrix} = \begin{bmatrix} * & 0 & 0 & 0\\ 0 & * & 0 & 0\\ 0 & 0 & * & 0\\ 0 & 0 & 0 & *\end{bmatrix}\begin{bmatrix}x_1(t)\\ x_2(t)\\ x_3(t)\\ x_4(t)\end{bmatrix} + \begin{bmatrix}B_1u_1(t)+w_1(t)\\ B_2u_2(t)+w_2(t)\\ B_3u_3(t)+w_3(t)\\ B_4u_4(t)+w_4(t)\end{bmatrix}$$

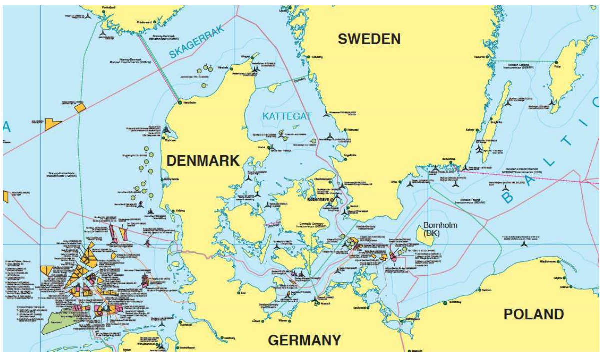
The objective is to make $\mathbf{E}|Cx_{i+1} - Cx_i|^2$ small for $i = 1, \dots, 4$.

Example 2: A supply chain for fresh products

Fresh products degrade with time:

$$\begin{bmatrix}\dot{x}_1(t)\\ \dot{x}_2(t)\\ \dot{x}_3(t)\\ \dot{x}_4(t)\end{bmatrix} = \begin{bmatrix} * & 0 & 0 & 0\\ 0 & * & 0 & 0\\ 0 & 0 & * & 0\\ 0 & 0 & 0 & *\end{bmatrix}\begin{bmatrix}x_1(t)\\ x_2(t)\\ x_3(t)\\ x_4(t)\end{bmatrix} + \begin{bmatrix}-u_1(t)+w_1(t)\\ u_1(t)-u_2(t)\\ u_2(t)-u_3(t)\\ u_3(t)+w_4(t)\end{bmatrix}$$

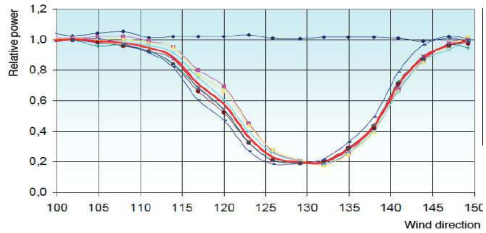
Example 3: A Wind Farm



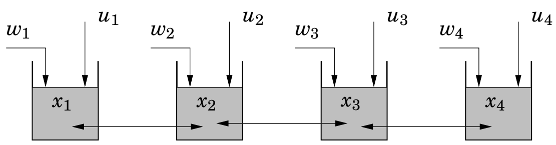
Example 3: A Wind Farm

$$\begin{bmatrix}\dot{x}_1(t)\\ \dot{x}_2(t)\\ \dot{x}_3(t)\\ \dot{x}_4(t)\end{bmatrix} = \begin{bmatrix} * & 0 & 0 & 0\\ * & * & 0 & 0\\ 0 & * & * & 0\\ 0 & 0 & * & *\end{bmatrix}\begin{bmatrix}x_1(t)\\ x_2(t)\\ x_3(t)\\ x_4(t)\end{bmatrix} + \begin{bmatrix}B_1u_1+w_1\\ B_2u_2+w_2\\ B_3u_3+w_3\\ B_4u_4+w_4\end{bmatrix}$$

Example 3: A Wind Farm



Example 4: Irrigation Channels



$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \\ \dot{x}_4(t) \end{bmatrix} = \begin{bmatrix} * & * & 0 & 0 \\ * & * & * & 0 \\ 0 & * & * & * \\ 0 & 0 & * & * \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix} + \begin{bmatrix} u_1(t) + w_1(t) \\ u_2(t) + w_2(t) \\ u_3(t) + w_3(t) \\ u_4(t) + w_4(t) \end{bmatrix}$$

Note: Off-diagonal elements are typically positive!

