The zero dynamics of a nonlinear system: from the origin to the latest progresses of a long successful story

Alberto Isidori

30^{th} Chinese Control Conference
23 July 2011

Part I : A successful story (1980 - present)

- The Origins of a Concept
- Zero Dynamics and High-Gain Feedback
- Zero Dynamics and Feedback Linearization
- Zero Dynamics and Stable Non-interacting Control
- Zero Dynamics and Output Regulation
- Zero Dynamics and Passivity
- Zero Dynamics and Limits of Performance

The Origins of a Concept (1980-84)

All begun in 1979, when the notion of controlled invariant subspace, (a subspace rendered invariant by feedback) introduced in the late 1960's by Basile-Marro-Wonham-Morse, was extended to nonlinear systems.

In linear systems, zeros are the eigenvalues of the unobservable part of a system rendered maximally unobservable by feedback.

the space of all state zero directions, i.e. all initial conditions x^0 for which there exists an input which zeroes the output. This is precisely $\mathcal{V}^*$, the maximal (A, B) invariant subspace contained in the kernel of C

Recall a subspace $\mathcal{V}$ is (A,B) invariant if

(2.4) $A\mathcal{V} \subset \mathcal{V} + \mathcal{B}$

Equivalently there exists mxn matrix F such that

(2.5) $(A + BF)\mathcal{V} \subset \mathcal{V}$

The Origins of a Concept (1980-84)

CDC 1980

NONLINEAR ZERO DISTIBUTIONS

TP3-5:00

A J Krener
Department of Mathematics
University of California
Davis, California 95616

A Isidori
Instituto di Automatica
Università di Roma
Via Eudossiana 18
00184 Roma

Introduction

This paper generalizes the concept of a zero from a linear to nonlinear system. We start by briefly reviewing the concept of zero for a linear system. This is quite a complicated idea and many different definitions have been proposed. Of necessity we cannot discuss them all and we refer the reader to the excellent survey paper of MacFarlane and Karcanias [1] for a full treatment. The particular definition that we shall adopt is from Desoer and Schulman [2].

A distribution Δ is a null observable distribution if it is (f,g) invariant and contained in kernel dh.

That the output is constant over equivalence classes is the nonlinear generalization of the output being zero. The above construction is the nonlinear version of loss of observability via state feedback.

no emphasis on the dynamics of the unobservable part

The Origins of a Concept (1980-84)

Associated with the construction of Δ*, there is a special choice of (local) coordinates in which the maximally unobservable part is highlighted.

IEEE TRANSACTIONS ON AUTOMATIC CONTROL, VOL. AC-26, NO. 2, APRIL 1981 331

Nonlinear Decoupling via Feedback: A Differential Geometric Approach

ALBERTO ISIDORI, MEMBER, IEEE, ARTHUR J. KRENER, MEMBER, IEEE, CLAUDIO GORI-GIORGI, AND SALVATORE MONACO

System is rendered maximally observable by a control

$$u = \alpha(x) + \beta(x) v$$

with α(x) and β(x) satisfying

$$L_g L_f^{\rho_i} h_i(x) \alpha_i(x) = L_f^{\rho_i + 1} h_i(x)$$

$$L_g L_f^{\rho_i} h_i(x) \beta_i(x) = \text{constant matrix.}$$

The Origins of a Concept (1980-84)

Following the construction of Theorem 5.1, we define

$$\xi_i(x) = (h_i(x), \cdots, L_f^{\rho_i} h_i(x)).$$

In the local coordinates $\xi(x) = (\xi_1(x), \cdots, \xi_m(x), \xi_{m+1}(x))$, the system becomes almost linear, i.e., each input/output channel (5.2) is of the form

$$\dot{\xi}_i = A_i \xi_i + B_i v_i$$

$$y_i = C_i \xi_i$$

The rest of the system, which is unobservable, is nonlinear:

$$\dot{\xi}_{m+1} = \tilde{f}_{m+1}(\xi) + \tilde{g}_{m+1}(\xi) v.$$

This is related to work of Brockett [3], on when a nonlinear system may be modified by feedback so as to be locally diffeomorphic to a linear system.

The Origins of a Concept (1980-84)

Proceedings of 23rd Conference
on Decision and Control
Las Vegas, NV, December 1984

FP2 - 4:00

A FREQUENCY DOMAIN PHILOSOPHY FOR NONLINEAR SYSTEMS,
WITH APPLICATIONS TO STABILIZATION AND TO ADAPTIVE CONTROL

A relative degree 1 system, in local coordinates can be expressed as

$$\dot{z} = f_1(z,y) \tag{2.4}$$

$$\dot{y} = L_f h(z,y) + uL_g h(z,y) .$$

In this setting, the zero dynamics is the (n-1)-th order system

$$\dot{z} = f_1(z,0) . \tag{2.5}$$

Definition 2.4.

The system (2.1) is globally minimum phase on M provided it is minimum phase and the zero dynamics (2.6) is globally asymptotically stable.

The Origins of a Concept (1984-86)

JEAN-PIERRE AUBIN

Consider the evolution of a control system with (multivalued) feedba

$$\begin{cases} \text{i)} & x'(t) = f(x(t), u(t)) \\ \text{ii)} & u(t) \in U(x(t)) \end{cases}$$

A subset K enjoys the *viability property* (for the control system described by f and U) if for every initial state $x_0 \in K$, there exists at least one solution to the system starting at x_0 which is *viable* in the sense that

$$\forall t \in [0, T], \quad x(t) \in K.$$

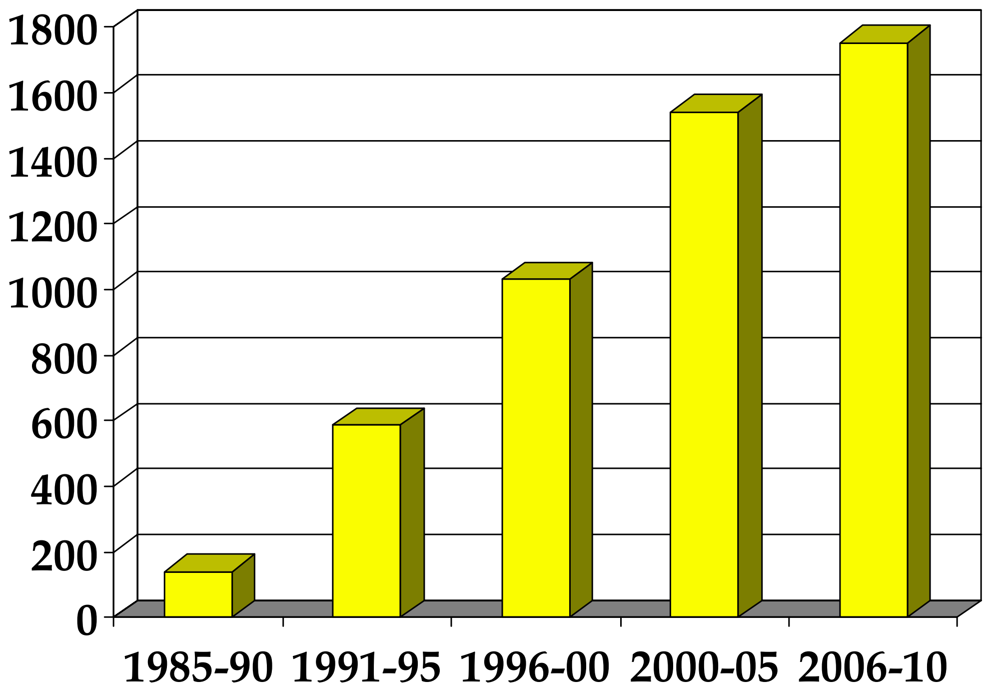
For linear control systems, this property has been introduced under the name of "controlled invariance" in [11], [90], [120]. See also [117]-[119] for instance. This property has then been extended to nonlinear systems in [26]-[30], [70], [75], [74], [116].

When a closed subset K is not a viability domain, we can state that *there exists a largest closed viability domain contained in* K. This domain will be denoted Viab(K) and called the *viability kernel*[12] of K. It may be empty.

The Origins of a Concept (1980-84)

Despite various attempts to reintroduce the same concept under different names

the terminology "Zero Dynamics" became standard



Google scholar beta

Zero Dynamics and High Gain Feedback (1984)

Proceedings of 23rd Conference
on Decision and Control
Las Vegas, NV, December 1984

FP2 - 4:00

A FREQUENCY DOMAIN PHILOSOPHY FOR NONLINEAR SYSTEMS,
WITH APPLICATIONS TO STABILIZATION AND TO ADAPTIVE CONTROL

Theorem 3.1. Suppose the system (3.1) on R^n is globally minimum phase and satisfies (H1)-(H2). Consider the output feedback law

$$u = -ky \tag{3.1}$$

For any bounded open set U R^n there exists k_U such that for all $k > k_U$ the closed-loop system (2.1)-(3.1) is locally and globally asymptotically stable to x_0 on U.

The result was incomplete as stated. In fact, for the Theorem to be true, the zero dynamics needs to be locally exponentially stable. Otherwise there might be nontrivial limit sets.
But it stresses the interest in the concept of “stabilization with guaranteed region of attraction”.

Zero Dynamics and High Gain Feedback (1984)

The problem of stabilization with guaranteed region of attraction was addressed, for a special class of systems, in

IEEE TRANSACTIONS ON AUTOMATIC CONTROL, VOL. AC-31, NO. 10, OCTOBER 1986

Potentially Global Stabilizability

A. BACCIOTTI

***Abstract*—In this note we consider a particular class of nonlinear systems with a controllable linear part. We prove that by an appropriate choice of a feedback law, it is possible to transform the system in such a way that the origin is asymptotically stable and the region of attraction is arbitrarily large.**

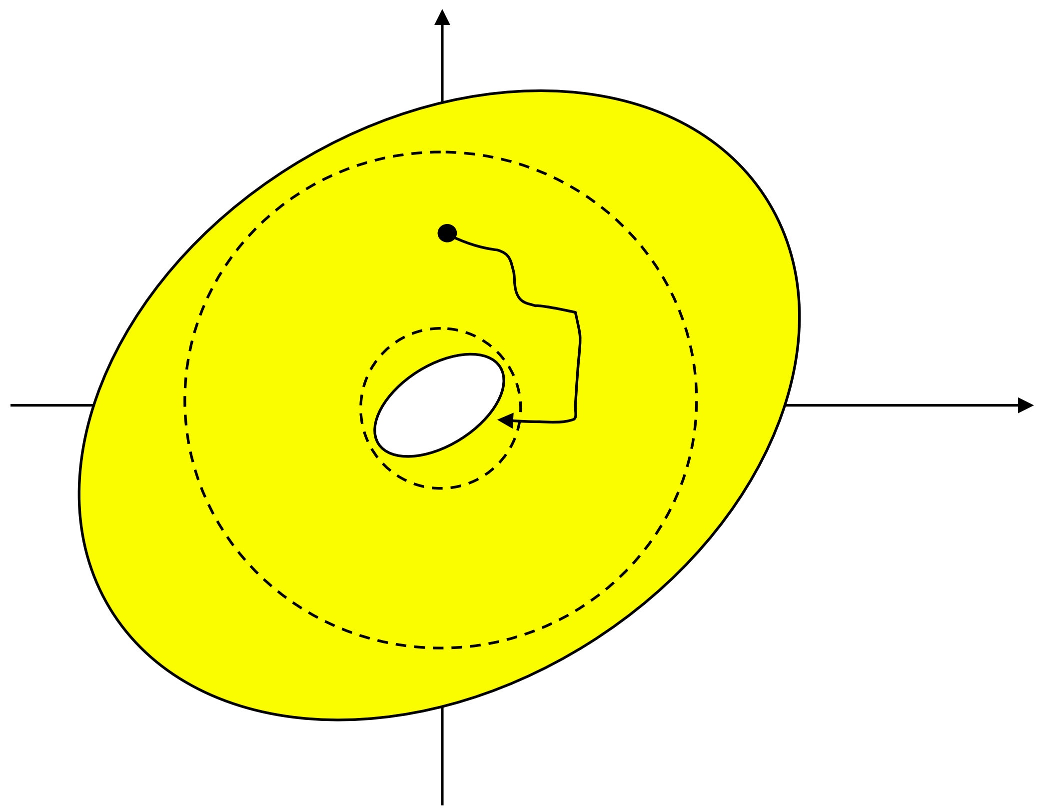
Bacciotti, in a later paper (NOLCOS 1992) provided what is commonly acknowledged to be the most elegant argument proving that, for systems possessing a globally stable zero dynamics, high gain output feedback yields semiglobal practical stability.