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Positive systems have nonnegative impulse response

If the matrices A, B and C have nonnegative coefficients except possibly for the diagonal of A, then the system

$$\frac{dx}{dt} = Ax + Bu$$
$$y = Cx$$

has non-negative impulse response $Ce^{At}B$.

Examples:

- Ecological system with x_k the population of species k.
- Chemical reaction with x_k the concentration of reactant k.
- Economic system with x_k the quantity of commodity k.
- Probabilistic model with x_k the probability of state k.

Positive Systems and Nonnegative Matrices

Classics:

- Perron (1907) and Frobenius (1912)
- Leontief (1936)
- Hirsch (1985)

Books:

- Gantmacher (1959)
- Berman and Plemmons (1979)
- Luenberger (1979)

Recent control related work:

- Angeli and Sontag (2003)
- Moreau (2004)

Stability of Positive systems

Suppose the matrix A has nonnegative off-diagonal elements. Then the following conditions are equivalent:

(i) The system $\frac{dx}{dt} = Ax$ is exponentially stable.

(ii) There exits a vector $x > 0$ such that $Ax < 0$. (The vector inequalities are elementwise.)

(iii) There is a *diagonal* matrix $P \succ 0$ such that $PA^T + AP \prec 0$

Stability can be Tested in a Distributed Way

Stability of $\dot{x} = Ax$ follows from existence of $x_k > 0$ such that

$$\underbrace{\begin{bmatrix} a_{11} & a_{12} & 0 & 0 \\ a_{21} & a_{22} & a_{23} & 0 \\ 0 & a_{32} & a_{33} & a_{32} \\ 0 & 0 & a_{43} & a_{44} \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} < \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

The first node verifies the inequality of the first row.

The second node verifies the inequality of the second row.

and so on ...

Performance of Positive systems

Suppose the matrices A, B and C have nonnegative coefficients except for the diagonal of A. Suppose A is Hurwitz. Then the following conditions are equivalent:

(i) $\max_\omega |C(i\omega I - A)^{-1}B| < \gamma$

(ii) $|CA^{-1}B| < \gamma$

(iii) There exits $x > 0$ such that $Cx < \gamma$, $Ax + B = 0$.

(iv) There is a *diagonal* matrix $P \succ 0$ such that

$$PA^T + AP + PC^TCP + \gamma^{-2}BB^T \prec 0$$

Note: The linear inequalities (iii) can be tested row by row.

Distributed Control Synthesis

Suppose the matrix

$$\begin{bmatrix} a_{11}+\ell_1 & a_{12} & 0 & 0 \\ a_{21}-\ell_1 & a_{22}+\ell_2 & a_{23} & 0 \\ 0 & a_{32}-\ell_2 & a_{33} & a_{32} \\ 0 & 0 & a_{43} & a_{44} \end{bmatrix}$$

is nonnegative for all $\ell_1, \ell_2 \in [0, 1]$. For stabilizing gains ℓ_1, ℓ_2, find $0 \le u_k \le x_k$ such that

$$\begin{bmatrix} a_{11} & a_{12} & 0 & 0 \\ a_{21} & a_{22} & a_{23} & 0 \\ 0 & a_{32} & a_{33} & a_{32} \\ 0 & 0 & a_{43} & a_{44} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ -1 & 1 \\ 0 & -1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} < \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

and set $\ell_1 = u_1/x_1$ and $\ell_2 = u_2/x_2$. Every row gives a local test.

Note: *Positivity assumed a priori. What if $\ell_1, \ell_2 \in \mathbf{R}$?*

Synthesizing Positive Systems

$$A + BL = \begin{bmatrix} a_{11}+\ell_1 & a_{12} & 0 & 0 \\ a_{21}-\ell_1 & a_{22}+\ell_2 & a_{23} & 0 \\ 0 & a_{32}-\ell_2 & a_{33} & a_{32} \\ 0 & 0 & a_{43} & a_{44} \end{bmatrix}$$

is stable and nonnegative if and only if $p_k \ge 0$ and

$$(A+BL)P = \begin{bmatrix} (a_{11}+\ell_1)p_1 & a_{12}p_2 & 0 & 0 \\ (a_{21}-\ell_1) & (a_{22}+\ell_2)p_2 & a_{23}p_3 & 0 \\ 0 & (a_{32}-\ell_2)p_2 & a_{33}p_3 & a_{32}p_4 \\ 0 & 0 & a_{43}p_3 & a_{44}p_4 \end{bmatrix}$$

make $(A+BL)P + P(A+BL)^T$ negative definite with nonnegative off-diagonal elements.
Solve using convex optimization in the pair (P, PL)!