Zero Dynamics and High Gain Feedback (1984)

Bacciotti's argument :

$$\begin{aligned}\dot{z} &= f(z,y)\\ \dot{y} &= a(z,y)+b(z,y)u\end{aligned}$$

Let $V(z)$ be a Lyapunov function for $\dot{z}=f(z,0)$. Pick $W(x)=V(z)+y^2$. Then, by high-gain, $\dot{W}(x)$ can be rendered negative on



Zero Dynamics and High-Gain Feedback (1984)

A similar result (i.e. under high-gain output feedback, the output approaches zero, while the other states approach trajectories of the dynamics of *f(x)+g(x)α(x)* restricted to Δ*) appeared, simultaneously and independently, in

INT. J. CONTROL, 1985, VOL. 42, NO. 6, 1369–1385

High-gain feedback in non-linear control systems†

RICCARDO MARINO‡

High-gain state and output feedback are investigated for non-linear control systems with a single additive input by using singular perturbation techniques.

Classical approximation results (Tihonov-like theorems) in singular perturbation theory are extended to non-linear control systems by defining a composite additive control strategy, a control-dependent fast equilibrium manifold and non-linear change of coordinates.

The analysis is only local, though, and the possible occurrence of nontrivial limit sets (due to lack of hyperbolicity) is not addressed.

Zero Dynamics and Semiglobal Stabilization (1990-94)

In the subsequent five years, precise results on this subject became gradually available.

An important preliminary step was the development of a global normal form, as done in

1122 IEEE TRANSACTIONS ON AUTOMATIC CONTROL, VOL. 36, NO. 10, OCTOBER 1991

Asymptotic Stabilization of Minimum Phase Nonlinear Systems

Zero Dynamics and Semiglobal Stabilization (1990-94)

In 1990, it was finally proven that globally minimum-phase systems in strict normal form can be semiglobally stabilized by high-gain partial state feedback, $u = kH\xi$.

Strict triangularity made possible to use a simple recursive argument

"adding an integrator"

In order to complete the proof in case some of the r_i's are larger than 1, it suffices to show that if a system

$$\dot{x} = f_0(x, u_1, \cdots, u_m) \tag{6.3}$$

is globally asymptotically stabilizable by smooth state feedback, so is also the system

$$\dot{x} = f_0(x, u_1, \cdots, u_{i-1}, \xi, u_{i+1}, \cdots, u_m) \tag{6.4a}$$

$$\dot{\xi} = v. \tag{6.4b}$$

To this purpose, let $u_1(x), \cdots, u_m(x)$ be a feedback which stabilizes (6.3), choose $u_j = u_j(x)$, $1 \le j \le m$ and $j \ne i$, in (6.4a) and set

$$w = \xi - u_i(x). \tag{6.4c}$$

Zero Dynamics and Semiglobal Stabilization (1990-94)

In 1990, it was finally proven that globally minimum-phase systems in strict normal form can be semiglobally stabilized by high-gain partial state feedback, $u = kH\xi$.

Strict triangularity made possible to use a simple recursive argument

Semiglobal stabilization via high-gain output feedback, $u=k(e)$, with $k(.)$ possibly nonlinear, was understood only at a later time, as a byproduct of the nonlinear version of the small gain theorem for input-to-state stable systems.

Zero Dynamics and Semiglobal Stabilization (1990-94)

SIAM J. CONTROL OPTIM.
Vol. 33, No. 5, pp. 1443 1488, September 1995

TOOLS FOR SEMIGLOBAL STABILIZATION BY PARTIAL STATE AND OUTPUT FEEDBACK*

ANDREW TEEL† AND LAURENT PRALY‡

Abstract. We develop tools for uniform semiglobal stabilization by partial state and output feedback. We show, by means of examples, that these tools are useful for solving a variety of problems. One application is a general result on semiglobal output feedback stabilizability when global state feedback stabilizability is achievable by a control function that is uniformly completely observable. We provide more general results on semiglobal output feedback stabilization as well. Globally minimum phase input–output linearizable systems are considered as a special case. Throughout our discussion we demonstrate the usefulness of considering local convergence separate from boundedness of solutions. For the former we employ a sufficient small gain condition guaranteeing convergence. For the latter we rely on Lyapunov techniques.

The Bible of semiglobal stabilization

Semiglobal stabilization via high-gain output feedback, *u=k(e),* with *k(.)* possibly nonlinear, was understood only at a later time, as a byproduct of the nonlinear version of the small gain theorem for input-to-state stable systems.

Zero Dynamics and Feedback Linearization (1986)

The absence of zero dynamics together with the possibility of achieving relative degree via dynamic feedback (Descusse-Moog) imply the existence of a feedback and coordinates change which transform the system into a fully linear and controllable one.

Proceedings of 25th Conference
on Decision and Control
Athens, Greece • December 1986

A SUFFICIENT CONDITION FOR FULL LINEARIZATION
VIA DYNAMIC STATE FEEDBACK

WA8 - 11:45

A. Isidori*, C.H. Moog**, A. De Luca***

* Visiting Professor, Department of Electrical
and Computer Engineering
Arizona State University
Tempe, Arizona

** Laboratoire d'Automatique
E.N.S.M.-Unite' Associee au C.N.R.S.
Nantes, France

*** Dipartimento di Informatica e Sistemistica
Universita di Roma "La Sapienza"
Roma, Italy

Abstract

The purpose of this paper is to show that any square invertible nonlinear system whose inverse is "state-free" can be turned into a fully linear controllable and observable system by means of dynamic state-feedback and coordinates transformations. A nonlinear system has an inverse which is "state-free" if the value of the input (at time t) can be expressed as a function of the values (at t) of the output and a finite number of its derivatives.

It is known that feedback can induce unobservability if and only if Δ^* (the so-called maximal controlled invariant distribution contained in the differential of h) is nonzero. Thus, the issue is to consider systems which have $\Delta^* = 0$ and to be sure that such a condition is not lost along the decoupling procedure.

Based on a recent understanding of the nonlinear equivalents of the notion of "transmission zeros," described in [6], it is shown here that the right class

Zero Dynamics and Feedback Linearization (1988)

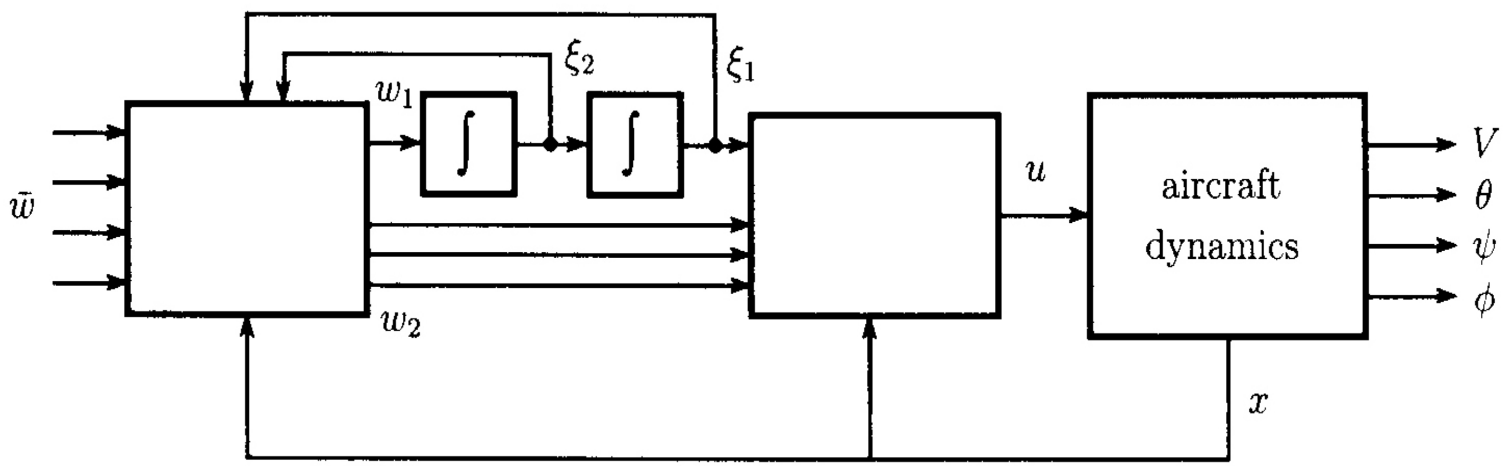
One example for all: feedback linearization of an aircraft dynamics

Automatica, Vol. 24, No. 4, pp. 471–483, 1988
Printed in Great Britain.

0005–1098/88 \$3.00 + 0.00
Pergamon Press plc.

Flight Control Design Using Non-linear Inverse Dynamics*

STEPHEN H. LANE† and ROBERT F. STENGEL†



Zero Dynamics and Feedback Linearization (1988)

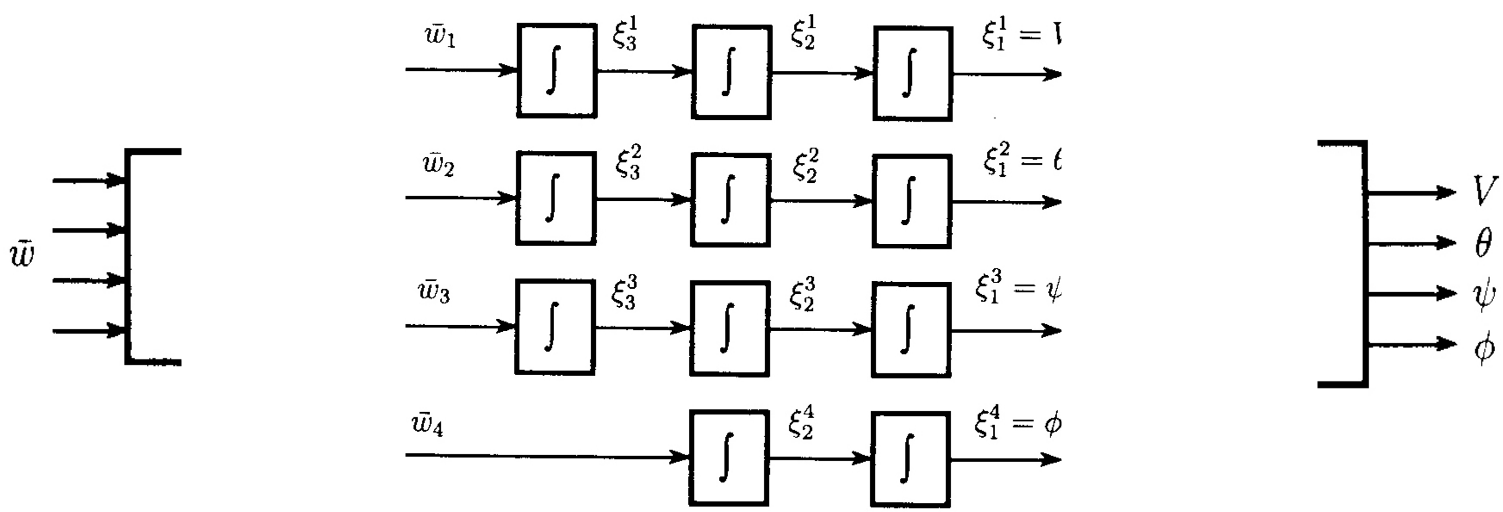
One example for all: feedback linearization of an aircraft dynamics

Automatica, Vol. 24, No. 4, pp. 471–483, 1988
Printed in Great Britain.