[Tanaka and Langbort, ACC 2010]

Positivity versus Passivity

- Passivity can be described naturally in frequency domain.
- Positivity can be described naturally in time-domain.
- Negative feedback loops preserve passivity.
- Positive feedback loops preserve positivity.
- Parallel connections preserve both passivity and positivity.
- Series connections preserves positivity, but not passivity.

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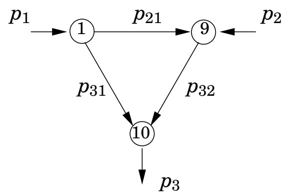
Example A: Electrical Power Transmission

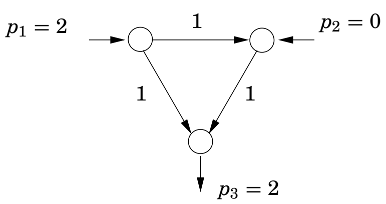
Two generators with generation cost 1 and 9 respectively.
One load willing to buy $p_3 = 2$ at the price 10:



Maximize profit:	$10p_3 - 9p_2 - p_1$
subject to capacity constraints:	$\lvert p_{jk}\rvert \le 1,\ p_1 \ge 0,\ p_2 \ge 0,\ p_3 \ge 2$
and conservation laws:	$p_1 = p_{21} + p_{31}$
	$p_{32} = p_{21} + p_2$
	$p_3 = p_{31} + p_{32}$

Optimal Allocation for Example A

Both transmission lines serving the load need to be used at full capacity to meet the demand $p_3 = 2$.

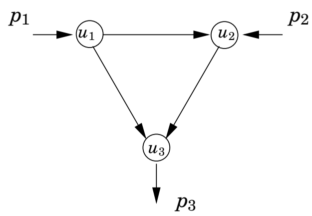


Optimal profit: $10p_3 - p_1 = 18$

In real power networks, electrons flow according to Kirchhoff's laws.
The allocation above is not feasible when all three lines are identical.
Why?

Example B: Optimal Potential Flow

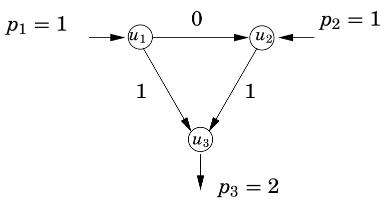
Power flow is driven by potential differences:



Maximize profit:	$10p_3 - 9p_2 - p_1$
subject to capacity constraints:	$\lvert u_j - u_k\rvert \le 1,\ p_j \ge 0,\ p_3 \ge 2$
and conservation laws:	$p_1 = (u_1 - u_2) + (u_1 - u_3)$
	$p_2 = (u_2 - u_1) + (u_2 - u_3)$
	$p_3 = (u_1 - u_3) + (u_2 - u_3)$

Optimal Allocation for Example B

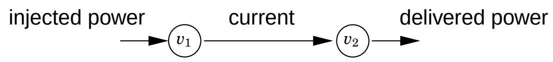
Both transmission lines serving the load need to be used at full capacity to meet the demand $p_3 = 2$. Hence $u_1 = u_2$ and there is no flow between node 1 and node 2!



The optimal profit is much smaller: $10p_3 - p_1 - 9p_2 = 10$

When transmission lines operate near capacity limits, losses are big. Can we take losses into account in the optimization?

Power Losses in a DC Transmission Line



For a DC transmission line with admittance y, input voltage v_1 and output voltage v_2, we have:

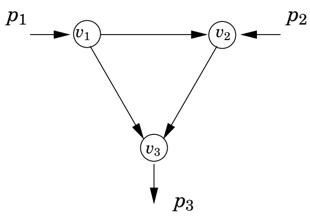
Line current:	$i = y(v_1 - v_2)$
Injected power:	$p_1 = yv_1(v_1 - v_2)$
Delivered power:	$p_2 = yv_2(v_1 - v_2)$
Power loss:	$p_1 - p_2 = y(v_1 - v_2)^2$

If the voltages are bounded from above by $\overline{v}$, there is an upper bound on how much power the transmission line can deliver:

$$p_2 = yv_2(v_1 - v_2) \leq yv_2(\overline{v} - v_2) \leq y\overline{v}^2/4$$

At the capacity limit, the power loss equals the delivered power.