0005–1098/88 \$3.00 + 0.00
Pergamon Press plc.

Flight Control Design Using Non-linear Inverse Dynamics*

STEPHEN H. LANE† and ROBERT F. STENGEL†



Zero D. and Stable Noninteracting Control (1987-91)

A trivial way to achieve non-interacting control via static state feedback is to set

$$u = \alpha(x) + \beta(x) v$$

with $\alpha(x)$ and $\beta(x)$ satisfying

$$L_g L_f^{\rho_i} h_i(x) \alpha_i(x) = L_f^{\rho_i+1} h_i(x)$$

$$L_g L_f^{\rho_i} h_i(x) \beta_i(x) = \text{constant matrix.}$$

In this way, though, the system is made maximally unobservable and the “zero dynamics” become internal dynamics of the non-interactive system.
A more refined analysis reveals that, no matter how non-interacting control via static state feedback is achieved, there is always a fixed internal dynamics (a sub-dynamics of the zero dynamics).

Zero D. and Stable Noninteracting Control (1987-91)

IEEE TRANSACTIONS ON AUTOMATIC CONTROL, VOL. 33, NO. 10, OCTOBER 1988 907

Fixed Modes and Nonlinear Noninteracting Control with Stability

We have shown by this theorem that any regular static-feedback $u = \alpha(x) + \beta(x)v$ achieving noninteracting control around x^e is such that the vector field $\tilde{f} = f + g\alpha$, which governs the unforced internal behavior of the closed-loop system, possesses in invariant manifold whose associated dynamics *do not depend upon the particular decoupling control law used*.

In a linear system, the sub-dynamics in question can be suppressed if the feedback is dynamically extended in a suitable way.
Thus, any square invertible linear system can always be rendered interactive with internal stability by dynamic feedback.
This is not the case, though, if the system is nonlinear.
In fact, there is an invariant manifold whose associated dynamics is independent of the (possibly dynamically extended) feedback used.
This is a sheer nonlinear phenomenon.

Zero D. and Stable Noninteracting Control (1987-91)

The dynamics in question was identified in:

SIAM J. CONTROL AND OPTIMIZATION
Vol. 29, No. 3, pp. 609-622, May 1991

006

NONLINEAR NONINTERACTION WITH STABILITY BY DYNAMIC STATE FEEDBACK*

K. G. WAGNER†

Abstract. The dynamic feedback noninteraction with stability problem for square affine systems with a nonsingular decoupling matrix is investigated. First, a characterization of the class of dynamic feedbacks resulting in a noninteractive system is given. It is then shown that these feedbacks induce a certain subdynamic, which is determined by the given system alone and independent of the particular feedback used. The stability of this subdynamic is necessary for the achievement of noninteraction with stability. This condition extends an analogous result on the static feedback noninteraction with stability problem obtained recently by Isidori and Grizzle.

while the synthesis on the noninteractive control law was found in:

IEEE TRANSACTIONS ON AUTOMATIC CONTROL, VOL. 36, NO. 9, SEPTEMBER 1991 1033

A Sufficient Condition for Nonlinear Noninteracting Control with Stability Via Dynamic State Feedback

Stefano Battilotti

Abstract—**In this paper we give a sufficient condition to solve the problem of achieving local noninteracting control with asymptotic stability via dynamic state feedback for a nonlinear affine system. Our condition, which is automatically satisfied in the case of linear systems, although only sufficient, is indeed helpful in overcoming some of the geometric obstructions recently pointed out in the literature.**

scalar output depends on only the ith scalar input is generally known as the nonlinear noninteracting control problem. Historically, necessary and sufficient conditions for solving the nonlinear noninteracting control problem (without stability) by means of regular static state feedback were given in [4], [5], [6], [8], [11], [12], [17]. Dynamic extension has also been investigated in [7], [13], [18], [19].

Zero Dynamics and Output Regulation (1990-2007)

Consider a problem of output regulation

$$\begin{aligned}
\dot{w} &= s(w) \qquad w \in W, \text{ a compact set} \\
\dot{x} &= f(w, x, u) \\
e &= h(w, x) \\
y &= h_{\mathrm{m}}(w, x)
\end{aligned}$$

Suppose $\omega(W) = W$. If the problem of output regulation is solvable, there exists an upper semicontinuous set-valued map

$$\pi \quad : \quad w \in W \quad \mapsto \quad \pi(w) \subset \mathbb{R}^n$$

whose graph $\mathcal{A} = \{(w, x) \in \mathbb{R}^n \times W : x \in \pi(w)\}$ is compact and satisfies:

- $\mathcal{A} \subset \{(w, x) : h(w, x) = 0\}$

- for each $(w, x) \in \mathcal{A}$ the set of all $u \in \mathbb{R}^m$ such that

$$\begin{pmatrix} f(w, x, u) \\ s(w) \end{pmatrix} \in T_{\mathcal{A}}(w, x) \qquad \text{is not empty}$$

Zero Dynamics and Output Regulation (1990-2007)

The set-valued map

$$c \;:\; (w,x) \in \mathcal{A} \;\to\; c(w,x) \subset \mathbb{R}^m$$

defined as

$$c(w,x) = \{u \in \mathbb{R}^m : \begin{pmatrix} f(w,x,u) \\ s(w) \end{pmatrix} \in T_{\mathcal{A}}(w,x)\}$$

is the regulation map of $\mathcal{A}$.

Zero Dynamics and Output Regulation (1990-2007)

If $\pi(w)$ and $c(w)$ are single valued and $\pi(w)$ is differentiable, they satisfy the so-called FBI equations

$$
\begin{aligned}
\frac{\partial \pi}{\partial w} s(w) &= f(w, \pi(w), c(w)) \\
0 &= h(w, \pi(w))
\end{aligned}
$$

If the zero dynamics are globally asymptotically stable, the problem of output regulation can always be solved, in a "semiglobal setting"

SIAM J. CONTROL OPTIM.
Vol. 45, No. 6, pp. 2277–2298

OUTPUT STABILIZATION VIA NONLINEAR LUENBERGER OBSERVERS*

L. MARCONI†, L. PRALY‡, AND A. ISIDORI§

Zero Dynamics and Passivity (1991)

1228 IEEE TRANSACTIONS ON AUTOMATIC CONTROL, VOL. 36, NO. 11, NOVEMBER 1991

Passivity, Feedback Equivalence, and the Global Stabilization of Minimum Phase Nonlinear Systems

Abstract—**In this paper, we derive conditions under which a nonlinear system can be rendered passive via smooth state feedback and we show that, as in the case linear systems, this is possible if and only if the system in question has relative degree 1 and is weakly minimum phase. Then, we prove that weakly minimum phase nonlinear systems having relative degree 1 can be globally asymptotically stabilized by smooth state feedback, provided that suitable controllability-like rank conditions are satisfied. This result incorporates and extends a number of stabilization schemes recently proposed in the literature for global asymptotic stabilization of certain classes of nonlinear systems.**

Zero Dynamics and Limits of Performance (1999)

IEEE TRANSACTIONS ON AUTOMATIC CONTROL, VOL. 44, NO. 4, APRIL 1999

Feedback Limitations in Nonlinear Systems: From Bode Integrals to Cheap Control

M. M. Seron, J. H. Braslavsky, P. V. Kokotović, and D. Q. Mayne

Abstract—Feedback limitations of nonlinear systems are investigated using the cheap control approach. The main result is that in the limit, when the control effort is free, the smallest achievable L_2 norm of the output is equal to the least amount of control energy (L_2 norm) needed to stabilize the unstable zero dynamics. This nonlinear result is structurally similar to an earlier linear result by Qiu and Davison (1993), which, in turn, is connected with a Bode-type integral derived by Middleton (1991).

Zero Dynamics and Limits of Performance (1999)

The cheap control problem

consists of finding a stabilizing state feedback control which minimizes the functional

$$J_\varepsilon = \frac{1}{2}\int_0^\infty [y^T(t)y(t) + \varepsilon^2 u^T(t)u(t)]\,dt \qquad (2)$$

when $\varepsilon > 0$ is small.

As $\varepsilon \to 0$, the optimal value J_ε^* tends to J_0^*, the *ideal performance*.

If the zero dynamics are stable, the ideal performance is zero. If they are unstable, the ideal performance is the minimal energy needed to stabilize the zero dynamics.

$$\begin{aligned} \dot{y} &= f(y,z) + g(y,z)u, \qquad y, u \in R^m \\ \dot{z} &= f_0(z) + g_0(z)y, \qquad z \in R^{n-m} \end{aligned} \qquad (18)$$

In a more recent paper, it was shown that the minimal “gain” needed to stabilize the zero dynamics determines a lower bound on the frequency of a sinusoidal exogenous input to be asymptotically rejected/tracked.

Part II : Challenges and Open Problems (2010 - ?)

- Seeking a Coordinate-free and Global Approach
- Output Redesign
- Steady-State Performance and Regulation
- Where are Multivariable Systems Gone ?

Seeking a Coordinate-free and Global Approach

Systems whose zero dynamics have an asymptotically stable equilibrium, under appropriate assumptions, can be stabilized with a guaranteed region of attraction using (possibly dynamic) output feedback. One of the problems of current interest is how to enhance the theory to the purpose of achieving global stability.

Instrumental, in this context, is the concept of strongly minimum phase system [Liberzon, et al.], introduced to the purpose of providing a formal, and coordinate-free, characterization of the class of systems in which the state is bounded by a function of the outputs and its first r-1 derivatives (r being the relative degree), modulo a decaying term depending on the initial conditions.

Seeking a Coordinate-free and Global Approach

If a system possesses a globally defined normal form, the property in question is simply the property that the inverse dynamics is input-to-state stable (with the respect to the output and its $r-1$ derivatives viewed as inputs).

We briefly summarize hereafter how this notion can be used to the purpose of achieving global stabilization, in a coordinate-free setting. For the sake of more generality, we address the problem of output stabilization, which contains problems of tracking and/or regulation as special cases .

Seeking a Coordinate-free and Global Approach

The definition which follows extends the Definition of Liberzon et al., of a strongly minimum-phase system, to the case of nontrivial compact limit sets.

Instead of seeking estimates of the (euclidian) norm $|x(t)|$ of the state $x(t)$, the definition seeks estimates of the distance $d_{\mathcal{A}}(x(t))$ of $x(t)$ from a compact set $\mathcal{A}$.

Let $\| \cdot \|_{[a,b]}$ denote the supremum norm of a signal restricted to an interval $[a, b]$ and let $\mathbf{y}^{r-1}$ denote the $\mathbb{R}^r$-valued signal

$$\mathbf{y}^{r-1} \triangleq \mathrm{col}(y, y^{(1)}, \ldots, y^{(r-1)}) \; .$$

Finally, let Z denote the set

$$Z \triangleq \{x \in \mathbb{R}^n | h(x) = L_f h(x) = \cdots = L_f^{r-1}(x) = 0\} \; .$$

Seeking a Coordinate-free and Global Approach

The definition that follows expresses, the property that the distance of $x(t)$ from $\mathcal{A}$ is bounded by a suitable function of the output and its first $r-1$ derivatives, modulo a decaying term depending on the initial conditions.

Definition. Consider a relative degree r system and let $\mathcal{A}$ be a compact subset of Z. This system is **strongly minimum-phase with respect to $\mathcal{A}$**, if there exist $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}_\infty$ such that for every initial state $x(0) \in \mathbb{R}^n$ and every admissible input $u(\cdot)$ the corresponding solution $x(t)$ satisfies

$$d_{\mathcal{A}}(x(t)) \leq \max\{\beta(d_{\mathcal{A}}(x(0)), t), \gamma(\|\mathrm{y}^{r-1}\|_{[0,t]})\}$$

as long as it exists.