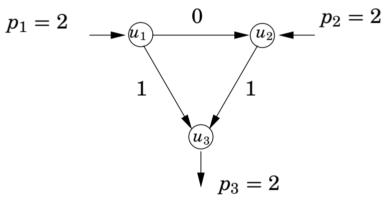
Example C: Optimal Power Flow with Losses



Maximize profit:	$10p_3 - 9p_2 - p_1$
subject to capacity constraints:	$0 \leq v_j \leq 2$
and conservation laws:	$p_1 = v_1(v_1 - v_2) + v_1(v_1 - v_3)$
	$p_2 = v_2(v_2 - v_1) + v_2(v_2 - v_3)$
	$p_3 = v_3(v_1 - v_3) + v_3(v_2 - v_3)$

Optimal Allocation for Example C

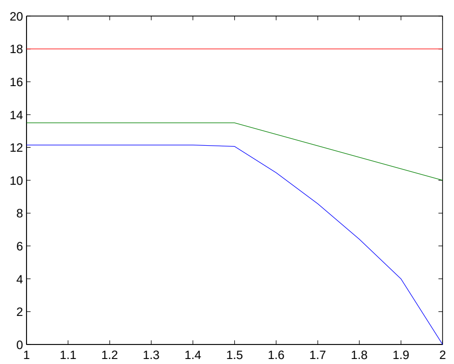
Both transmission lines serving the load need to be used at full capacity to meet the demand $p_3 = 2$. Hence $v_1 = v_2 = \overline{v}$ and there is no current between node 1 and node 2!



There is no room for profit: $10p_3 - p_1 - 9p_2 = 0$

Notice that half of the generated power is lost in transmission!

Profit Versus Power Demand



Analogies to Electric Power Flow

Water distribution systems: Electrical voltage corresponds to water pressure. Differences in pressure creates flow.

Gas diffusion: Electrical voltage corresponds to partial pressure. Gradients in partial pressure creates diffusion.

Exchange economy: Voltages correspond to *inverse* prices. Price differences drive commodity flows. Delivered electric power corresponds to delivered commodity volume.

Two kinds of flow of simultaneous interest.
In power transmission networks, electric current is conserved, but electric power is dissipated due to transmission losses.

In economic systems the commodity value is conserved, but the commodity volume is dissipated due to transportation losses.

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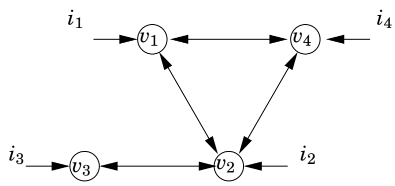
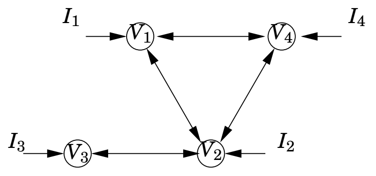
A General Power Transmission Network



$$\underbrace{\begin{bmatrix} i_1 \\ i_2 \\ i_3 \\ i_4 \end{bmatrix}}_{i} = \underbrace{\begin{bmatrix} Y_{12} + Y_{14} & -Y_{12} & 0 & -Y_{14} \\ -Y_{21} & Y_{21} + Y_{23} + Y_{24} & -Y_{23} & -Y_{24} \\ 0 & -Y_{32} & Y_{32} & 0 \\ -Y_{41} & -Y_{42} & 0 & Y_{41} + Y_{42} \end{bmatrix}}_{Y} \underbrace{\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix}}_{v}$$

An Optimal Flow Problem for AC Power

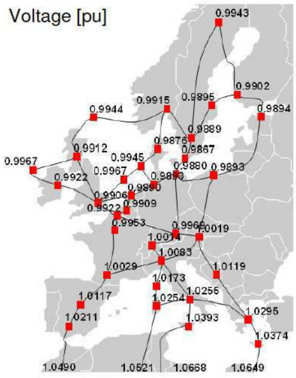


$I_k \in \mathbf{C}$
$V_k \in \mathbf{C}$

Minimize	Re $\sum_k I_k^* V_k$	
subject to $I = YV$ and	$\underline{P}_k \leq$ Re $(I_k^* V_k) \leq \overline{P}_k$	
	$\underline{Q}_k \leq$ Im $(I_k^* V_k) \leq \overline{Q}_k$	
	$\underline{v}_k \leq \lvert V_k \rvert \leq \bar{v}_k$	for $k = 1, \ldots, 4$

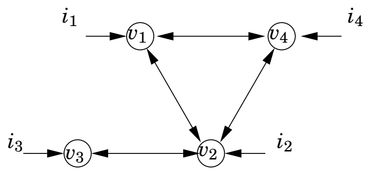
(Convex relaxation by Lavaei/Low inspired this talk.)

Future DC Power Transmission Network in Europe?



From Cigré Conference 2010, "Continental Overlay HVDC-Grid" by ABB

Optimizing DC Power Flow



$i_k \in \mathbf{R}$
$v_k \in \mathbf{R}$

Minimize	$\sum_k i_k v_k$	
subject to $i = Yv$ and	$i_k v_k \leq \bar{p}_k$	
	$(v_k - v_j)^2 \leq c_{kj}$	
	$\underline{v}_k \leq v_k \leq \bar{v}_k$	for all k, j

Notice: $\bar{p}_k$ negative at loads, positive at generators.

Positive Quadratic Programming

Given $A_0, \ldots, A_K \in \mathbf{R}^{n \times n}$ with nonnegative off-diagonal entries and $b_1, \ldots, b_K \in \mathbf{R}$, the following equality holds:

$$\begin{array}{lll} \text{max} & x^T A_0 x \\ \text{subject to} & x \in \mathbf{R}^n_+ \\ & x^T A_k x \geq b_k \\ & k = 1, \ldots, K \end{array} = \begin{array}{ll} \text{max} & \mathrm{trace}(A_0 X) \\ \text{subject to} & X \succeq 0 \\ & \mathrm{trace}(A_k X) \geq b_k \\ & k = 1, \ldots, K \end{array}$$

Proof

If $X = \begin{bmatrix} |x_1|^2 & & * \\ & \ddots & \\ * & & |x_n|^2 \end{bmatrix}$ maximizes the right hand side,

then $x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ maximizes the left.

[Goemans/Williamson (1994), Zhang (1999), Kim/Kojima (2003)]

Positive Quadratic Programming

Given $A_0, \ldots, A_K \in \mathbf{R}^{n \times n}$ with nonnegative off-diagonal entries and $b_1, \ldots, b_K \in \mathbf{R}$, the following equality holds:

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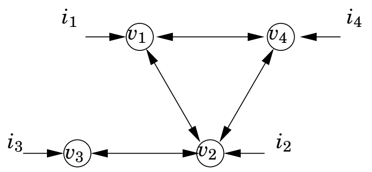
Proof

If $X = \begin{bmatrix} |x_1|^2 & & * \\ & \ddots & \\ * & & |x_n|^2 \end{bmatrix}$ maximizes the right hand side,

then $x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ maximizes the left.

Note: The problem is convex in $|v_1|^2, \ldots, |v_4|^2$!

Optimizing DC Power Flow



$i_k \in \mathbf{R}$
$v_k \in \mathbf{R}$

Minimize	$\sum_k i_k v_k$	
subject to $i = Yv$ and	$i_k v_k \leq \bar{p}_k$	
	$(v_k - v_j)^2 \leq c_{kj}$	
	$\underline{v}_k \leq v_k \leq \bar{v}_k$	for all k, j

Notice: All mixed terms have the right sign!

Dual Positive Quadratic Programming

Given $A_0, \ldots, A_K \in \mathbf{R}^{n \times n}$ with nonnegative off-diagonal entries and $b_1, \ldots, b_K \in \mathbf{R}$, the following equality holds:

$$\begin{array}{lll} \text{max} & x^T A_0 x \\ \text{subject to} & x \in \mathbf{R}^n_+ \\ & x^T A_k x \geq b_k \\ & k = 1, \ldots, K \end{array} = \begin{array}{ll} \text{min} & -\sum_k \lambda_k b_k \\ \text{subject to} & \lambda_1, \ldots, \lambda_K \geq 0 \\ & 0 \succeq A_0 + \sum_k \lambda_k A_k \\ & \end{array}$$

Interpretation:
In the power flow example, λ_k is the price of power at node k.