Seeking a Coordinate-free and Global Approach

Strongly minimum phase systems can be globally stabilized using output feedback. We consider first the case $r = 1$ for which we have:

Theorem 1. *Consider a relative degree* 1 *system and let* $\mathcal{A}$ *be a compact subset of* Z. *Suppose the system is strongly minimum-phase with respect to* $\mathcal{A}$. *Suppose that* $h(x) = 0$ *for all* $x \in \mathcal{A}$. *Then there exists a continuous feedback law* $u = \kappa(y)$ *such that, in the resulting closed-loop system, any* $x(0) \in \mathbb{R}^n$ *produces trajectory that is bounded on* $[0, \infty)$ *and* $\lim_{t\to\infty} d_{\mathcal{A}}(x(t)) = 0$.

In case $r > 1$, a similar result can be obtained by means of a feedback of the form $\kappa(\sum_{i=0}^{r-1} c_i L_f^i h(x) + L_f^{r-1} h(x))$.

Output redesign

Systems possessing a stable zero dynamics can be (robustly) globally/semiglobally stabilized using output feedback.

How can this appealing design paradigm be extended to system whose zero dynamics are unstable ?

A simple idea is to define a new dummy output, for which the zero dynamics would be stable, and use it to derive the feedback stabilizer.

Surprisingly, this naïve idea has applications.

Output redesign

Consider a system

$$\dot{x} = Ax + Bu \qquad\qquad y = Cx$$

with $b := CB \neq 0$ (i.e. its relative degree is 1) augmented by

$$\begin{aligned} \dot{\psi} &= L\psi + Mu \\ \tilde{y} &= y - N\psi \,. \end{aligned}$$

If $NM = 0$, the augmented system still has relative 1, and $V_{\mathrm{a}}^{*} = \{(x,\psi) : Cx = N\psi\}$. The redesigned zero dynamics are given by the restriction of

$$\begin{pmatrix} A & 0 \\ 0 & L \end{pmatrix} - \frac{1}{b}\begin{pmatrix} B \\ M \end{pmatrix}\begin{pmatrix} CA & -NL \end{pmatrix}$$

to V_{a}^{*}. Let P be the projection of $\mathbb{R}^n$ onto $\mathrm{Ker}(C)$ along $\mathrm{Im}(B)$. Then, an easy calculation shows that the redesigned zero dynamics are those of

$$\begin{aligned} (\dot{Px}) &= PA(Px) + \tfrac{1}{b}PABN\psi \\ \dot{\psi} &= L\psi - M\left[\tfrac{1}{b}CA(Px) + (\tfrac{1}{b})^2 CABN\psi + \tfrac{1}{b}NL\psi\right] \end{aligned}$$

Output redesign

The latter can be interpreted as

$$
\begin{aligned}
(\dot{Px}) &= PA(Px) + \tfrac{1}{b}PABu_{\mathrm{a}} \\
y_{\mathrm{a}} &= \tfrac{1}{b}\Big[CA(Px) + (\tfrac{1}{b})CABu_{\mathrm{a}} + \tfrac{1}{b}\dot{u}_{\mathrm{a}}\Big]
\end{aligned}
\qquad (1)
$$

controlled by

$$
\begin{aligned}
\dot{\psi} &= L\psi - My_{\mathrm{a}} \\
u_{\mathrm{a}} &= Ny_{\mathrm{a}}
\end{aligned}
\qquad (2)
$$

If (2) robustly stabilizes (1), a high-gain control

$$u = k(y - N\psi)$$

robustly stabilizes the original system.

Open questions (indeed, for the nonlinear version):

How to extend this (elementary) construction to a higher relative degree case ?

Can the construction be iterated ?

How to take advantage of extra controls and measurements ?

Output redesign

Examples of non-minimum phase systems that can be handled by this method have been recently discussed, in particular, dealing with robust regulation.

It can be shown that if the controlled plant has an arbitrary number of zeros at the origin while all remaining zeros have negative real part, the problem of output regulation can always be solved.

On the other hand, if the controlled plant has a zero with positive real part, the problem can be solved only if the frequencies which characterize the harmonic components of the exogenous input exceed a minimal value determined by the gain needed to stabilize the inner-loop. This is yet another manifestation of why unstable zero dynamics pose limits to the achievable performances.

Output redesign

The procedure described above addresses the special case in which $L_g h(x)=1$. If this is not the case, appropriate modifications are needed.

In this respect, a recently proposed technique for recovering the performance of a feedback linearized system by means of extended high-gain observers [Freidovich-Khalil] and its extension in the context of output redesign, provide a very appealing enhancement of this design paradigm.

Steady-State Performance and Regulation

In the problem of output regulation, a plant

$$\begin{aligned} \dot{w} &= s(w) \\ \dot{x} &= f(w, x, u) \\ y_1 &= h_1(w, x) := e \\ y_2 &= h_2(w, x) \end{aligned}$$

is controlled via

$$\begin{aligned} \dot{x}_c &= f_c(x_c, e, y_2) \\ u &= h_c(x_c, e, y_2). \end{aligned}$$

Consider the associated closed-loop (with output e)

$$\begin{aligned} \dot{w} &= s(w) \\ \dot{x} &= f(w, x, h_c(x_c, h_1(w, x), h_2(w, x))) \\ \dot{x}_c &= f_c(x_c, h_1(w, x), h_2(w, x))) \\ e &= h_1(w, x) \end{aligned}$$

and suppose it has a single-valued limit set, contained in the locus where $h_1(x, w) = 0$.

Steady-State Performance and Regulation

Let this set (the steady-state locus of the closed-loop system) be the graph of $x = \pi(w), x_c = \pi_c(w)$. Then, it is readily seen that, if the problem of regulation is solved, there exist maps $\pi(w)$ and $\pi_c(w)$ satisfying

$$\left.\begin{aligned} L_s\pi(w) &= f(w, \pi(w), \psi(w)) \\ 0 &= h(w, \pi(w)) \end{aligned}\right. \qquad \forall w \in W$$

$$\left.\begin{aligned} L_s\pi_c(w) &= f_c(\pi_c(w), 0, h_2(w, \pi(w))) \\ \psi(w) &= h_c(\pi_c(w), 0, h_2(w, \pi(w)))\,. \end{aligned}\right. \qquad \forall w \in W$$

The first two are known as the regulator, or FBI, equations.