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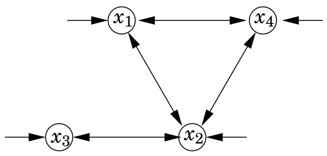
Primal Decomposition



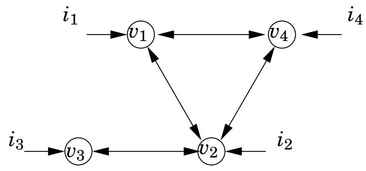
The convex problem

$$\min_{x_k}[V_1(x_1,x_2,x_4)+V_2(x_1,x_2,x_3,x_4)+V_3(x_2,x_3)+V_4(x_1,x_2,x_4)]$$

can be solved by the following distributed iteration:

$$\begin{cases} x_1^+ = \arg\min_{x_1}[V_1(x_1,x_2,x_4)+V_2(x_1,x_2,x_3,x_4)+V_4(x_1,x_2,x_4)] \\ x_2^+ = \arg\min_{x_2}[V_1(x_1,x_2,x_4)+V_2(x_1,x_2,x_3,x_4)+V_3(x_2,x_3)+V_4(x_1,x_2,x_4)] \\ x_3^+ = \arg\min_{x_2}[V_2(x_1,x_2,x_3,x_4)+V_3(x_2,x_3)] \\ x_4^+ = \arg\min_{x_2}[V_1(x_1,x_2,x_4)+V_2(x_1,x_2,x_3,x_4)+V_4(x_1,x_2,x_4)] \end{cases}$$

Finding Optimum by Distributed Control



Given power prices λ_k at each node, find the optimal allocation:

Minimize $\sum_{j,k}\lambda_k y_{kj} v_k(v_k-v_j)$ subject to $v_k \le \bar{v}_k$

Primal decomposition gives convergence to optimum:

$$v_k^+ = \arg\min_{v_k\le\bar{v}_k}\sum_j \Big[\underbrace{\lambda_k y_{kj} v_k(v_k-v_j)}_{\text{value into link } jk} - \underbrace{\lambda_j y_{kj} v_j(v_k-v_j)}_{\text{value out from link } jk}\Big]$$

The Distributed Control Law

The dynamics

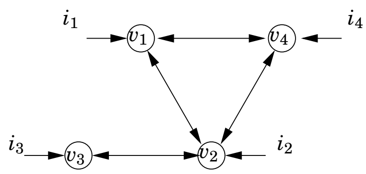
$$v_k^+ = \arg\min_{v_k\le\bar{v}_k}\sum_j \Big[\underbrace{\lambda_k y_{kj} v_k(v_k-v_j)}_{\text{value into link } jk} - \underbrace{\lambda_j y_{kj} v_j(v_k-v_j)}_{\text{value out from link } jk}\Big]$$

has the form

$$v^+ = \min\{\bar{v}\,,\,Av\}$$

where A has nonnegative coefficients.

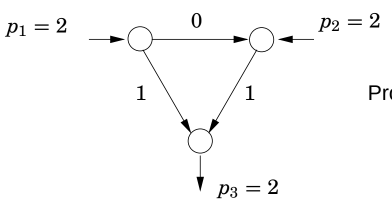
Can it pay off to disconnect a line?



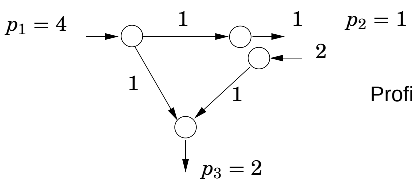
Given power prices λ_k at each node, find the optimal allocation:

Minimize $\sum_{j,k}\lambda_k y_{kj} v_k(v_k-v_j)$ subject to $v_k \le \bar{v}_k$, $y_{kj}\in[0,\bar{y}_{kj}]$

Yes, it can!



Profit$=0$



Profit$=7$

Summary

- Why Distributed Control?
- Optimizing Electrical Power Flow
- Positive Quadratic Programming
- Distributed Control of Positive Systems
- Finding Optimum by Distributed Control

To read:
Slides on www.control.lth.se/Staff/anders_rantzer.html
Extended abstract in Proceedings of CCC 2011
Upcoming paper in CDC 2011

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