The second equations represent constraints that are not as easy to meet as it was in the case of linear systems. The problem is that we cannot count (yet) on a nonlinear version of the fundamental Lemma used in the case of linear systems (which reposes on linear-algebraic arguments) and this, if $h_2(w, \pi(w))$ is not vanishing (and, even worse, uncertain) is a big problem.

Steady-State Performance and Regulation

Choosing a controller having the structure of Francis-Wonham controller, that is

$$\begin{aligned}\dot{\eta} &= \varphi(\eta) + Ge\\ \dot{\xi} &= \alpha(\eta,\xi) + \beta(e, y_2)\\ u &= \gamma(\eta,\xi) + \delta(e, y_2)\end{aligned}$$

does not help much. In fact, splitting

$$\pi_c(w) = \begin{pmatrix}\sigma(w)\\ \theta(w)\end{pmatrix},$$

the steady-state constraints become in this case

$$\begin{aligned}L_s\sigma(w) &= \varphi(\sigma(w))\\ L_s\theta(w) &= \alpha(\sigma(w),\theta(w)) + \beta(0, h_2(w,\pi(w)))\\ \psi(w) &= \gamma(\sigma(w),\theta(w)) + \delta(0, h_2(w,\pi(w)))\,.\end{aligned} \tag{1}$$

If $h_2(w,\pi(w))$ were identically zero, available results on the theory of output regulation would show that a suitable choice of φ, γ can ensure the existence of a solution $\pi(w)$ for any $\psi(w)$, but the existence of solutions in general is still a challenging open problem.

Steady-State Performance and Regulation

Related design problems:

How to handle the problem of robust model following ?

Important advances in this regard are the recent works dealing with the internal-model approach to problems of consensus and coordination. This is a promising new frontier of research.

How to take advantage of control parameters in the exosystem ? An approach of this kind has been successfully used in the control of an under-actuated “insect-like” flying robot.

Where are Multivariable Systems gone ?

In the late 1960's and early 1970's, analysis of the internal structure of a MIMO system, of how this structure can be affected by feedback and output injection and the use of the results of these analysis to solve problems of stabilization, tracking, decoupling, fault isolation all developed hand-in-hand.

Inexplicably, in the case of nonlinear systems, while most of the methods for the analysis of the internal structure have been conveniently extended (in the 1980's and early 1990's), from the mid 1990s, the study of problems of feedback design for MIMO nonlinear systems has come to a (almost complete) stall.

Why was this the case ?
Was the study of these problems irrelevant?
Or was it too difficult ?

Where are Multivariable Systems gone ?

Attention seemed to be focused only on SISO systems or, at best, on square MIMO systems having identical relative degrees for each output. The option of achieving relative degree by dynamic feedback was soon forgotten.

Apparently, the option of exploiting extra controls or extra measurements to achieve stability, or to solve tracking problems, has never been systematically investigated.

Zero dynamics has been developed as an extension of V^*, not of V^*/R^* and hence not immediately suitable for dealing with systems having more controls than outputs.

Zero structure at infinity affects the solvability of problems of model following. While the nonlinear equivalent of this notion was thoroughly studied, no design technique for feedback design followed.

Where are Multivariable Systems gone ?

The multi-input multi-output version of the (possibly global) stabilization paradigms described earlier is still a largely open domain of research. A recent interesting advance in this domain has been the contributed by Liberzon.

He considers input-affine systems having m inputs and $p \geq m$ outputs, which have the following property: for some integer N, there exist functions $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}_\infty$ such that for every initial state $x(0)$ and every admissible input $u(\cdot)$ the corresponding solution $x(t)$ satisfies

$$|x(t)| \leq \max\{\beta(|x(0)|, t), \gamma(\|\mathrm{y}^{N-1}\|_{[0,t]})\}$$

as long as it exists.

A single-input single-output system having relative degree r which is strongly minimum phase with respect to $\{0\}$ has this property, with $N = r$. Thus, the property in question is a possible extension to multivariable systems of the property of being **strongly minimum phase**.

Where are Multivariable Systems gone ?

Then, Liberzon assumes that the system is **globally left invertible**, in the sense that (the global version of) Singh's inversion algorithm terminates at a stage $k^* \leq m$ in which the input $u(t)$ can be uniquely recovered from the output $y(t)$ and a finite number of its derivatives.

Under this (and another technical) assumption it is shown that a static feedback law $u = \alpha(x)$ exists that globally stabilizes the system. The role of this law is essentially to guarantee that – in the associated closed-loop system – the individual components of the output obey linear differential equations whose characteristic polynomials are Hurwitz.

This result is very promising, and is the more general result available to date dealing with global stabilization of multivariable systems possessing a (strongly) stable zero dynamics. The feedback law proposed, though, is a static state feedback law. The problem of finding a UCO (in the sense of Teel-Praly) feedback law is still open.

Where are Multivariable Systems gone ?

There are classes of MIMO systems, though, in which the design paradigm based on high-gain feedback (from the output and their higher derivatives) is applicable. These can be found, for instance, in the class of those systems in which $L_g h(x)$ is nonsingular. In this case, in fact, if in addition there exist a matrix K and a number $b_0 > 0$ such that

$$[L_g h(x)]^{\mathrm{T}} K + K L_g h(x) \geq b_0 I$$

and if the property outlined above holds for $N = 1$, the global stabilization paradigm described in the previous section is applicable.

In this respect, it should be observed that the property that $L_g h(x)$ is nonsingular could be achieved via a transformation of the type

$$\tilde{y} = \phi(y, y^{(1)}, \ldots, y^{(k)}),$$

A special case of systems for which such transformation exists are the systems whose input-output behavior can be rendered linear via static state-feedback.

Conclusions

The theory of feedback design for MIMO nonlinear systems deserves much more ….

In summary, there are plenty of intellectually challenging and practically relevant problems out there waiting for a response.

Recalling that nothing is more practical than a good theory, don’t let the funding agencies prescribe you what to investigate. Go ahead and take the challenge !

Thank you for your attention !